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Impulsive predator-prey model

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Abstract. In this paper we considered a predator-prey model with state-dependent impulse. We determine periodic solutions for the model without impulses and we prove the existence of nontrivial periodic solution in the case of impulse depending on the state of the model.

1. Introduction

In this work we consider a prey-predator model with impulse effects depending on the state of the model. The impulsive differential equations appear generally in the description of phenomena submitting jumps in the state variables for short time, these big changes are modeled by discrete equations called impulse effects (see [1–3] and [5]).

In this work, we consider a model inspired from [4] where two populations of insects denoted by x and y, respectively, are treated by chemical spray. The evolution of the two populations is governed by a predator-prey model and the effect of the chemical spray is described by impulse effects. The impulse effects in this model are depending on the size of the population density x, when it reaches some threshold h_2 the treatment is used in order to reduce x insects, but the effect of the treatment eliminate also a fraction of the population y, the reduction by treatment on x and y are px and py, respectively.

More specifically, we consider the following model

$$\begin{cases} \dot{x} = x(a - by), & x \neq h_2, \\ \dot{y} = y(-d + \frac{\lambda bx}{1+bhx}), & x \neq h_2, \\ \Delta x = -px, & x = h_2, \\ \Delta y = -qy, & x = h_2, \end{cases} \tag{1}$$

where x and y represent the population densities at time t, the parameters a, b, λ, h and d are positive constants and $p, q \in (0, 1)$.

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2. Analysis of the model without impulses

In the case without impulse effects system (1) is reduced to the following predator-prey model

$$\begin{cases} \dot{x} = x(a - by), \\ \dot{y} = y(-d + \frac{\lambda bx}{1+bhx}), \end{cases} \tag{2}$$

where
$x(t)$: density of the prey at time t,
$y(t)$: density of the predators at time t,
a: growth rate of the prey in the absence of predator,
b: predation rate of the predator on the prey per unit of time,
d: death rate of the predator in the absence of prey,
λ: time to search for prey,
h: time to capture prey.

2.1 Existence of solutions of (2)

Let $z = \begin{pmatrix} x \\ y \end{pmatrix}$, the system (2) is equivalent to

$$\begin{cases} z'(t) = f(z(t)), \\ z(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \end{cases}$$

where

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(a - by) \\ y(-d + \frac{\lambda bx}{1+bhx}) \end{pmatrix}.$$

The function f is locally Lipschitz. By using the Cauchy-Lipschitz theorem we obtain the local existence of solutions.

2.2 Positivity of solutions of (2)

From system (2) we have

$$x(t) = x_0 \exp \int_0^t (a - by(s))ds$$

and

$$y(t) = y_0 \exp \int_0^t \left(-d + \frac{\lambda bx(s)}{1 + bhx(s)}\right) ds.$$

Proposition 2.1: *1.* $(0, 0)$ *is a trivial equilibrium of (2).*
2. If $x_0 > 0$ *(resp.* $y_0 > 0$*) then for all* $t > 0$, $x(t) > 0$ *(resp.* $y(t) > 0$*).*

Remark 2.2: *For* $x_0 = 0$ *(resp.* $y_0 = 0$*), (2) has a semi trivial solution* $(0, y_0 \exp(-dt))$ *(resp.* $(x_0 \exp(at), 0)$*).*

2.3 Global existence

For global existence, we use the following proposition.

Proposition 2.3: *Let* $H : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ *defined for* $x,\ y > 0$ *by*

$$H(x, y) = \frac{\lambda}{h} \ln(1 + bhx) - d \ln x + by - a \ln y. \tag{3}$$

Then H *is the first integral for the system (2), i.e. if* (x, y) *is a solution of (2), then*

$$\forall t > 0,\ H(x(t), y(t)) = cste.$$

Proof: We have

$$\frac{\dot{x}}{\dot{y}} = \frac{x(a - by)}{y\left(-d + \frac{\lambda bx}{1+bhx}\right)}, \text{ for } x \neq x^* \text{ and } y \neq 0.$$

Then

$$-d\frac{\dot{x}}{x} + \frac{\lambda b\dot{x}}{1 + bhx} = a\frac{\dot{y}}{y} - b\dot{y}.$$

