Set-Valued Stochastic Equation with Set-Valued Square Integrable Martingale

Jun-Gang Li [1] , Shi-Qing Zheng [2]

[1]*Department of Statistics North China University of Technology, Beijing 100144, China.*
[2]*Department of Statistics North China University of Technology, Beijing 100144, China.*
[1]*jgli@ncut.edu.cn,* [2] *1416922582 @qq.com*

Abstract: In this paper, we shall introduce the stochastic integral of a stochastic process with respect to set-valued square integrable martingale. Then we shall give the Aumann integral measurable theorem, and give the set-valued stochastic Lebesgue integral and set-valued square integrable martingale integral equation. The existence and uniqueness of solution to set-valued stochastic integral equation are proved. The discussion will be useful in optimal control and mathematical finance in psychological factors.

1. Introduction

Set-valued theory is used in optimal control(cf.[1]), mathematical finance(cf. [2]), fixed point theory(cf. [3]). Set-valued and fuzzy set-valued theory can be used to account for psychological factors(cf.[4,5]).In [6], stochastic control problems are discussed by stochastic integral with respect to set-valued square integral martingales. M. Malinowski *et al.* discussed the set-valued stochastic integral driven by semi-martingale and the set-valued stochastic differential equations driven by semi-martingale in [7]. J. Li *et al.* discussed set-valued stochastic Lebesgue integral and set-valued stochastic differential equaton in [8,9,10]. Puri and Ralescu defined fuzzy set-valued martingales and proved convergence theorems of fuzzy set-valued martingales in [11]. J. Li *et al.* discussed the space of fuzzy set-valued square integrable martingales in [12] and fuzzy set-valued stochastic Lebesgue integral in [13].

In this paper, we shall give the set-valued stochastic integral equation :

$$F(t) = F(0) + (L)\int_0^t b(s, F(s))ds + (M)\int_0^t \sigma(s, F(s))dG(s)$$

the first integral is set-valued stochastic Lebesgue integral (see [9,10]), the second integral is set-valued square integrable martingale integral (see [6]). Aumann integral measurable theorem of the second integral shall be given. The existence and uniqueness of the solution to the equation shall be proved.

We organize our paper as follows: in Section 2, we shall introduce some necessary notations, definitions, and results about set-valued stochastic variables and martingales. Furthermore, we shall give the Aumann integral measurable theorem. In Section 3, we shall give the set-valued stochastic integral equation with respect to set-valued square integrable martingale, and prove the existence and uniqueness of the solution to the set-valued stochastic integral equation.

2. Preliminary on Set-Valued Random Variables and Martingales

Throughout this paper, assume that $(\Omega, \mathcal{A}, \mu)$ is a complete probability space, the σ-field filtration $\{\mathcal{A}_t : t \in I\}$ satisfies the usual conditions (i.e. Containing all nullsets, non-decreasing and right continuous), $I = [0, T]$ with $T > 0$, R is the set of all real numbers, N is the set of all natural numbers, R^d is the d-dimensional Euclidean space with usual norm $\|\cdot\|$, $\mathfrak{B}(E)$ is the Borel field of the space E. Let $K(R^d)$ be the family of all nonempty, closed subsets of R^d. Let $K_c(R^d)$ (resp. $K_k(R^d)$, $K_{kc}(R^d)$) be the family of all nonempty closed convex (resp. compact, compact convex) subsets of R^d. For any x Î R^d, A, B are nonempty subsets of R^d, define the distance between x and A,

$$d(x, A) = \inf_{y \in A} \| x - y \|,$$

$$A + B = \{a + b : a \in A, b \in B\},$$

$$A \ominus B = \{x \in R^d : x + B \subset A\}.$$

The *Hausdorff metric* on $K(R^d)$ is defined as

$$d_H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$$

for $A, B \in K(R^d)$. For $B \in K(R^d)$, define

$$\|B\|_K = d_H(0,B) = \sup_{a\in B}\|a\|.$$

If $F:(\Omega,\mathcal{A}) \to K(R^d)$ satisfies that for any open set $O\subset R^d$, $F^{-1}(O)=\{\omega\in\Omega: F(\omega)\cap O\neq\Phi\}\in\mathcal{A}$, then F is called $\mathcal{A}$-measurable (or a set-valued random variable, random set, multivalued function.).

