Some properties of *k*-gamma and *k*-beta functions

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Abstract: In this paper, firstly, we introduce the several definitions of classic gamma function and beta function, and the several definitions of k-gamma function and k-beta function. Secondly, we discuss the relations between the several definitions and properties of gamma function, beta function, k-gamma function and k-beta function. Finally, we prove these properties by mathematical induction and integral transformation method.

1 Definitions of Gamma and Beta Functions

The gamma function is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer,

$$\Gamma(n) = (n-1)!. \tag{1}$$

There are many forms on the definition of gamma function. Euler's integral form

$$\Gamma(x) = \int_0^\infty v^x e^{-v} dv , \tag{2}$$

or

$$\Gamma(x) = 2\int_0^\infty v^{x^2} e^{-2v-1} dv , \tag{3}$$

or

$$\Gamma(x) = \int_0^1 \left[\ln\frac{1}{v}\right]^{x-1} dv . \tag{4}$$

The problem of extending the factorial to non-integer arguments was apparently first considered by Daniel Bernoulli and Christian Goldbach in the 1720s, and was solved at the end of the same decade by Leonhard Euler (see [1-4]). Euler gave two different definitions: the first was not his integral but an infinite product,

$$n! = \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^n}{1+\frac{n}{k}} . \tag{5}$$

In the 19th century, Carl Friedrich Gauss rewrote Euler's product as (see [1])

$$\Gamma(x) = \lim_{x\to\infty} \frac{n^x (n-1)!}{x(x+1)...(x+n-1)}. \tag{6}$$

Karl Weierstrass further established the role of the gamma function in complex analysis, starting from yet another product representation. The gamma function can also be defined by an infinite product form (Weierstrass form) (Krantz 1999, p. 157; Havil 2003, p. 57) (see [3,4]).

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}, \tag{7}$$

where $\gamma \approx$ =0.577216 is Euler's constant.
The Euler limit form is

$$-\ln\Gamma(z) = \ln z + \gamma z + \sum_{n=1}^{\infty}\left[\ln\left(1+\frac{z}{n}\right) - \frac{z}{n}\right]. \tag{8}$$

In 1933, Egan defined the gamma function as (see [5])

$$\ln\Gamma(x) = (x-\frac{1}{2})\ln x - x + c + \phi(x), \tag{9}$$

where

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$$\phi(x) = \Delta(x) + \Delta(x+1) + \Delta(x+2) + \ldots, c = 1 - \phi(1),$$

and

$$\Delta(x) = \int_x^{x+1} \ln s ds - \frac{1}{2}\ln x - \frac{1}{2}\ln(x+1). \quad (10)$$

The Beta function is defined as

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} du. \quad (11)$$

Let $u = \frac{1}{1+v}$, then

$$B(x,y) = \int_0^\infty v^{y-1}(1+v)^{-x-y} dv. \quad (12)$$

The definitions of gamma function given by Euler, Carl Friedrich Gauss, Karl Weierstrass and Egan are equivalent to each other.

2 Properties of Gamma and Beta Functions

By the definitions of the gamma function and beta function, their properties can be derived as the following (see [5]):

$\ln\Gamma(x)$ is a convex function on $(0,\infty)$,

$$\Gamma(1) = 1, \quad (13)$$

$$x\Gamma(x) = \Gamma(x+1), \quad (14)$$

$$\lim_{y\to\infty} \frac{\Gamma(y+x)}{y^x\Gamma(y)} = 1, \quad (15)$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad (16)$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (17)$$

$$\Gamma(x) = \lim_{y\to\infty}(y+a)^x B(y+b,x). \quad (18)$$

In (16), x is non integer complex number. The proof of these properties may be found in literatures [1-5] and the references cited therein.

3 Definitions of *k*-gamma and *k*-beta Functions

The motivation to introduce k-gamma function comes from the repeated appearance of expressions of the form

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k) \quad (19)$$

in a variety of contexts, such as the combination of creation and annihilation operators (see [6, 7]) and the perturbation computation of Feynman integrals(see [8]). The function of variable x given by formula (19) will be called the Pochhammer k-symbol. Setting $k = 1$ one obtains the usual Pochhammer symbol $(x)_n$, also known as the raising factorial (see [9, 10]).