Therefore

$$-d \ln x + \frac{\lambda}{h} \ln(1 + bhx) = a \ln y - by + \text{cste}.$$

$\square$

2.4 Boundeness of solutions of (2)

Using Eq. (3) we have

Lemma 2.4: *The solution* $(x(t), y(t))$ *is bounded.*

Proof:
Let $\tilde{f}(x) = f(x) - \frac{1}{2}\left(\frac{\lambda}{h} - d\right) \ln(1 + bhx)$ and $\tilde{g}(x) = g(x) - \frac{b}{2} y$,
where $f(x) = \frac{\lambda}{h} \ln(1 + bhx) - d \ln x$ and $g(x) = by - a \ln y$.
We have $\tilde{f}'(x) = -\frac{d}{x} + \frac{\frac{1}{2}\left(\frac{\lambda}{h}+d\right)hb}{1+bhx} = \frac{\left(\frac{\lambda}{h}-d\right)hbx-2d}{2x(1+bhx)}$.
Then there exist $A > 0$ such that $\forall x > A,\ \tilde{f}(x) > 0$ *i.e.*

$$\forall x > A,\ f(x) > \frac{1}{2}\left(\frac{\lambda}{h} - d\right) \ln(1 + bhx).$$

Similarly, $\tilde{g}'(y) = \frac{b}{2} - \frac{a}{y} = \frac{by-2a}{2y}$.
Then there exist $B > 0$ such that $\forall y > B,\ \tilde{g}(y) > 0$ *i.e.*

$$\forall y > B,\ g(y) > \frac{b}{2}y.$$

Thus if (x, y) is outside the compact $[0, A] \times [0, B]$,

$$H(x, y) = f(x) + g(x) > \frac{1}{2}\left(\frac{\lambda}{h} - d\right) \ln(1 + bhx) + \frac{b}{2}y.$$

We deduce that

$$0 < x(t) < \max\left\{A, \frac{\exp\left(\frac{2hH(x_0,y_0)}{\lambda-dh}\right) - 1}{bh}\right\}$$

and

$$0 < y(t) < \max\left\{B, \frac{2}{b}H(x_0, y_0)\right\}.$$

□

2.5 Stability equilibria of (2)

The system (2) admits a trivial equilibrium $E_0 = (0,0)$. In addition, for $\lambda > dh$, there exist another nontrivial equilibrium $E^* = \left(\frac{d}{\lambda b - dbh}, \frac{a}{b}\right)$.

Theorem 2.5:
1. The trivial equilibrium point E_0 is unstable.
2. The nontrivial equilibrium point E^* is stable.

Proof:

1. Let $f_1(x, y) = x(a - by)$, $f_2(x, y) = y(-d + \frac{\lambda bx}{1+bhx})$, and $f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_1(x, y) \end{pmatrix}$.
   We have
   $$Df(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by & -bx \\ \frac{\lambda yb}{(1+bhx)^2} & -d + \frac{\lambda bx}{1+bhx} \end{pmatrix}$$
   and $Df(0,0) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}$.
   The eigenvalues of this matrix are $\lambda_1 = a > 0$ and $\lambda_2 = -d < 0$. Therefore the equilibrium point $(0,0)$ is unstable.
2. We have $Df(\frac{d}{\lambda b - dbh}, \frac{a}{b}) = \begin{pmatrix} 0 & -\frac{db}{\lambda b - dbh} \\ \frac{a(\lambda b - dbh)^2}{\lambda b^2} & 0 \end{pmatrix}$.
   Since $\lambda > dh$, the matrix admits two eigenvalues $\lambda_1 = i\sqrt{\frac{ad(\lambda b - dbh)}{\lambda}}$ and $\lambda_2 = -i\sqrt{\frac{ad(\lambda b - dbh)}{\lambda}}$ with real part equal to zero. In this case we can't deduce the stability of E^*.
   Let V be a Lyapunov function defined by
   $$V(x, y) = H(x, y) - H\left(\frac{d}{b(\lambda - dh)}, \frac{a}{b}\right).$$
   Then $V(x, y) > 0$ for all $x, y \in \mathbb{R}_+ / \left\{\left(\frac{d}{b(\lambda - dh)}, \frac{a}{b}\right)\right\}$, $V\left(\frac{d}{b(\lambda - dh)}, \frac{a}{b}\right) = 0$ and $V'(x, y) = 0$.
   Therefore, we conclude neutral stability of the equilibrium E^*. □