Let $f=\{f(t), \mathcal{A}_t, t\in I\}$ be a R^d-valued adapted stochastic process. Let $L^p[\Omega,\mathcal{A}_t,\mu,R^d]$ be the set of R^d-valued $\mathcal{A}_t$-measurable random variable x with $E[\|x\|^p]<\infty$ $(1\leq p<\infty)$, write $\|x\|_p=[E[\|x\|^p]]^{1/p}$.

Let $S_F^p(\mathcal{A})=\{f\in L^p[\Omega,\mathcal{A},\mu:R^d]: f(\omega)\in F(\omega), a.e.\ \omega\in\Omega\}$

For short, denoted by S_F^p. A set-valued random variable $F:\Omega\to K(R^d)$ is called *integrable* if S_F^1 is non-empty. F is L^p-bounded if and only if the real-valued random variable $\|F\|_K\in L^p[\Omega;R]$ If F is L^1-bounded, then F is also called *integrably bounded*. Let $L^p[\Omega,\mathcal{F},\mu,K(R^d)]$ be the family of all $K(R^d)$-valued L^p-bounded $\mathcal{F}$-measurable random variables. Similarly, we have notations $L^p[\Omega,\mathcal{F},\mu,K_c(R^d)]$, $L^p[\Omega,\mathcal{F},\mu,K_{kc}(R^d)]$, $L^p[\Omega,K(R^d)]$ and so on. A set-valued stochastic process $F=\{F(t),\mathcal{A}_t,t\in I\}$ is called a set-valued martingale if

(i) $F=\{F(t),\mathcal{A}_t,t\in I\}$ is adapted and for any $t\in I$, $F(t)$ is L^1-bounded;

(ii) for any $t\geq s$, $t\in I$, $E[F(t)|\mathcal{A}_s]=F(s)$, a.e. (μ).

A set-valued martingale $F=\{F(t),\mathcal{A}_t,t\in I\}$ is call-ed square integrable, if $\sup_{t\in I}E[\|F(t)\|_K^2]<\infty$.

Denote $CMS(F)$ as the set of R^d-valued continuous martingale selections of set-valued square integrable martingale F.

Concerning more notations, definitions and more results of set-valued random variable, set-valued martingales, readers could refer to [6,14].

Definition (cf.[6]) Assume $F=\{F(t),\mathcal{A}_t,t\in I\}$ is a separable square integrable set-valued martingale and for any fixed $\omega\in\Omega$, $F(\omega,t)$ is lower semicontinuous with $F(0)=0$ a.e., g is a predictable bounded stochastic process. For any $\omega\in\Omega$, $t\in I$, define

$$(A)\int_0^t g(s,w)dF(s,w)$$
$$=\{\int_0^t g(s,w)df(s,w): f\in CMS(F)\},$$

then $(A)\int_0^t g(s,w)dF(s,w)$ is said to be the *Aumann type stochastic integral* of g with respect to the set-valued square integrable martingale F.

By Therom 4.4 and Therom 4.7 in [6], we have the following Aumann integral measurable theorem:

Theorem 1 Assume that $F=\{F(t),\mathcal{A}_t,t\in I\}$ is a separable square integrable set-valued martingale and for any fixed $\omega\in\Omega$, $F(w,t)$ is lower semicontinuous with $F(0)=0$ a.e., g is a predictable bounded stochastic process. For any $\omega\in\Omega$, $t\in I$, we have that

$$cl\{(A)\int_0^t g(s,w)dF(s,w)\}$$
$$=cl\{\int_0^t g(s,w)df(s,w): f\in CMS(F)\}$$

is a set-valued random variable.

The above theorem is useful to discuss the theory of Aumann integral, etc.

