

# Mathematical model of an integrated circuit cooling through cylindrical rods

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**Abstract.** One of the main challenges in integrated circuits development is to propose alternatives to handle the extreme heat generated by high frequency of electrons moving in a reduced space that cause overheating and reduce the lifespan of the device. The use of cooling fins offers an alternative to enhance the heat transfer using combined a conduction-convection systems. Mathematical model of such process is important for parametric design and also to gain information about temperature distribution along the surface of the transistor. In this paper, we aim to obtain the equations for heat transfer along the chip and the fin by performing energy balance and heat transfer by conduction from the chip to the rod, followed by dissipation to the surrounding by convection. Newton's law of cooling and Fourier law were used to obtain the equations that describe the profile temperature in the rod and the surface of the chip. Ordinary differential equations were obtained and the respective analytical solutions were derived after consideration of boundary conditions. The temperature along the rod decreased considerably from the initial temperature (in contact with the chip surface). This indicates the benefit of using a cylindrical rod to distribute the heat generated in the chip.

## 1 Introduction

Integrated circuits are integral part of a diversity of electronics and electro-mechanical instruments. [1] The development of electronic devices is currently pushing the time rate of energy transfer per unit volume. Transistors are widely used as integral part of several electronic components consuming high amounts of electric power. In general, the failure rate of these devices is halved for each 10°C reduction in the junction operating temperature, consequently, the operating temperature at which each electronic devices work has to be kept below specific levels to minimize the risk of failure. As a result, new adjustments in element design are performed to facilitate the dissipation of energy, reduction of temperature and to provide alternatives for cooling [2]. Alternatively, schemes including single-phase high heat flux cooling along with hybrid microchannel/jet-impingement modules are also considered [3, 4]. Therefore it is important to take into account not only the electrical parameters but also thermal. Both of these parameters have equivalent laws that are used to perform calculations and develop dynamic models and numerical simulation. One of the most important parameters in thermal management of an electronic device and cooling is the temperature at the transistor chip surface. Precise information of this parameter is critical because functional properties of the

device are diminished when care of temperature control is not considered [5]. Many integrated circuits are constituted from silicon. Technologies for embedding thinned integrated circuits in polyimide sheets are being performed to increase thermal resistance of chips; however it is inevitable that the operational life of the product is diminished when the device is exposed to temperature increments. Integrated circuits are normally attached to materials including copper, ceramics or metal matrix composites. Due to differences in dilatation coefficients of the elements or compounds, differential thermal expansion occurs when the chip reaches higher temperatures, affecting the performance of the transistor. For these reasons, thermal management solutions are intensively sought [6], including engineering of fin geometry [7], i.e. optimization of heat transfer density is extensively investigated for the design of thin-film electronic circuits. Accordingly, natural and forced convection are commonly used to achieve a device's cooling requirements [8]. In this work, we analyzed the cooling of a transistor surface studying the mathematical model of the heat transfer by conduction from the surface to the fin and further dissipation to the surrounding by convection.

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## 2 Description of the mathematical model

The analysis of Newton's law of cooling gives important information to increase the rate of heat transfer. In general, this can be achieved by two different methods. The first one is to raise the coefficient of convection heat transfer by means, for example, of a fan. The second possibility is to increase the surface area, which can be achieved by connecting the surface with extended surfaces made of highly conductive materials, which are usually named fins. [9]

An alternative to achieve convective removal of heat from a surface is by using extensions on the surface of the transistor to increase the area [10, 11]. There exist several designs of fin, namely triangular, rectangular, cylindrical, circular, trapezoidal, concave parabolic, radial or even in the form of spiral tubes [12]. A fan is used to supply air to allow uniform dissipation of the heat that goes to the integrated circuits. Figure 1 shows a general scheme of the system under study. For simplification purposes, only one rod is represented, but in real practice, the surface is covered by several cylindrical rods. Cooling fins are generally used to improve heat transfer by increasing the surface area available for convection. Many geometric shapes can be used but the most common are rectangular, cylindrical and triangular. Initially, the surface of the chip is at temperature  $T_s$ , which will be normally greater than the surrounding temperature  $T_0$ . The length of the rod ( $L$ ) is cooled by a fan [11]. The heat it conducted through the surface and dissipated to the surroundings by convection.