2.6 Periodicity solutions of (2)

Theorem 2.6: *All solutions of system (2) are periodic.*

Proof: From Fig. 1, it shows four zones, denoted by I, II, III and IV, in which x and y are monotone.
Our proof consists to follow a trajectory through these areas to show its periodicity (see Fig. 2).
We can prove easily that all trajectories pass through the four zones successively.
We can prove that $x_0(t_4) = x_0(t_0)$ (see Fig. 2), which prove that trajectories are periodic. □

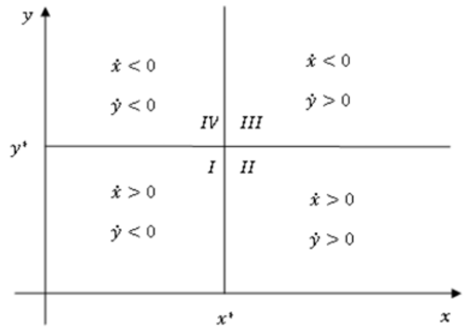

Figure 1. Phase portraits for Lotka Volterra system (2).

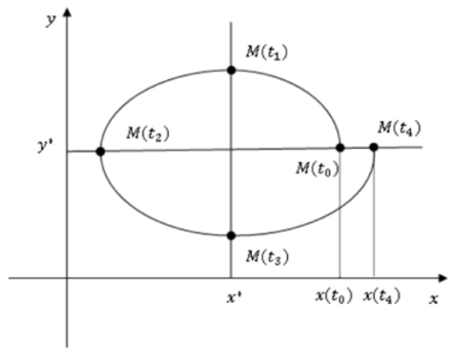

Figure 2. Trajectory of the solution in phase space.

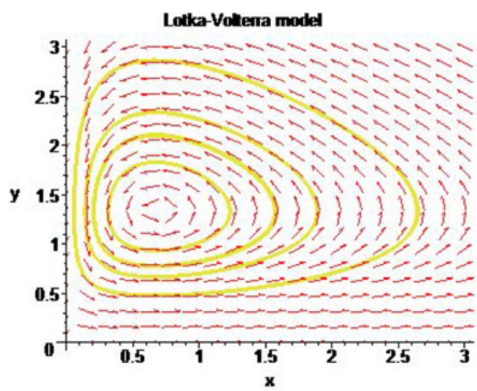

Figure 3. Closed phase plane trajectories for the Lotka-Volterra system (2), with $a = 0.8$, $b = 0.6$, $\lambda = 0.5$, $d = 0.2$, $h = 0.02$.

2.7 Numerical simulations

When $x(t)$ and $y(t)$ are plotted individuals *versus* t, we see that periodic variation of the predator population $y(t)$ lags slightly behind the prey population $x(t)$ (see Fig. 4).

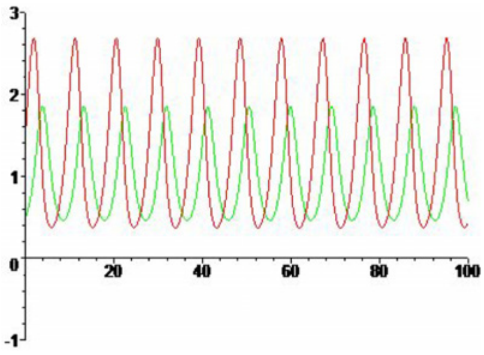

Figure 4. Periodic solutions for the prey $x(t)$ and the predator $y(t)$ for the Lotka-Volterra system (2) with initial conditions $x(0) = 1.5$, $y(0) = 0.5$, $\lambda = 2$, $a = b = d = h = 1$.

