An Efficient Lattice-based Distributed IBE

Lu YIN, Quan ZHANG, and Rui-Lin LI

School of Electronic Science and Engineering, National University of Defense Technology, NUDT, Changsha, China

ruruyin@163.com, quanzhang@nudt.edu.cn, securitylr@163.com

Abstract—This paper improves the lattice-based secret sharing method presented by Bansarkhani and Meziani and combines the method with the IBE based on the standard LWE problem constructing an efficient lattice-based distributed decryption IBE. Our construction avoids the weakness in the traditional method like Lagrange interpolation, Blakley's space geometry method or Chinese remainder theorem method. Through theoretical analysis, compared with the bilinear pairing based distributed IBE, our scheme refrains from the complex pairing operation and has less calculation cost. A proof of security holds in the standard model.

1 Introduction

In 1984, Shamir proposed the concept of Identity-Based Cryptography(IBC)[1]. The characteristic of IBC is that a user's public key can be calculated by an open algorithm according to some public information like the user's name, e-mail address or phone number, etc. The property simplifies public key distribution. In this aspect, the IBC has an advantage over traditional public key cryptosystem. Since the user's private key must be created by a trusted third party PKG, it results in the inherent key escrow problem and restricts the adoption of IBC.

In 2003, Boneh and Franklin designed the first practical Identity-Based Encryption scheme (IBE)[2] using bilinear maps on elliptic curves and proposed an approach to mitigate the key escrow problem. In this approach, the IBE system's master key is distributed to multiple PKGs using technique of Shamir (t, n) threshold secret sharing[3]. The BasicIdent scheme is briefly described below:

Let G_1 be an additive cyclic group of order p and G_T be a multiplicative group of order p , P is a generator of G_1 , $\hat{e}: G_1 \times G_1 \rightarrow G_T$ is a bilinear map.

Let H_1 and H_2 be two cryptographic hash functions $H_1: \{0,1\}^* \rightarrow G_1$, $H_2: G_T \rightarrow \{0,1\}^{l_m}$, where l_m represents the bit length of the message.

1) Setup. The master PKG chooses a random master key $s \in \mathbf{Z}_p^*$ and sets $P_{pub} = sP$. $Q_{id} = H_1(id)$ is derived from the user's public key id. The master PKG then distributes to n PKGs in a Shamir (t, n) threshold secret sharing fashion, each PKG_i gets one share s_i .

2) Encrypt. The sender picks a random $r \in \mathbf{Z}_p^*$ and encrypts the plaintext under the public key id, setting the ciphertext to be

$$C = (U, V) = (rP, m \oplus H_2(\hat{e}(Q_{id}, rP_{pub})))$$

3) Decrypt. After receiving the ciphertext C , user i sends decryption request $R = (id, U)$ to PKG_i . When responding to the request more than t (t is the threshold value) PKG_i

sends $\hat{e}(s_i Q_{id}, U)$. User i can then construct plaintext m via $m = V \oplus H_2(\prod_{i=1}^t \hat{e}(\lambda_i s_i Q_{id}, U)) = V \oplus H_2(\hat{e}(s Q_{id}, U))$ where

the λ_i 's are the Lagrange coefficients.

In this scheme, each distributed PKG can either generate the decryption share $\hat{e}(s_i Q_{id}, U)$ online or provide the partial private key $s Q_{id}$ for the user. Of course, the complete private key can be obtained when the collusion of more than t PKGs happens. Thus the honesty of these entities required

in the application is comparable to PKG, which is the essential reason for these entities to be PKGs. Distributed PKGs required to be involved in the decryption online all the time, it violates the basic requirement of IBE, PKG can be offline after generating the private keys for the users in its domain, that Shamir proposed[1].

To solve this problem, in PKC 2004, Baek and Zheng Yuliang [4] gave a better approach to share a private key associated with an identity among n entities which appointed by user rather than sharing a master key. So these n entities are downgraded to pure decryption servers without the function of PKG. Because these servers have users' trust and the online/offline state controlled by the user, the problem of distributed PKGs online all the time in the decryption period can be solved.

However, due to the distributed decryption IBE in [4] is based on bilinear pairing and Shamir threshold secret sharing, considering the complex computation of pairing operation, it affects the overall efficiency of the protocol. In addition, from the point of view of security, their scheme is hard to resist quantum attacks.

In recent years, constructing IBE from some hard problems in lattice attracts much attention. In 2008, under the random oracle model Gentry, Peikert and Vaikuntanathan[5] constructed the first IBE based on the LWE problem in the lattice. Subsequently, Cash, Hofheinz and Kiltz[6], Agarwal, Boneh and Boyen[7] proposed lattice based IBE in the standard model. Since the operation on the lattice is mostly linear and the calculation speed is fast, so we combine the idea of distributed decryption presented by Baek and Zheng Yuliang[3] with efficient lattice IBE[7] to construct an efficient lattice-based distributed IBE.