3. Set-Valued Stochastic Integral Equation

We consider the following set-valued stochastic integral equation

$$F(t)=F(0)+(L)\int_0^t b(s,F(s))ds+(M)\int_0^t\sigma(s,F(s))dG(s), \tag{1}$$

Where for any $t\in I$, $F\in L^2(K(R^d))$ with initial condition $F(0)$ being an L^2-bounded set-valued random variable, $b:I\times K(R^d)\to K(R^d)$ is measurable, $\sigma:I\times K(R^d)\to R^d\otimes R^m$ is bounded predictable measurable, $G(s)$ is a set-valued square integral martingale. $(L)\int_0^t b(s,F(s))ds$ is set-valued stochastic Lebesgue integral (see [9]), $(M)\int_0^t\sigma(s,F(s))dG(s)$ is stochastic integral with respect to set-valued square integral martingale (see Definition 4.5 in [6]).

Theorem 2 Assume that $b(t,F), \sigma(t,F)$ satisfy the following conditions:

(1) Linear bound condition:

$$\|b(t,F)\|^2 + \|\sigma(t,F)\|^2 \leq a^2(1+\|F\|_K^2);$$

(2) Lipschitz continuous condition:

$$\|b(t,F_1(t)) - b(t,F_2(t))\| + \|\sigma(t,F_1(t)) - \sigma(t,F_2(t))\| \leq a\, d_H(F_1(t),F_2(t))$$

where $a > 0$ is a constant;

(3) Set-valued integral inequality:

$$Ed_H^2\left(\int_0^t \sigma(s,F_1(s))dG_s, \int_0^t \sigma(s,F_2(s))dG_s\right) \leq E\int_0^t d_H^2(F_1(s),F_2(s))ds$$

Then for any $F(0) \in L^2(K(R^d))$, there exists a unique solution to equation (1).

Proof: First, it is obviously the existence of b, σ which satisfies (1),(2),(3).

If $F(t) \in L^2(K(R^d))$, for $\forall t \in I$, then

$$E\|b(t,F(t))\|_K^2 + E\|\sigma(t,F(t))\|^2 \leq a^2(1+E\|F\|_K^2).$$

Without loss of generality, assume that $b \in L^2(K(R^d))$, $\sigma \in L^2(R^d \times R^m)$.

Step 1. $F_0(t) = F(0)$,

$$F_{n+1}(t) = F(0) + \int_0^t b(s,F_n(s))ds + \int_0^t \sigma(s,F_n(s))dG_s, n \geq 0$$

We shall prove that for any $n \geq 0$, F_n satisfies:

(a) $F_n \in L^2(K(R^d))$, for $\forall t \in I$

(b) $\lim_{s\to t} Ed_H^2(F_n(t),F_n(s)) = 0.$

For $n = 0$, it is obviously right. Suppose that F_n has properties $(a),(b)$ for any fixed n, we shall prove so does F_{n+1}.

Let $Y(t) := \int_0^t b(s_1,F_n(s_1))ds_1$ by Theorem 3.6 in [9], (a) holds to $Y(t)$.

$$Ed_H^2(Y(t),Y(s))$$
$$= Ed_H^2\left(cl(Y(s)+\int_s^t b(s_1,F_n(s_1))ds_1), Y(s)\right)$$
$$\leq Ed_H^2\left(\int_s^t b(s_1,F_n(s_1))ds_1, 0\right)$$
$$= E\left\|\int_s^t b(s_1,F_n(s_1))ds_1\right\|_K^2$$

By the Linear bound condition (1) and norm property (β) of set-valued Lebesgue integral, it satisfies

Let $Z(t) := \int_0^t \sigma(s_1,F_n(s_1))dG_{s_1}$ It is obviously that (a) holds.

$$Ed_H^2(Z(t),Z(s))$$
$$= Ed_H^2\left(\int_0^t \sigma(s_1,F_n(s_1))dG_{s_1}, \int_0^s \sigma(s_1,F_n(s_1))dG_{s_1}\right)$$
$$= Ed_H^2\left(\int_0^s \sigma(s_1,F_n(s_1))ds_1 + \left(\int_0^t \sigma(s_1,F_n(s_1))dG_{s_1} \ominus \int_0^s \sigma(s_1,F_n(s_1))dG_{s_1}\right), \int_0^s \sigma(s_1,F_n(s_1))dG_{s_1}\right)$$
$$\leq Ed_H^2\left(\int_0^t \sigma(s_1,F_n(s_1))dG_{s_1} \ominus \int_0^s \sigma(s_1,F_n(s_1))dG_{s_1}, 0\right)$$
$$\leq Ed_H^2\left(\int_s^t \sigma(s_1,F_n(s_1))dG_{s_1}, 0\right)$$

By Theorem 1 and Theorem 4.7 in [6], the set-valued integral $\int_s^t \sigma(s_1,F_n(s_1))dG_{s_1}$ is represented by Aumann integral.