3. The model with impulse effects

3.1 Existence and stability of trivial solution of (1)

Let $y(t) = 0$ for $t \in [0, \infty)$, then from system (1) we have

$$\begin{cases} \dot{x} = ax(t) & x \neq h_2 \\ \Delta x = x(t^+) - x(t) = -px(t) & x = h_2. \end{cases} \tag{4}$$

Let $x_0 = x(0) = (1 - p)h_2$, then the solution of equation $\frac{dx}{dt} = ax(t)$ is $x(t) = (1 - p)h_2 \exp(at)$. We have $x(T) = h_2 = (1 - p)h_2 \exp(aT) = h_2$, then $T = -a^{-1} \ln(1 - p)$. Since $x(T^+) + x(T) = -px(T) = (1 - p)x(T)$ we have $x(T^+) = x_0$.

This means that (1) has the following semi-trivial periodic solution

$$(x_s(t), y(t)) = ((1 - p)h_2 \exp(at), 0), \qquad \text{for } 0 \leq 0 < T. \tag{5}$$

We have the following result.

Theorem 3.1: *If* $0 < \mu = (1 - q)(1 - p)^{\frac{d}{a}} \left(\frac{1+bhh_2}{1+(1-p)bhh_2}\right)^{\frac{\lambda}{ah}} < 1$, *then* $(x_s, 0)$ *is orbitally asymptotically stable semi-trivial periodic solution of (1).*

Proof:

Consider the autonomous system with impulse effects

$$\begin{cases} \dot{x} = P(x, y) & \varphi(x, y) \neq 0, \\ \dot{y} = Q(x, y) & \varphi(x, y) \neq 0, \\ \Delta x = \xi(x, y) & \varphi(x, y) = 0, \\ \Delta y = \eta(x, y) & \varphi(x, y) = 0, \end{cases} \tag{6}$$

where $P(x, y)$ and $Q(x, y)$ are continuous differential functions defined on $\mathbb{R}^2$, and φ is a sufficiently smooth function with $grad\varphi(x, y) \neq 0$.

Let $(\alpha(t), \beta(t))$ be a positive T-periodic solution of (6). By Corollary 2 of Theorem 1 given in [5], we have the following Lemma.

Lemma 3.2: *[5] If the Floquet multiplier satisfies the condition $|\mu| < 1$, where*

$$\mu = \prod_{j=1}^{n} k_j \exp\left[\int_0^T \left(\frac{\partial P(\alpha(t), \beta(t))}{\partial x} + \frac{\partial Q(\alpha(t), \beta(t))}{\partial y}\right) dt\right]$$

with

$$k_j = \frac{P_+\left(\frac{\partial \eta}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial \eta}{\partial x}\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial x}\right) + Q_+\left(\frac{\partial \xi}{\partial x}\frac{\partial \varphi}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\right)}{\frac{\partial \varphi}{\partial x}P + \frac{\partial \varphi}{\partial y}Q}$$

and P, Q, $\frac{\partial \eta}{\partial y}$, $\frac{\partial \eta}{\partial x}$, $\frac{\partial \varphi}{\partial y}$, $\frac{\partial \varphi}{\partial x}$, $\frac{\partial \xi}{\partial x}$, $\frac{\partial \xi}{\partial y}$ are calculated at the point $(\alpha(\tau_j), \beta(\tau_j))$, $P_+ = P(\alpha(\tau_j^+), \beta(\tau_j^+))$, $Q_+ = Q(\alpha(\tau_j^+), \beta(\tau_j^+))$ and $\tau_j (j \in n)$ is the time of the j-th jump. Then, $(\alpha(t), \beta(t))$ is orbitally asymptotically stable.