If our construction follows the Lagrange polynomial interpolation used in Baek and Zheng Yuliang's scheme to distribute a secret n -dimensional vector, the communication cost will increase because the order of each vector component has to be recorded to recover the n -dimensional vector. Therefore, our construction uses the secret sharing approach presented by R. Bansarkhani and M. Mezziani[8] to distribute the user's private key which is an n -dimensional vector in the lattice based IBE. However we argue that there is a scalability problem in the secret sharing construction of R. Bansarkhani and M. Mezziani. We propose to employ a variable number of columns of a matrix to distribute the n -dimensional vector so that the number of decryption servers is available within a certain range.

The advantages of our lattice based distributed IBE are listed as follows:

- In the distribution and decryption share construction/reconstruction stage, the new secret sharing technique based on the variable matrix mitigates the scalability problem and avoids the weakness of recording the order of each vector component while recovering the whole n -dimensional vector using traditional methods like Lagrange polynomial interpolation[2], Blakley's geometry of space scheme[9] or Chinese remainder theorem method[10], etc.

- The secret shares and the reconstructed private key can be verified by a lattice-based hash function.

- The user makes a decryption request to each decryption server after receiving the ciphertext, then the decryption servers calculate decryption shares associated with the ciphertext and secret shares they possess. Through a secure channel, the corresponding decryption shares are delivered to the user. The whole encryption and decryption process including secret shares distribution, the generation of decryption shares or recovering the plaintext uses simple matrix operations. Through the theoretical analysis, compared with the distributed bilinear pairing-based IBE, our construction can not only resist quantum attacks, and also reduce the cost of calculation.

2 Preliminaries

2.1 Notation

The set of real numbers is defined as $\mathbf{R}$ and the integers as $\mathbf{Z}$. We denote a vector x and the i_{th} component of x is written as $x^{(i)}$.

Let S be a set of vectors $S = \{s_1, \dots, s_k\}$, where $s_1, \dots, s_k \in \mathbf{R}^m$. We define the L_2 length of the longest vector in S as $\|S\|$, namely $\|S\| := \max_{1 \leq i \leq k} \|s_i\|$. We let $\tilde{S} := \{\tilde{s}_1, \dots, \tilde{s}_k\}$ denote the Gram-Schmidt orthogonalization of the vectors $s_1, \dots, s_k$ and $\|\tilde{S}\|$ denote the Gram-Schmidt norm of S .

The statistical distance between two distributions X and Y over a finite set Ω is defined as:

$$\Delta(X; Y) := \frac{1}{2} \sum_{s \in \Omega} |\Pr[X = s] - \Pr[Y = s]| \quad (1)$$

Two distributions are statistically close if $\Delta(X; Y)$ is negligible. Let U_Ω be a uniform distribution over Ω . If $\Delta(X; U_\Omega) \leq \delta$, we say X is δ -uniform over Ω .

2.2 Integer Lattices

An integer lattice is a discrete subgroup of $\mathbf{Z}^m$. Our article will use the q -ary integer lattices to construct the lattice-based IBE. The q -ary integer lattices contain $q\mathbf{Z}^m$ as a sublattice where q is some integer. Let q be a prime, $A \in \mathbf{Z}^{n \times m}_q$ be arbitrary, $u \in \mathbf{Z}^n_q$, define the full-rank q -ary lattice:

$$\Lambda_q^\perp(A) := \{e \in \mathbf{Z}^m \mid Ae = 0 \pmod{q}\} \quad (2)$$

The coset or shifted lattice is defined to be:

$$\Lambda_q^u(A) := \{e \in \mathbf{Z}^m \mid Ae = u \pmod{q}\}. \quad (3)$$

$$\text{For } t \in \Lambda_q^u(A), \Lambda_q^u(A) = \Lambda_q^\perp(A) + t.$$

2.3 Discrete Gaussians

For any $c \in \mathbf{R}^m$, real $\sigma > 0$, the m -dimensional Gaussian function $\rho_{\sigma,c}(x)$ is defined by:

$$\forall x \in \mathbf{R}^m, \rho_{\sigma,c}(x) = \exp\left(-\pi \frac{\|x-c\|^2}{\sigma^2}\right). \quad (4)$$

For any lattice $\Lambda \subset \mathbf{R}^m$, $\rho_{\sigma,c}(\Lambda) = \sum_{x \in \Lambda} \rho_{\sigma,c}(x)$. We define the probability mass function of the discrete Gaussian distribution $D_{\Lambda,\sigma,c}$:

$$\forall x \in \Lambda, D_{\Lambda,\sigma,c}(x) = \frac{\rho_{\sigma,c}(x)}{\rho_{\sigma,c}(\Lambda)}. \quad (5)$$

Gentry etc.[5] proposed the Gaussian sampling algorithm that, given a basis of an arbitrary lattice, samples lattice points from a discrete Gaussian distribution.