Therefore,

$$\lim_{s\to t}\int_s^t \sigma(s_1,F_n(s_1))dG_{s_1} = \{0\}, \text{ a. e..}$$

Thus, we have $\lim_{s\to t} Ed_H^2(Z(t),Z(s)) = 0.$

Step 2. Now we shall prove that F_n convergences to F.

Let

$$F_0(t) = F(0),$$
$$F_{n+1}(t) = F(t_0) + \int_0^t b(s,F_n(s))ds + \int_0^t \sigma(s,F_n(s))dG_s, n \geq 0$$

We have

$$d_H(F_1(t),F_0(t))$$
$$= d_H\left(F_0(t) + \int_0^t b(s,F_0(s))ds + \int_0^t \sigma(s,F_0(s))dG_s, F_0(t)\right)$$
$$\leq d_H\left(\int_0^t b(s,F_0(s))ds, 0\right) + d_H\left(\int_0^t \sigma(s,F_0(s))dG_s, 0\right)$$
$$= \left\|\int_0^t b(s,F_0(s))ds\right\|_K + \left\|\int_0^t \sigma(s,F_0(s))dG_s\right\|_K$$

For the first part, we have

$$E\left\|\int_0^t b(s,F_0(s)ds\right\|_K^2 \leq E\left(\int_0^t \|f(s,F_0(s)\|_K\, ds\right)^2$$
$$\leq tE\left(\int_0^t \|b(s,F_0(s))\|_K^2\, ds\right)$$
$$= t\int_0^t E\|b(s,F_0(s))\|_K^2\, ds$$
$$\leq ta^2\int_0^t (1+E\|F_0(s)\|_K^2)ds$$
$$\leq A^2 t,$$

For the second part, by inequality (3), we have

$$E\|\int_0^t \sigma(s,F_0(s))dG_s\|_K^2$$
$$=Ed_H^2(\int_0^t \sigma(s,F_0(s))dG_s,0)$$
$$\leq E\int_0^t d_H^2(F_0(s),0)ds$$
$$=tE\|F(0)\|_K^2$$

Let $B^2=E\|F(0)\|_K^2$,

$$Ed_H^2(F_1(0),F_0(t))$$
$$\leq 2E\|\int_0^t b(s,F_0(s))ds\|_K^2+2E\|\int_0^t \sigma(s,F_0(s))dG_s\|_K^2$$
$$\leq 2(A+B)^2t.$$

By the same way, we have

$$Ed_H^2(F_{n+1}(t),F_n(t))$$
$$=E(d_H(F(0)+\int_0^t b(s,F_n(s))ds+\int_0^t \sigma(s,F_n(s))dG_s,F(0)$$
$$+\int_0^t b(s,F_{n-1}(s))ds+\int_0^t \sigma(s,F_{n-1}(s))dG_s$$
$$\leq Ed_H(F(0),F(0))+Ed_H(\int_0^t b(s,F_n(s))ds,\int_0^t b(s,F_{n-1}(s))ds)$$
$$+Ed_H(\int_0^t \sigma(s,F_n(s))dG_s,\int_0^t \sigma(s,F_{n-1}(s))dG_s)$$
$$\leq 2Ed_H^2(\int_0^t b(s,F_n(s))ds,\int_0^t b(s,F_{n-1}(s))ds)$$
$$+2Ed_H^2(\int_0^t \sigma(s,F_n(s))dG_s,\int_0^t \sigma(s,F_{n-1}(s))dG_s)$$
$$\leq 2tE\int_0^t d_H^2(b(s,F_n(s)),b(s,F_{n-1}(s)))ds$$
$$+2E\int_0^t d_H^2(F_n(s),F_{n-1}(s))ds$$
$$\leq 2ta^2E\int_0^t d_H^2(F_n(s),F_{n-1}(s))ds$$
$$+2E\int_0^t d_H^2(F_n(s),F_{n-1}(s))ds$$
$$=2(ta^2+1)E\int_0^t d_H^2(F_n(s),F_{n-1}(s))ds$$