Definition 1(see [11]) For k > 0, the k-gamma function is given by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad (20)$$

where x is a complex number.

Definition 2(see [11]) For k > 0, the k-beta function is given by

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \quad (21)$$

4 Properties of *k*-gamma Function

Proposition 1(see [11]) For a positive real number k and a complex number x with $Re\ x>0$, the k-Gamma function is given by the integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt. \quad (22)$$

Proof: We observe that

$$\int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt = \lim_{n\to\infty} \int_0^{(nk)^{1/k}} \left(1 - \frac{t^k}{nk}\right)^n t^{x-1} dt. \quad (23)$$

Let $A_{n,i}(x), i = 0,1,2,\ldots,n,$ be given by

$$A_{n,i}(x) = \int_0^{(nk)^{1/k}} \left(1 - \frac{t^k}{nk}\right)^i t^{x-1} dt. \quad (24)$$

The following recursive formula is proven using integration by parts

$$A_{n,i}(x) = \int_0^{(nk)^{1/k}} \left(1 - \frac{t^k}{nk}\right)^i t^{x-1} dt$$

$$= \frac{1}{x}\left(1 - \frac{t^k}{nk}\right)^i t^x \Big|_0^{(nk)^{1/k}}$$

$$+\frac{i}{x}\int_0^{(nk)^{1/k}}\left(1-\frac{t^k}{nk}\right)^{i-1}\frac{t^{k-1}}{n}t^x dt$$

$$=\frac{i}{nx}\int_0^{(nk)^{1/k}}\left(1-\frac{t^k}{nk}\right)^{i-1}t^{x+k-1}dt$$

$$=\frac{i}{nx}A_{n,i-1}(x+k). \tag{25}$$

When i=0, we have

$$A_{n,0}(x)=\int_0^{(nk)^{1/k}}t^{x-1}dt=\frac{(nk)^{x/k}}{x}. \tag{26}$$

When *i=1,2,...,n,* we get

$$A_{n,1}(x)=\frac{1}{nx}A_{n,0}(x+k)=\frac{1}{nx}\frac{(nk)^{\frac{x+k}{k}}}{x+k}, \tag{27}$$

$$A_{n,2}(x)=\frac{2}{nx}A_{n,1}(x+k)=\frac{1}{nx}\frac{2}{n(x+k)}\frac{(nk)^{\frac{x+2k}{k}}}{x+2k}, \tag{28}$$

$$A_{n,n}(x)=\frac{n}{nx}A_{n,n-1}(x+k)$$

$$=\frac{1}{nx}\frac{2}{n(x+k)}\cdots\frac{n}{n(x+(n-1)k)}\frac{(nk)^{\frac{x+nk}{k}}}{x+nk}$$

$$=\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}(1+\frac{x}{nk})}. \tag{29}$$

By Definition 1, we have

$$\lim_{n\to\infty}A_{n,n}(x)=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}(1+\frac{x}{nk})}$$

$$=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}=\Gamma_k(x). \tag{30}$$

From (23), (24) and (30), we obtain

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt.$$

This completes the proof of **Proposition 1**.

Proposition 2(see [11]) For a complex number *x* and a positive real number *k*, the *k*-gamma function and the classical Gamma function have the relation

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right). \tag{31}$$

Proof: Let $s=t^k/k$, then $ds=t^{k-1}dt$ and

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt=\int_0^\infty t^{x-k}e^{-\frac{t^k}{k}}t^{k-1}dt$$

$$=\int_0^\infty (t^k)^{\frac{x}{k}-1}e^{-\frac{t^k}{k}}t^{k-1}dt=k^{\frac{x}{k}-1}\int_0^\infty (s)^{\frac{x}{k}-1}e^{-s}ds$$

$$=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right). \tag{32}$$

The proof of **Proposition 2** is completed.