2.4 Learning with Errors

For an integer q , a positive integer n and a distribution $\mathcal{X}$ on $\mathbf{Z}_q^n$, learning with errors problem $LWE_{q,\mathcal{X}}$ aims to distinguish between the distribution $A_{s,\mathcal{X}}$ ($s \in \mathbf{Z}_q^n$ is some uniform secret value) and the uniform distribution on $\mathbf{Z}_q^n \times \mathbf{Z}_q$. The instance of LWE problem[7] contains access to an unspecified oracle $\mathcal{O}$ ($\mathcal{O}_s$ or $\mathcal{O}_\mathcal{S}$). $\mathcal{O}_s$ denotes a noisy pseudo-random sampler, it outputs samples $(u_i, v_i) = (u_i, u_i^T s + x_i) \in \mathbf{Z}_q^n \times \mathbf{Z}_q$, where $x_i \in \mathbf{Z}_q$ taken freshly from $\mathcal{X}$ and u_i is uniform in $\mathbf{Z}_q^n$. $\mathcal{O}_\mathcal{S}$ represents a truly random sampler whose output is some truly uniform random samples from $\mathbf{Z}_q^n \times \mathbf{Z}_q$.

We say A algorithm can decide the LWE problem, if $LWE - adv[A] := |\Pr[A^{\mathcal{O}_s} = 1] - \Pr[A^{\mathcal{O}_\mathcal{S}} = 1]|$ is non-negligible for a random $s \in \mathbf{Z}_q^n$.

Regev[11] proved that for certain Gaussian noise distributions $\mathcal{X}$ (denoted $\bar{\Psi}_\alpha$) on $\mathbf{Z}_q^n$, $LWE_{q,\bar{\Psi}_\alpha}$ is as hard as using a quantum algorithm to solve the worst-case SIVP and GapSVP.

2.5 Trapdoor Generation Algorithm

In our construction, we use the TrapGen algorithm proposed by Alwen and Peikert[12] to generate a uniform matrix A , together with a good trapdoor basis T of $\Lambda_q^\perp(A)$.

Theorem 1.[12] Let odd $q \geq 3$, $m \geq \lceil 6n \log q \rceil$, $\delta = 1/3$. There is a probabilistic polynomial time algorithm $\text{TrapGen}(q, n)$ outputs $(A \in \mathbf{Z}_q^{n \times m}, T_A \in \mathbf{Z}_q^{m \times m})$ such that A is statistically close to a uniform matrix, T_A is a basis of $\Lambda_q^\perp(A)$ satisfying $\|T_A\| = O(\sqrt{n \log q})$, $\|T_A\| \leq O(n \log q)$.

2.6 SampleLeft and SampleRight Algorithm

Let matrixes $A, B \in \mathbf{Z}_q^{n \times m}$ and R be a matrix in $\{-1, 1\}^{m \times m}$, matrix $F = (A \mid AR + B) \in \mathbf{Z}_q^{n \times 2m}$. In our construction, we have to sample short vectors in $\Lambda_q^u(A)$ for $u \in \mathbf{Z}_q^n$ as the user's private key using SampleLeft algorithm or SampleRight algorithm for security game.

SampleLeft(A, M_1, T_A, u, σ)[7]: Input a rank n matrix $A \in \mathbf{Z}_q^{n \times m}$, $M_1 \in \mathbf{Z}_q^{n \times m_1}$, a short basis T_A of $\Lambda_q^\perp(A)$, a Gaussian parameter $\sigma > \|\tilde{T}_A\| \cdot \omega(\sqrt{\log(m+m_1)})$ and a vector $u \in \mathbf{Z}_q^n$, let $F_1 = (A \mid M_1)$. This algorithm outputs a vector $e \in \mathbf{Z}_q^{m+m_1}$ satisfying $F_1 \cdot e = u \pmod{q}$, e is sampled from a distribution statistically close to $D_{\Lambda_q^u(F_1), \sigma}$.

SampleRight(A, B, R, T_B, u, σ)[7]: Input $A \in \mathbf{Z}_q^{n \times k}$ and a rank n matrix $B \in \mathbf{Z}_q^{n \times m}$, a matrix $R \in \mathbf{Z}_q^{k \times m}$ (Let $s_R := \|R\| = \sup_{\|x\|=1} \|Rx\|$), T_B is a basis of $\Lambda_q^\perp(B)$ and $u \in \mathbf{Z}_q^n$, the Gaussian parameter $\sigma > \|\tilde{T}_B\| \cdot s_R \omega(\sqrt{\log m})$. This algorithm outputs a vector $e \in \mathbf{Z}_q^{m+k}$, let F_2 be $(A \mid AR + B)$, e is sampled from a distribution statistically close to $D_{\Lambda_q^u(F_2), \sigma}$.

2.7 Lattice-based Hash function

Definition [13] For security parameter λ , we choose a random matrix $M \in \mathbf{Z}_q^{\lambda \times m}$, M_i denotes the i th column of M , for the input $x = (x_1, x_2, \dots, x_m) \in \mathbf{Z}_q^m$, $h_M: \mathbf{Z}_q^m \rightarrow \mathbf{Z}_q^\lambda$ is defined as

$$H_M(x) = Mx \pmod{q} = \sum_{i=1}^m x_i M_i \pmod{q}. \quad (6)$$

Let μ be polynomial in λ , denoted as $\mu = \text{poly}(\lambda)$, q be a λ -bits modulo, $m = 2\lambda^2$. When the input $x \in \{0, 1, \dots, \mu\}^m$, the LBH is collision-free and its security can be reduced to $GAPSPV_{14\pi\sqrt{\lambda}\beta}$.