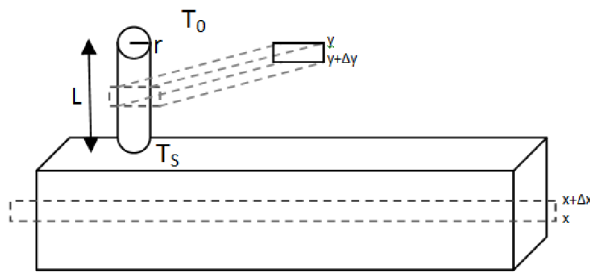


Fig. 1. Schematic diagram of the physical system under study

Fundamental assumptions are contemplated to propose the mathematical model. We consider an isotropic material with three constant parameters, namely the physical properties of the material (conduction of heat, density, specific heat), the generation of heat in the dispositive, and finally the rate of heat energy transfer through the surface of contact per unit time and temperature difference [13].

We proceed to model the transfer of heat removal around the chip at position  $y$  in the rod by considering Newton's law of cooling as represented in the steady

state heat balance equations (1) and (2) describing the heat in the chip and the heat dissipated by the rod to the surroundings. [14]

$$Sq|_y - Sq|_{y+\Delta y} - q_L S' = 0 \quad (1)$$

$$\pi r^2 q|_y - \pi r^2 q|_{y+\Delta y} - (2\pi r \Delta y)h(T_1 - T_0) = 0 \quad (2)$$

where  $S$  is the cross sectional area,  $S'$  is the lateral external area,  $r$  is the radius of the rod,  $q$  corresponds to the heat flux in  $y$  direction ( $W/m^2$ ),  $q_L$  is the convection heat transfer from the rod to the surrounding  $h$  the convection heat transfer coefficient,  $T_r$  and  $T_0$  are the temperature in the rod and the surrounding respectively. Dividing equation (2) by  $\pi r^2 \Delta y$  and taking the limit when  $\Delta y \rightarrow 0$  leads to equations (3) and (4) respectively. Accordingly, considering the Fourier law, equation (4) is transformed into (5)

$$\frac{q|_y - q|_{y+\Delta y}}{\Delta y} - \frac{2h(T_r - T_0)}{r} = 0 \quad (3)$$

$$-\frac{dq}{dy} - \frac{2h(T_r - T_0)}{r} = 0 \quad (4)$$

$$k_r \frac{d^2 T_r}{dy^2} - \frac{2h(T_r - T_0)}{r} = 0 \quad (5)$$

Additionally, we can transform the previous equation into dimensionless variables of temperature, distance and heat transfer coefficient. Introducing the dimensionless variables represented in (6), equation (7) is obtained:

$$\xi = \frac{y}{L}, \text{ for } \epsilon \in (0,1); \alpha = \sqrt{\frac{2hL^2}{rk_r}} \quad (6)$$

$$\frac{d^2 T}{d\xi^2} - \alpha(T_r - T_0) = 0 \quad (7)$$

We can change variables according to the temperature difference  $T_r - T_0$  to have  $d^2 \theta / d(\xi)^2 - \alpha \theta = 0$ .

Solving the previous second ordinary differential equation by the method of characteristics, the characteristic roots for the heat balance equation are  $m = \pm \alpha$ . Therefore, the solution of the equation leads to  $\theta = Ae^{\alpha \xi} + Be^{-\alpha \xi}$ . Boundary conditions are required to evaluate the arbitrary constants  $A$  and  $B$ . the heat loss occurs at the rod's lateral surface, and the flux from the top surface is small as expressed in boundary condition (8), whereas, the surface temperature at length corresponds to the exterior hot surface as expressed in (9)

$$\frac{d\theta}{d\xi}(0) = 0 \quad (8)$$

$$\theta(1) = T_h \quad (9)$$

To evaluate the arbitrary constants  $A$  and  $B$ , the boundary conditions (8) and (9) is used. However, for the purpose of applying boundary conditions, it is often convenient to represent the solution of equation (7) using hyperbolic functions, i.e.  $\theta = A \cosh \alpha \xi + B \sinh \alpha \xi$ . As a result  $B=0$ , and hence  $\theta = A \cosh \alpha \xi$ . The value of  $A$

can be obtained using boundary condition (9) which corresponds to  $A = T_h / \cosh \alpha$ . Substitution of the previous values of  $A$  and  $B$  obtained leads to equation (10)

$$\theta = \frac{T_h}{\cosh \alpha} \cosh \alpha \zeta \quad (10)$$

And returning to original variables,

$$T_r - T_0 = \frac{T_h}{\cosh \sqrt{\frac{2hL^2}{rk_r}}} \cosh \sqrt{\frac{2hL^2}{rk_r}} \frac{y}{L} \quad (11)$$