To obtain μ corresponding to (1), we do the following computations.
We have

$$k_1 = \frac{(-q+1)(1-p)ah_2}{ah_2} = (1-p)(1-q).$$

Therefore

$$\begin{aligned} I &= \int_0^T \left(\frac{\partial f_1((1-p)h_2\exp(at), 0)}{\partial x} + \frac{\partial f_2((1-p)h_2\exp(at), 0)}{\partial y}\right)dt \\ &= -\ln(1-p) + \tfrac{d}{a}\ln(1-p) + \tfrac{\lambda}{ah}\ln\left(1 + bh(1-p)h_2\tfrac{1}{1-p}\right) - \tfrac{\lambda}{ah}\ln(1 + bh(1-p)h_2) \end{aligned}$$

Then

$$\mu = (1-q)(1-p)^{\frac{d}{a}}\left(\frac{1 + bhh_2}{1 + bhh_2(1-p)}\right)^{\frac{\lambda}{ah}}.$$

So, $|\mu| < 1$ is equivalent to

$$(1-q)(1-p)^{\frac{d}{a}}\left(\frac{1 + bhh_2}{1 + bhh_2(1-p)}\right)^{\frac{\lambda}{ah}} < 1,$$

which gives

$$q > 1 - \frac{1}{(1-p)^{\frac{d}{a}}\left(\frac{1+bhh_2}{1+bhh_2(1-p)}\right)^{\frac{\lambda}{ah}}}. \tag{7}$$

From (7) we have $|\mu| < 1$, then $(x_s(t), 0)$ is orbitally asymptotically stable. □

3.2 Analysis of exponential stability of $(x_s, 0)$

Let Φ be the flow associated to (1), we have

$$X(t) = \Phi(t, X_0), \quad 0 < t \le T_1, \tag{8}$$

where $X_0 = X(0)$. We assume that the flow Φ applies up time T_1. So, $X(T) = \Phi(T, X_0)$.

We denotes $X(T^+)$ the state of the population after the chemical treatment, then $X(T^+)$ is determined in terms of $X(T)$. We have $X(T^+) = \theta(X(T)) = \theta(\Phi(T, X_0))$.

Let Ψ be operator defined by

$$\Psi(T, X) = (\Psi_1(T, X), \Psi_2(T, X)) = \theta(\Phi(T, X)) \tag{9}$$

and denote by $D_X\Psi$ the derivative of Ψ with respect to X.

Then, X is a $T-$periodic solution of (1) if and only if

$$\Psi(T, X_0) = X_0,$$

that is X_0 is a fixed point de $\Psi(T, .)$. It is exponentially stable if and only if the spectral radius $\rho(D_X\Psi(T, .))$ is strictly less than 1.

Denote by $x_0 = x_s(0)$, then $(x_0, 0)$ is the initial condition of $(x_s, 0) = \nu$ and $\nu(0) = (x_0, 0)$.

So, $X(T) = \Phi(T, X_0)$, where $X_0 = ((1 - p)h_2, 0)$. Then

$$\Phi(T, X_0) = \Phi(T, ((1 - p)h_2, 0)).$$

Denote by $X(T^+)$ the state of the population after the chemical treatment. We have $X(T^+) = \theta(X(T)) = \theta(\Phi(T, X_0))$, *i.e.*

$$\theta(\Phi(T, X_0)) = \theta(\Phi(T, ((1 - p)h_2, 0)).$$

Let Ψ be the operator defined by

$$\Psi(T, X) = (\Psi_1(T, X), \Psi_2(T, X)) = \theta(\Phi(T, X)) \tag{10}$$

with $\Phi(t) = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$, $\theta(t) = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$, $\theta_1 = (1 - p)x$ and $\theta_2 = (1 - q)y$.

We have

$$D_X(\theta(\Phi(T, X))) = \begin{pmatrix} \frac{\partial\theta_1}{\partial x} & \frac{\partial\theta_1}{\partial y} \\ \frac{\partial\theta_2}{\partial x} & \frac{\partial\theta_2}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_1}{\partial y} \\ \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_2}{\partial y} \end{pmatrix}.$$

Then

$$\frac{\partial}{\partial t}(D_X(\theta(\Phi(T, (1 - p)h_2, 0)))) = \begin{pmatrix} a & -bx \\ 0 & -d + \frac{\lambda bx}{1+bhx} \end{pmatrix} \begin{pmatrix} \frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_1}{\partial y} \\ \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_2}{\partial y} \end{pmatrix}.$$