2.8 Full-rank Differences Function

In our construction, we first use a collision resistant hash to convert the identity in $\{0, 1\}^*$ to elements in $\mathbf{Z}_q^n$. Then use the full-rank difference function encoding the $id = (d_1, d_2, \dots, d_{n-1}) \in \mathbf{Z}_q^n$ to a matrix in $\mathbf{Z}_q^{n \times n}$. Let the polynomial $g_d(X)$ be $g_d(X) = \sum_{i=0}^{n-1} d_i x^i \in \mathbf{Z}_q[X]$, $\text{coeffs}(g_d)$ means the n -dimensional vector of coefficients of g_d , $f \in \mathbf{Z}_q[X]$ be some polynomial of degree n , the Full-Rank Differences function (FRD) $H(id)$ as the definition in [7],

$$H(id) = \begin{bmatrix} \text{coeffs}(g_d) \\ \text{coeffs}(X \cdot g_d \bmod f) \\ \text{coeffs}(X^2 \cdot g_d \bmod f) \\ \dots \\ \text{coeffs}(X^{n-1} \cdot g_d \bmod f) \end{bmatrix} \in \mathbf{Z}_q^{n \times n}. \quad (7)$$

3 The Main Construction

3.1 Description of Generic Distributed IBE

The standard IBE includes four algorithms: Setup, Extract, Encrypt and Decrypt. Our construction introduces three new algorithms Distribution, Construct Dec and VerS.

The PKG runs the algorithm Setup to generate the master key, public key and all the public parameters.

The PKG runs the Extract algorithm after it receives the user's private key extraction request to generate the private key related with the user's identity.

In the application, the user can choose the number of decryption servers. Then the PKG or other authorized party that possesses the private key from PKG can run the Distribution algorithm to distribute the private key into k decryption servers Γ_i for $1 \leq i \leq k$ ($k \leq 2m$ in our distributed IBE). The k decryption servers use a broadcast channel in common, each secret share is kept secretly by the decryption server. In the Distribution algorithm, an improved secret sharing method will be applied to generate shares of the private key and verify the validity of shares.

The i th server Γ_i receives its secret share and runs the algorithm VerS to verify the validity of the secret share.

Any user who wants to send message to others can run the algorithm Encrypt after obtaining the public parameter and the receiver's identity. The legal receiver gets the ciphertext

and sends a decryption request to the decryption servers. Then the decryption server Γ_i implements the ConstructDec algorithm to generate corresponding decryption share and sends the decryption share to the user. The user can run the Decrypt algorithm combining the received valid decryption shares to recover the plaintext.

3.2 Our Distributed IBE

-Setup(n): On input n as a security parameter, set parameter, set q, m, σ, α (parameters set will be specified in the next section). First, this algorithm runs TrapGen(q, n) generating a uniformly random matrix $A_0 \in \mathbf{Z}_q^{n \times m}$ altogether with a short basis $T_{A_0} \in \mathbf{Z}_q^{m \times m}$ for $\Lambda_q^\perp(A_0)$. T_{A_0} is the master key. Then it chooses two uniformly random matrixes $A_1, B \in \mathbf{Z}_q^{n \times m}$ and a uniformly random vector $u \xleftarrow{R} \mathbf{Z}_q^n$. Finally, it returns the public parameters $PP = (A_0, A_1, B, u)$ while keeping the master key $MK = (T_{A_0})$ secretly.

-Extract(id, MK, PP): Given an identity $id \in \mathbf{Z}_q^n$, a master key MK and public parameters PP , this algorithm uses $\text{SampleLeft}(A_0, A_1 + H(id)B, T_{A_0}, u, \sigma)$ to sample a vector $e \in \mathbf{Z}_q^{2m}$ as the private key SK_{id} . H is a full-rank differences map. Let $F_{id} := (A_0 \parallel A_1 + H(id)B)$, clearly the private key satisfies $F_{id} \cdot e = u \pmod{q}$.

-Distribution($SK_{id}, k, q, 2m$): Given a private key SK_{id} and the number of decryption servers k , this algorithm first uses function TrapGen by choosing n', q and $2m \geq [6n' \log q]$ as input to obtain a basis $T \in \mathbb{Z}_q^{2m \times 2m}$. Then it selects arbitrary k linearly independent column-vectors of T forming a matrix $Q := \{q_1, \dots, q_k\} \in \mathbb{Z}_q^{2m \times k}$ satisfying $\text{Rank}(Q) = \text{Rank}([Q \parallel SK_{id}])$. Then, it computes a vector v from $SK_{id} = Q \cdot v$ and selects k linearly independent vectors $\eta_i \xleftarrow{\$} \{0, 1\}^k$ composing $\eta := \{\eta_1, \dots, \eta_k\}$ (the Hamming weight of η_i satisfying $1 < \text{wt}(\eta_i) < [k/2]$).