Proposition 3(see [11]) For a non-integer complex number *x* and positive real numbers *k, s*, the following identities hold

$$(x)_{n,s}=\left(\frac{s}{k}\right)^n\left(\frac{kx}{s}\right)_{n,k}, \tag{33}$$

$$\Gamma_s(x)=\left(\frac{s}{k}\right)^{\frac{x}{s}-1}\Gamma_k\left(\frac{kx}{s}\right). \tag{34}$$

Proof: By the definition of the Pochhammer *k*-symbol, we have

$$(x)_{n,s}=x(x+s)(x+2s)...(x+(n-1)s)$$

$$=\frac{s}{k}\frac{kx}{s}\left(\frac{s}{k}\frac{kx}{s}+s\right)\left(\frac{s}{k}\frac{kx}{s}+2s\right)...\left(\frac{s}{k}\frac{kx}{s}+(n-1)s\right)$$

$$=\left(\frac{s}{k}\right)^n\frac{kx}{s}\left(\frac{kx}{s}+k\right)\left(\frac{kx}{s}+2k\right)...\left(\frac{kx}{s}+(n-1)k\right)$$

$$=\left(\frac{s}{k}\right)^n\left(\frac{kx}{s}\right)_{n,k}. \tag{35}$$

By Definition 1, we have

$$\Gamma_s(x)=\lim_{n\to\infty}\frac{n!s^n(ns)^{\frac{x}{s}-1}}{(x)_{n,s}}$$

$$= \lim_{n\to\infty} \frac{n!k^n \left(\frac{s}{k}\right)^n \left(\frac{s}{k}\right)^{\frac{x}{s}-1} (nk)^{(\frac{kx}{s})/k-1}}{\left(\frac{s}{k}\right)^n \left(\frac{kx}{s}\right)_{n,k}}$$

$$= \left(\frac{s}{k}\right)^{\frac{x}{s}-1} \Gamma_k\left(\frac{kx}{s}\right). \tag{36}$$

The proof of Proposition 3 is completed.

Proposition 4(see [11]) For a positive real number k, the k-gamma function satisfies the following properties

$$\Gamma_k\left(x+k\right) = x\Gamma_k\left(x\right); \tag{37}$$

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}; \tag{38}$$

$$\Gamma_k\left(k\right) = 1; \tag{39}$$

$$\frac{1}{\Gamma_k(x)} = xk^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left(\left(1+\tfrac{x}{nk}\right) e^{-\frac{x}{nk}} \right), \tag{40}$$

where

$$\gamma = \lim_{n\to\infty}(1+\frac{1}{2}+\ldots\frac{1}{n}-\ln(n)); \tag{41}$$

$$\Gamma_k(x) = a^{\frac{x}{k}} \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}a} dt; \tag{42}$$

where a and x are real numbers;

$$\Gamma_k(x)\Gamma_k(k-x) = \frac{\pi}{k\sin(\frac{\pi x}{k})}. \tag{43}$$

where x is a non-integer complex number;

For a real number $x, \Gamma_k\left(x\right)$ is logarithmically convex.

Proof 1: From (31), we have

$$\Gamma_k(x+k) = k^{\frac{x+k}{k}-1} \Gamma\left(\frac{x+k}{k}\right) = k^{\frac{x}{k}} \frac{x}{k} \Gamma\left(\frac{x}{k}\right)$$

$$= xk^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right) = x\Gamma_k(x). \tag{44}$$

2. By **Definition 1** of the k-gamma function, we have

$$\Gamma_k(x+nk) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{x+nk}{k}-1}}{(x+nk)_{n,k}}$$

$$= \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \frac{(x)_{n,k}(nk)^n}{(x+nk)_{n,k}}$$

$$= \Gamma_k(x) \lim_{n\to\infty} \frac{(x)_{n,k}(nk)^n}{(x+nk)_{n,k}} = \Gamma_k(x)(x)_{n,k}. \tag{45}$$

It implies that

$$(x)_{n,k} = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}. \tag{46}$$

3. By **Definition 1** of the k-gamma function, we have

$$\Gamma_k(k) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{k}{k}-1}}{(k)_{n,k}} = \lim_{n\to\infty} \frac{n!k^n}{(k)_{n,k}} = 1. \tag{47}$$

4. From (7) and (32), we obtain the required formula:

$$\frac{1}{\Gamma_k(x)} = \frac{1}{k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)} = xk^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{nk}\right) e^{-\frac{x}{nk}} \right). \tag{48}$$