We obtain that

$$\begin{cases} \dfrac{\partial\phi_1'}{\partial x} = a\dfrac{\partial\phi_1}{\partial x} - b((1 - p)h_2\exp(at))\dfrac{\partial\phi_2}{\partial x}, & \dfrac{\partial\phi_1}{\partial x}(0, (1 - p)h_2, 0) = 1, \\ \dfrac{\partial\phi_1'}{\partial y} = a\dfrac{\partial\phi_1}{\partial y} - b((1 - p)h_2\exp(at))\dfrac{\partial\phi_2}{\partial y}, & \dfrac{\partial\phi_1}{\partial y}(0, (1 - p)h_2, 0) = 0, \\ \dfrac{\partial\phi_2'}{\partial x} = \left[-d + \dfrac{\lambda b((1 - p)h_2\exp(at))}{1 + bh((1 - p)h_2\exp(at))}\right]\dfrac{\partial\phi_2}{\partial x}, & \dfrac{\partial\phi_2}{\partial x}_{21}(0, (1 - p)h_2, 0) = 0, \\ \dfrac{\partial\phi_2'}{\partial y} = \left[-d + \dfrac{\lambda b((1 - p)h_2\exp(at))}{1 + bh((1 - p)h_2\exp(at))}\right]\dfrac{\partial\phi_2}{\partial y}, & \dfrac{\partial\phi_2}{\partial y}(0, (1 - p)h_2, 0) = 1. \end{cases} \tag{11}$$

for $0 < t < T$.

Then

$$\begin{cases} \dfrac{\partial \phi_1}{\partial x} = \exp(at), & \dfrac{\partial \phi_1}{\partial x}(0,(1-p)h_2,0) = 1, \\[2ex] \dfrac{\partial \phi_1}{\partial y} = \left(\displaystyle\int_0^t -b(1-p)h_2 \exp(-ds) \right. & \dfrac{\partial \phi_1}{\partial y}(0,(1-p)h_2,0) = 0, \\ \qquad \left. \left(\dfrac{1+bh(1-p)h_2 \exp(as)}{1+bh(1-p)h_2} \right)^{\frac{\lambda}{ah}} ds \right) \exp(at), & \\[2ex] \dfrac{\partial \phi_2}{\partial x} = 0, & \dfrac{\partial \phi_2}{\partial x}(0,(1-p)h_2,0) = 0, \\[2ex] \dfrac{\partial \phi_2}{\partial y} = \exp(-dt) \left(\dfrac{1+bh(1-p)h_2 \exp(at)}{1+bh(1-p)h_2} \right)^{\frac{\lambda}{ah}}, & \dfrac{\partial \phi_2}{\partial y}(0,(1-p)h_2,0) = 1. \end{cases} \tag{12}$$

We have

$$\begin{aligned} D_X(\theta(\Phi(T,X))) &= \begin{pmatrix} (1-p)\dfrac{\partial \phi_1}{\partial x}(T) & (1-p)\dfrac{\partial \phi_1}{\partial y}(T) \\[2ex] (1-q)\dfrac{\partial \phi_2}{\partial x}(T) & (1-q)\dfrac{\partial \phi_2}{\partial y}(T) \end{pmatrix} \\ &= \begin{pmatrix} (1-p)\exp(aT) & K(T) \\[2ex] 0 & (1-q)\exp(-dT)\left(\dfrac{1+bh(1-p)h_2 \exp(aT)}{1+bh(1-p)h_2} \right)^{\frac{\lambda}{ah}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & K(T) \\[2ex] 0 & (1-q)\exp(-dT)\left(\dfrac{1+bh(1-p)h_2 \exp(aT)}{1+bh(1-p)h_2} \right)^{\frac{\lambda}{ah}} \end{pmatrix}. \end{aligned}$$

We have the following results

Theorem 3.3: *The semi trivial periodic solution* $(x_s, 0)$ *is not exponentially stable.*

4. Conclusion

In this work we have studied the stability of some periodic solutions of an impulsive prey-predator model. We have found sufficient conditions for orbital stability of semi trivial periodic solution and proved that it is not exponentially stable. It will be interesting to see the eventual bifurcation of periodic solutions to study their stability and to give some numerical simulations to illustrate the results obtained.

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