It then computes each decryption server's secret share $d_i = Q \cdot \eta_i \in \mathbb{Z}_q^{2m}$. Subsequently, it confidentially sends the secret share d_i to decryption server Γ_i . By selecting a random matrix $M \in \mathbb{Z}_q^{k \times 2m}$ to pick a hash function $h_M: \mathbb{Z}_q^{2m} \rightarrow \mathbb{Z}_q^k$, it computes $h_M(q_i)$ ($1 \leq i \leq k$) and $h_M(SK_{id})$. The distributor makes common parameter $cp = (v, \eta, h_M, h_M(q_i), h_M(SK_{id}))$ public and keeps Q and SK_{id} secretly.

-VerS($h_M(d_i), \eta, h_M(q_i)$): This algorithm computes $h_M(d_i)$ and checks if $h_M(d_i) = \sum_{j=1}^k \eta_{ij} \cdot h_M(q_j)$, then accepts the secret share d_i and returns an acknowledgement, otherwise outputs "Invalid shares".

-Encrypt(PP, id, b): On input a message $b \in \{0, 1\}$, an identity id , public parameters PP , this algorithm picks a uniformly random $s \xleftarrow{R} \mathbf{Z}_q^n$, a uniformly random matrix

$R \leftarrow \{-1, 1\}^{m \times m}$ and noise vectors $x \leftarrow \mathbb{Z}_q^\alpha$, $y \leftarrow \mathbb{Z}_q^m$, let $z = R^T y$, and subsequently computes $c_0 \leftarrow u^T s + x + b \lfloor q/2 \rfloor \in \mathbb{Z}_q$, $c_1 \leftarrow F_{id}^T s + \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{Z}_q^{2m}$ and returns a ciphertext $C = (c_0, c_1)$.

-ConstructDec(η, v, c_1, d_j): On input η, v, c_1 , this algorithm computes $E = [\eta_1, \dots, \eta_k] \cdot A_2^T$ (E denotes $k \times k$ unit matrix) to obtain the matrix $A_2 = [a_1, \dots, a_k]$. The decryption server Γ_j computes decryption share W_j , $W_j = [(\sum_{i=1}^k v_i \cdot a_i^{(1)}) \cdot d_j^{(1)}, \dots, (\sum_{i=1}^k v_i \cdot a_i^{(i)}) \cdot d_j^{(2m)}]$. Then it computes $h_M(W_j^T)$ and outputs ζ_{j,c_1} , $\zeta_{j,c_1} = (W_j \cdot c_1, h_M(W_j^T))$.

-Decrypt(ζ_{j,c_1}, PP, C): Given ζ_{j,c_1} , PP , $C = (c_0, c_1)$, it computes $\sum_{j=1}^k h_M(W_j^T)$ and checks if $\sum_{j=1}^k h_M(W_j^T) = h_M(SK_{id})$. If the test holds, accept the decryption shares and do as follows: compute $\omega \leftarrow c_0 - \sum_{j=1}^k (W_j \cdot c_1) \in \mathbb{Z}_q$ and compare ω and $\lfloor q/2 \rfloor$, if $|\omega - \lfloor q/2 \rfloor| < \lfloor q/4 \rfloor$, output 1, otherwise output 0.

3.3 Correctness

Due to $\sum_{j=1}^k (W_j \cdot c_1) = (\sum_{j=1}^k W_j) \cdot c_1$, we can obtain $\sum_{j=1}^k W_j = (Q \cdot v)^T$ from the deduction for $\sum_{j=1}^k W_j$, as follows:

$$\begin{aligned} \sum_{j=1}^k W_j &= \begin{bmatrix} (\sum_{i=1}^k v_i \cdot a_i^{(1)}) \cdot d_1^{(1)} + (\sum_{i=1}^k v_i \cdot a_i^{(2)}) \cdot d_1^{(2)} + \dots + (\sum_{i=1}^k v_i \cdot a_i^{(k)}) \cdot d_1^{(k)} \\ (\sum_{i=1}^k v_i \cdot a_i^{(1)}) \cdot d_1^{(2)} + (\sum_{i=1}^k v_i \cdot a_i^{(2)}) \cdot d_1^{(3)} + \dots + (\sum_{i=1}^k v_i \cdot a_i^{(k)}) \cdot d_1^{(k)} \\ \vdots \\ (\sum_{i=1}^k v_i \cdot a_i^{(1)}) \cdot d_1^{(2m)} + (\sum_{i=1}^k v_i \cdot a_i^{(2)}) \cdot d_1^{(2m)} + \dots + (\sum_{i=1}^k v_i \cdot a_i^{(k)}) \cdot d_1^{(2m)} \end{bmatrix}^T \\ &= \begin{bmatrix} (\sum_{i=1}^k v_i \cdot a_i^{(1)}), (\sum_{i=1}^k v_i \cdot a_i^{(2)}), \dots, (\sum_{i=1}^k v_i \cdot a_i^{(k)}) \end{bmatrix} \cdot [d_1, d_2, \dots, d_k]^T \\ &= v^T \cdot [a_1, a_2, \dots, a_k] \cdot [d_1, d_2, \dots, d_k]^T \\ &= ([d_1, d_2, \dots, d_k] \cdot [a_1, a_2, \dots, a_k]^T \cdot v)^T \\ &= ([q_1, q_2, \dots, q_k] \cdot [\eta_1, \eta_2, \dots, \eta_k] \cdot [a_1, a_2, \dots, a_k]^T \cdot v)^T \\ &= ([q_1, q_2, \dots, q_k] \cdot E \cdot v)^T \\ &= (Q \cdot v)^T. \end{aligned}$$