5. Let $t = a^{\frac{1}{k}} s$, from (22) we get

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt = a^{\frac{1}{k}} \int_0^\infty (a^{\frac{1}{k}} s)^{x-1} e^{-\frac{s^k}{k}a} ds$$

$$= a^{\frac{x}{k}} \int_0^\infty s^{x-1} e^{-\frac{s^k}{k}a} ds. \tag{49}$$

6. From (16) and (22), we obtain

$$\Gamma_k(x)\Gamma_k(k-x) = k^{\frac{x}{k}-1} \Gamma\left(\frac{x}{k}\right) k^{\frac{k-x}{k}-1} \Gamma\left(\frac{k-x}{k}\right)$$

$$= k^{-1} \Gamma\left(\frac{x}{k}\right) \Gamma\left(\frac{k-x}{k}\right) = \frac{\pi}{k\sin(\frac{\pi x}{k})}. \tag{50}$$

7. Let $\varphi(k,x) = \ln\Gamma_k(x)$. We obtain

$$\varphi(k,x) = -\ln x + \frac{x}{k}\ln(k) - \frac{x}{k}\gamma$$

$$-\sum_{n=1}^{\infty}\left(\ln\left(1+\frac{x}{nk}\right)-\frac{x}{nk}\right), \tag{51}$$

$$\frac{\partial\varphi(k,x)}{\partial x}=-\frac{1}{x}+\frac{\ln(k)-\gamma}{k}-\sum_{n=1}^{\infty}\left(\frac{1}{x+nk}-\frac{1}{nk}\right), \tag{52}$$

and

$$\frac{\partial^2\varphi(k,x)}{\partial x^2}=\sum_{n=0}^{\infty}\frac{1}{(x+nk)^2}. \tag{53}$$

It implies that $\Gamma_k(k)$ is logarithmically convex on $(0,\infty)$.

The proof of **Proposition 4** is completed.

5 Properties of *k*-beta Function

Proposition 5(see [11]) For a positive real number *k*, the *k*-beta function satisfies the following properties

$$B_k(x,y)=\frac{1}{k}B(\frac{x}{k},\frac{y}{k}). \tag{54}$$

$$B_k(x,y)=\int_0^{\infty}t^{x-1}(1+t^k)^{-\frac{x+y}{k}}dt. \tag{55}$$

$$B_k(x,y)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt. \tag{56}$$

$$B_k(x,y)=\frac{x+y}{xy}\prod_{n=0}^{\infty}\frac{nk(nk+x+y)}{(nk+x)(nk+y)}. \tag{57}$$

Proof: From (21) and (31), we have

$$B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$$

$$=\frac{k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)k^{\frac{y}{k}-1}\Gamma\left(\frac{y}{k}\right)}{k^{\frac{x+y}{k}-1}\Gamma\left(\frac{x+y}{k}\right)}=\frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right). \tag{58}$$

From (12) and (58), we get

$$B_k(x,y)=\frac{1}{k}B(\frac{x}{k},\frac{y}{k})=\frac{1}{k}\int_0^{\infty}(1+v)^{-\frac{x}{k}-\frac{y}{k}}v^{\frac{y}{k}-1}dv$$

$$=\int_0^{\infty}(1+t^k)^{-\frac{x+y}{k}}t^{x-1}dt\,. \tag{59}$$

From (11) and (58), we obtain

$$B_k(x,y)=\frac{1}{k}B(\frac{x}{k},\frac{y}{k})=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt. \tag{60}$$

From (21) and (40), we have

$$B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}=\frac{x+y}{xy}\prod_{n=1}^{\infty}\frac{nk(nk+x+y)}{(nk+x)(nk+y)}. \tag{61}$$

The proof of **Proposition 5** is completed.

Summary

In this paper, firstly, we introduce the several definitions of classic gamma function and beta function, and discuss the relations between the several definitions and properties of gamma function and beta function by mathematical induction and integral transformation method. Secondly, we introduce the several definitions of *k*-gamma function and *k*-beta function, discuss the relations between *k*-gamma function, *k*-beta function, classic gamma function and beta function, and also discuss properties of *k*-gamma function and *k*-beta function by method of mathematical deduction. Finally, we prove these properties.

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