As a result, the temperature in the rod can be derived from previous equation (11). The next step, is to describe the transfer of heat around the computer chip at the position  $x$  with a thickness  $\Delta x$ . This is expressed in equation (12):

$$Sq|_x - Sq|_{x+\Delta x} + (S\Delta x)Q = 0 \quad (12)$$

where  $S$  is cross-sectional surface area of the chip,  $Q$  is the heat generated per unit volume and  $\Delta x$  the thickness. Dividing equation (12) by  $S\Delta x$  we obtain:

$$\frac{q|_x - q|_{x+\Delta x}}{\Delta x} + Q = 0 \quad (13)$$

Taking the limit as  $\Delta x \rightarrow 0$  gives the differential equation that represents the heat flux.

$$-\frac{dq}{dx} + Q = 0 \quad (14)$$

If the chip is homogenous, we can consider the vector form of Fourier's law of heat conduction which describes that the heat flux along the axis is proportional to the gradient in temperature,  $q = -k\nabla T$ . Therefore, equation (14) yields (15)

$$k_c \frac{d^2 T_c}{dx^2} + Q = 0 \quad (15)$$

where  $k_c$  and  $T_c$  corresponds to the thermal conductivity and temperature of the chip respectively.

The boundary conditions are set in equations (16) and (17)

$$\frac{dT_c}{dx}(0) = 0 \quad (16)$$

$$T_c(\delta) = T_h \quad (17)$$

Solution of equation (15) is represented in (18),

$$T_c = \frac{1}{k_c} \left( Cx + D - \frac{1}{2} Qx^2 \right) \quad (18)$$

After application of the boundary condition (16) we obtain  $C=0$  and hence  $T_c = (1/k_c)(D - Qx^2/2)$ . The value of  $D$  can be obtained using boundary condition (17) which corresponds to  $D = k_c T_h + Q\delta^2/2$ . Substitution of the previous values of  $C$  and  $D$  leads to equation (19)

$$T_c = \frac{1}{k_c} \left( k_c T_h + \frac{Q\delta^2}{2} - \frac{1}{2} Qx^2 \right) \quad (19)$$

And simplifying (19), the temperature in the chip surface can be expressed as :

$$T_c = T_h + \frac{Q}{2k_c} (\delta^2 - x^2) \quad (20)$$

From equation (20) it is evident that for the case of  $x=\delta$ , the temperature in the chip equals the exterior hot surface ( $T_c=T_h$ ).

## 4 Discussion

Thermal management in electronic devices is important to protect the correct functioning and operation of the dispositive. Accordingly, the knowledge of temperature distribution across the chip surface is a key factor also for operational safety reasons. In Fig. 1 we present two cases of temperature distribution. The first case is when the temperature at the surface of the chip is 75°C and the second is 50°C. In both examples, the final temperature (near the surface that is in contact with the surrounding at  $y=0$ ) approximates to 30°C. The information obtained from the plot is also important for the design and determination of rod geometry, i.e. to know the required length that is necessary to achieve a specific temperature. In the present example, if the temperature level is acceptable at 40°C, the required dimensionless length is 0.4 when the temperature in the chip surface is 75°C, while in the second case (temperature chip surface of 50°C) is 0.2. This represents a numerical and graphical example for which the formula obtained for the temperature profile in the rod was found to be useful. As a result, the use of cylindrical rods provided to be useful to aids in heat transfer by increasing the surface of the chip and allowing better heat distribution

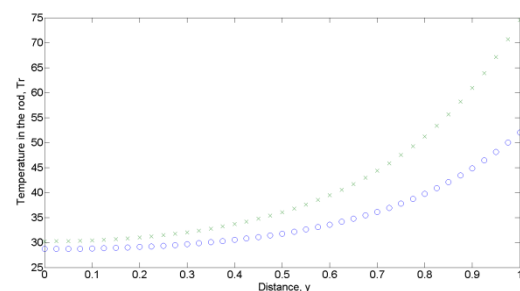


Fig.1 Temperature profile in the rod according to distance from the upper part (contact with surrounding).  $T_h=75^\circ\text{C}(\times)$ ,  $T_h=50^\circ\text{C}(\text{O})$ .

## 3 Conclusions

The development of microelectronic devices that are advanced in technology and smaller in size confront considerable challenges in thermal design aiming to dissipate the heat generated inside the device that could reduce reliability of an equipment, malfunctioning and problem of operation. An analysis of combined conduction-convection system was performed in the

present study to describe the cooling of a transistor surface by the use of a cooling rod. Mathematical model of these phenomena was performed to obtain the profile of temperature in the rod and the transistor surface.

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