When $\sum_{j=1}^k W_j = (Q \cdot v)^T = (SK_{id})^T$ is easily gotten, we have ω as follows:

$$\begin{aligned} \omega &= c_0 - (SK_{id})^T c_1 \\ &= u^T s + x + b \lfloor q/2 \rfloor - SK_{id}^T (F_{id}^T s + \begin{bmatrix} y \\ z \end{bmatrix}) \\ &= b \lfloor q/2 \rfloor + x - SK_{id}^T \begin{bmatrix} y \\ z \end{bmatrix}. \end{aligned}$$

Setting $(x - e_{id}^T \begin{bmatrix} y \\ z \end{bmatrix})$ as the error term, the decryption operates successfully when the error term is less than $q/4$.

With high probability. By Lemma 22 in [8], the norm of error term is bounded by $q\sigma m\alpha\omega(\sqrt{\log m} + O(\sigma m^{3/2}))$.

3.4 Parameters

To make the system work correctly, following conditions have to be satisfied:

1) Let $m > 6n\log q$, so the function TrapGen can operate.

2) Set the Gaussian parameter $\sigma > m\omega(\sqrt{\log m})$, so SampleLeft and SampleRight can operate.

3) Let $q > 2\sqrt{n}/\alpha$, so Regev's reduction holds.

4) Set $\alpha < [\sigma m\omega(\sqrt{\log m})]^{-1}$ and $q = \Omega(\sigma m^2)$ for the error term is less than $q/5$.

5) Let $\mu = O(n\log q)$ and $\mu > m\omega(\sqrt{\log m})$, $\lambda^2 = m$ to satisfy that the input for lattice-based hash function is collision-free.

We assume that $n^\delta > \lceil \log q \rceil = O(\log n)$, n is the security parameter, we set the parameters $(m, q, \sigma, \alpha, \mu, \lambda)$ as follows:

$$m = 6n^{1+\delta}, q = m^{2.5} \cdot \omega(\sqrt{\log n}), \sigma = m \cdot \omega(\sqrt{\log n})$$

$$\alpha = [m^2 \cdot \omega(\sqrt{\log n})]^{-1}, \mu = mO(\log n), \lambda = \sqrt{m}.$$

We set λ and m to the nearest larger integer, q to the nearest larger prime.

4 Security Reduction

We define the security notion for our distributed IBE. IND-r-sIDD-CPA is the notion of indistinguishability of ciphertext under a selective identity chosen plaintext attack for the distributed IBE. IND-r denotes indistinguishable from random, that means the challenge ciphertext is indistinguishable from random elements in the ciphertext space.

Definition. Let $\mathcal{A}$ be an IND-r-sIDD-CPA PPT adversary. Consider the following Game 0 in which the adversary $\mathcal{A}$ interacts with the Challenger.

Initial: The adversary $\mathcal{A}$ outputs a target identity id^* .

Setup: Then Challenger runs algorithm Setup(n), n is a security parameter, and sends the resulting public system parameters $PP = (A_0, A_1, B, u)$ to $\mathcal{A}$. However, it keeps the master key MK secretly.

Phase 1: $\mathcal{A}$ corrupts $k - 1$ decryption servers.

Phase 2: The Challenger runs the Extract algorithm to obtain a private key d_{id^*} corresponding to the target identity id^* . Subsequently, the Challenger runs the Distribution algorithm on d_{id^*} with the input parameter k . It then obtains a set of secret shares $\{d_i | 1 \leq i \leq k\}$ and sends the $k - 1$ secret shares of corrupted decryption servers with the corresponding $h_M(d_i)$ to $\mathcal{A}$. Besides, the secret share of the uncorrupted server is kept secretly.

Phase 3: $\mathcal{A}$ issues adaptive private key extraction queries $q_1, \dots, q_m$, where q_i denotes the i^{th} query on

$id_i (id_i \neq id^*)$. Then the Challenger runs Extract algorithm on id_i to obtain a corresponding private key e_i to respond the query q_i . It sends e_i to $\mathcal{A}$.

Challenge: After $\mathcal{A}$ finishes the Phase 3, it outputs a plaintext $\mathcal{M}$ to be challenged. The Challenger chooses a random bit $r \in \{0,1\}$ and a random ciphertext $\mathcal{C} \in \mathbb{Z}_q \times \mathbb{Z}_q^{2m}$. If $r = 0$ it gives the challenge ciphertext $\mathcal{C}^* = \text{Encrypt}(\text{PP}, id^*, \mathcal{M})$ to $\mathcal{A}$. If $r = 1$, it sends the random ciphertext $\mathcal{C}^* = \mathcal{C}$ to $\mathcal{A}$.

Phase 4: $\mathcal{A}$ issues arbitrary adaptive private key extraction queries $q_{m+1}, \dots, q_n$, where q_i denotes the i^{th} query on $id_i (id_i \neq id^*)$. The Challenger responds as in Phase 3.

Guess: $\mathcal{A}$ outputs a guess $r' \in \{0,1\}$. If $r = r'$, it wins the game.

We define the advantage of an INDr-sIDD-CPA adversary $\mathcal{A}$ in attacking the distributed IBE $\mathcal{E}$ as $\text{Adv}_{\mathcal{E}, \mathcal{A}}(n) = |\Pr[r = r'] - 1/2|$.

Theorem 2. In the standard model, our distributed IBE with parameters setting as in Section parameters is INDr-sIDD-CPA secure if the $(\mathbb{Z}_q, n, \bar{\Psi}_\alpha)$ -LWE assumption holds.

Proof.

Game 0. It is the original INDr-sIDD-CPA game as previously mentioned.

Game 1. In Game 1, we change slightly the way to generate A_1 which is one of public parameters in PP. Let the Challenger in Game 1 select a random matrix $R^* \in \{-1,1\}^{m \times m}$ at the Setup phase, id^* is the target identity that $\mathcal{A}$ wants to attack. A_1 is constructed as $A_1 = A_0 R^* - H(id^*)B$. The Challenger use SampleRight algorithm to generate e_{id} to respond to a private key extraction query. The rest of the game is the same as Game 0.

By Lemma 13 in [7], Game 1 and Game 0 is statistically indistinguishable.

Game 2. We change the way to generate A_0 and B in PP. The Challenger chooses a random matrix in $\mathbb{Z}_q^{n \times m}$ as A_0 and runs the TrapGen(q, n) to generate a uniformly random matrix $B \in \mathbb{Z}_q^{n \times m}$ altogether with a trapdoor T_B . The construction of A_1 is the same as Game 1.

The Challenger runs SampleRight($A_0, (H(id) - H(id^*))B, R^*, T_B, u, \sigma$) to sample a vector $e \in \mathbb{Z}_q^{2m}$ and sends $SK_{id} = e$ to $\mathcal{A}$ to respond to the private key query.

Let $F_{id} = (A_0 | A_0 R^* + (H(id) - H(id^*)) \cdot B)$, according to Theorem 19 in [7], when $\sigma > \|\tilde{T}_B\|_{s_R} \omega(\sqrt{\log m})$, the generated e is distributed statistically close to $D_{\Lambda_q^u(F_{id}), \sigma}$, as in Game 1.

So Game 2 and Game 1 is statistically indistinguishable.

Game 3. In Game 3, the challenge ciphertext $\mathcal{C}^* = (c_0^*, c_1^*)$ is selected by random independent elements in $\mathbb{Z}_q \times \mathbb{Z}_q^{2m}$. The rest of Game 3 is the same as Game 2. The advantage of $\mathcal{A}$ in this game is zero.

We can refer to the reduction from the LWE problem in [7] to show Game 2 and Game 3 are indistinguishable for $\mathcal{A}$.

The reduction from the LWE problem shows as follows:

We assume that $\mathcal{A}$'s advantage in distinguishing Game 2 and Game 3 is non-negligible and construct an LWE algorithm $\mathcal{B}$ using $\mathcal{A}$. $\mathcal{B}$ receives LWE samples $(u_i, v_i) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$, $i = 0, \dots, m$ from an oracle $\mathcal{O}$ as mentioned in Section 2.4.

Initial: The adversary $\mathcal{A}$ outputs a target identity id^* to $\mathcal{B}$.

Setup: $\mathcal{B}$ constructs public parameters $\text{PP} = (A_0, A_1, B, u_0, R^*)$, where $A_0 = [u_1, u_2, \dots, u_m] \in \mathbb{Z}_q^{n \times m}$, the construction of A_1 and B is the same as in Game 2, u_0 is the 0^{th} LWE sample, $R^* \in \{-1,1\}^{m \times m}$.

Queries: $\mathcal{B}$ responds to the private key extraction queries as in Game 2.

Challenge: $\mathcal{A}$ outputs a plaintext $b^* \in \{0,1\}$, then $\mathcal{B}$ sends a challenge ciphertext $\mathcal{C}^* = (c_0^*, c_1^*)$ for the target

id^* to $\mathcal{A}$. We set $v^* = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{Z}_q^m$, $c_0^* = v_0 + b^*[q/2] \in \mathbb{Z}_q$ and $c_1^* = \begin{bmatrix} v^* \\ (R^*)^T v^* \end{bmatrix} \in \mathbb{Z}_q^{2m}$.

When the LWE oracle $\mathcal{O} = \mathcal{O}_s$ (which is a pseudorandom sampler), then $\mathcal{C}^*$ is distributed the same as the challenge ciphertext in Game 2. When $\mathcal{O} = \mathcal{O}_\$$, the challenge ciphertext is the same as in Game 3.

Guess: $\mathcal{A}$ guesses if it interacts with a Game 2 Challenger or Game 3 Challenger after issuing arbitrary adaptive private key extraction queries $q_{m+1}, \dots, q_n$. then $\mathcal{B}$ outputs the guess of $\mathcal{A}$ to answer the LWE challenge.

The advantage of $\mathcal{B}$ to solve LWE problem is the same as the advantage of $\mathcal{A}$ to distinguish Game 2 and Game 3.

5 Efficiency Analysis

We show the comparison of our lattice-based distributed IBE and Baek and Zheng Yuliang's pairing-based distributed IBE in calculation cost.

Table1:Comparison of Our Scheme And Bz Scheme

comparison		
	<i>Baek's scheme</i>	<i>Our scheme</i>
	Calculation cost	Calculation cost
Private key distribution	kM_1	$(k+1)\text{Mul}$
Secret share generation	tP	$\text{Mul} + 2m(kM + (k-1)A)$
Verification of secret share	$(n + 4t + 4)P + (t + 2)H$	$(2k+1)H$
Combination	$tL + tM_1 + (t - 1)M_T$	$k\text{Mul}$

Table2:Notation Illustration

Notation	Operation	Notation	Operation
L	Lagrange interpolation operation	H	Hash operation
P	pairing	Mul	matrix

	operation		multiplication
M_1	the point multiplication in G_1	M	multiplication operation in $\mathbb{Z}$
M_T	multiplication in G_T	A	addition operation in $\mathbb{Z}$

The k represents the number of decryption servers, t represents the threshold.

Table II explains the meaning of the notations in Table I. For example, it shows that at private key distribution phase, Baek's scheme costs kM_1 (k times point multiplication in G_1) and it costs $(k+1)$ times matrix multiplication. The advantages in our scheme appears in secret share generation and verification of secret share phase. As we know, the pairing operation is much more complicate than simple matrix multiplication and addition. Our distributed IBE avoids the complex pairing operation, so our scheme reduces the calculation cost and improves the efficiency.

Conclusion

In this paper, we improve the lattice-based secret sharing method and propose a lattice-based distributed IBE provably secure against selective identity chosen-plaintext attack. Through theoretical analysis, compared with the Baek and Zheng Yuliang's scheme, our scheme reduces calculation cost.

Acknowledgement

I would like to express my gratitude to all those who helped me for finishing this paper, especially to Professor Zhang Quan, my supervisor, for his encouragement and guidance.

References

- [1] Shamir A. Identity-based cryptosystems and signature schemes[C]//Advances in cryptology. Springer Berlin Heidelberg, 1984: 47-53.
- [2] Boneh D, Franklin M. Identity-based encryption from the Weil pairing[C]//Advances in Cryptology-CRYPTO 2001. Springer Berlin Heidelberg, 2001: 213-229.
- [3] Shamir A. How to share a secret[J]. Communications of the ACM, 1979, 22(11): 612-613.
- [4] Baek J, Zheng Y. Identity-based threshold decryption[M]//Public Key Cryptography-PKC 2004. Springer Berlin Heidelberg, 2004: 262-276.
- [5] Gentry C, Peikert C, Vaikuntanathan V. Trapdoors for hard lattices and new cryptographic constructions[C]//Proceedings of the fortieth annual ACM symposium on Theory of computing. ACM, 2008: 197-206.
- [6] Cash D, Hofheinz D, Kiltz E, et al. Bonsai trees, or how to delegate a lattice basis[J]. Journal of cryptology, 2012, 25(4): 601-639.
- [7] Agrawal S, Boneh D, Boyen X. Efficient lattice (H) IBE in the standard model[M]//Advances in Cryptology-EUROCRYPT 2010. Springer Berlin Heidelberg, 2010: 553-572.
- [8] El Bansarkhani R, Mezziani M. An efficient lattice-based secret sharing construction[M]//Information Security Theory and Practice. Security, Privacy and Trust in Computing Systems and Ambient Intelligent Ecosystems. Springer Berlin Heidelberg, 2012: 160-168.
- [9] Blakely G R. Safeguarding cryptographic keys[C]//Proc. AFIPS. 1979, 48: 313-317.
- [10] Asmuth C, Bloom J. A modular approach to key safeguarding[J]. IEEE transactions on information theory, 1983, 30(2): 208-210.
- [11] Regev O. On lattices, learning with errors, random linear codes, and cryptography[J]. Journal of the ACM (JACM), 2009, 56(6): 34.
- [12] Alwen J, Peikert C. Generating shorter bases for hard random lattices[J]. Theory of Computing Systems, 2011, 48(3): 535-553.
- [13] Zhou F. Lattice theory and cryptography [M]. Science Press, 2013