Iterating the above process, we get

$$Ed_H^2(F_{n+1}(t),F_n(t))\leq 2^{n+1}(ta^2+1)^{2n}(A+B)^2\frac{t^{n+1}}{(n+1)!}$$

i.e.

$$\Delta_n(t)=E\sup_{s\in[0,t]}d_H^2(F_{n+1}(s),F_n(s))$$
$$\leq 2^{n+1}(ta^2+1)^{2n}(A+B)^2\frac{t^{n+1}}{(n+1)!}$$

Thus,

$$\Delta_n(T)\leq 2^{n+1}(Ta^2+1)^{2n}(A+B)^2\frac{T^{n+1}}{(n+1)!}.$$

Therefore, we have

$$\sum_{n=1}^{\infty}\Delta_n(T)<\infty.$$

This ensures the existence of the strong solution.

Step 3. Let F , G be two solutions to equation (1). The uniqueness of the solution is the same to the proof of the existence.

Remark 3 Furthermore, fuzzy set-valued martingale integral, fuzzy set-valued stochastic Lebesgue integral [13], and fuzzy set-valued stochastic integral would be useful in the area.

Acknowledgment

The authors would like to thank reviewers for their valuable comments. This work was supported by youqing NCUT-XN132.

References

[1] M. Kisielewicz, Differential Inclusions and Optimal Control, Kluwer Academic Publishers, 1991.

[2] I. Karatzas, Lectures on the Mathematics of Finance, American Mathematical Society, Providence, RI, 1997.

[3] A. Fryszkowski, Fixed Point Theory for Decomposable Sets, Kluwer Academic Publishers, 2004.

[4] S. Li and L. Guan, "Fuzzy set-valued Gaussian process and Brownian motion, " Information Sciences, vol.177, pp. 3251–3259, August 2007.

[5] J. Li and J. Wang, "On a fluid queue with a fuzzy set-valued Gaussian process input, " unpublished.

[6] S. Li, J. Li and X. Li, "Stochastic integral w.r.t. set -valued square integrable martingale, " Journal of Mathematical Analysis and Applications vol.370, pp. 659-671, October 2010.

[7] M. Malinowski, "On a new set-valued stochastic integral with respect to semimartingales and its applications," Journal of Mathematical Analysis and Applications vol.408, pp.669-680, Decembe 2013.

[8] J. Li S. Li and Y. Ogura, "Strong solution of Ito type set-valued stochastic differential equation, " Acta Mathematica Sinica, English Series, vol.26, pp. 1739-1748, September 2010.

[9] J. Li and S. Li, "Set-valued stochastic Lebesgue integral and representation theorems, International Journal of Computational Intelligence Systems, " vol.1, pp. 177-187, May 2008.

[10] J. Li and S. Li, "Ito type set-valued stochastic differential equation, " Journal of Uncertain Systems, vol.3, pp.52-63, February 2009.

[11] M. L. Puri and D. A. Ralescu, "Convergence theorem for fuzzy martingales," Journal of Mathematical Analysis and Applications vol.160, pp.107-122, September 1991.

[12] J. Li, S. Li and Y. Xue, "The Space of Fuzzy Set-Valued Square Integrable Martingales, " IEEE

International Conference on Fuzzy Systems, pp. 872-876, August 2009.

[13] J. Li and J. Wang, "Fuzzy set-valued stochastic Lebesgue integral," Fuzzy Sets and Systems, vol.200, pp.48-64, August 2012.

[14] S. Li, Y. Ogura and V. Kreinovich, Limit Theorems and Applications of Set-Valued and Fuzzy Set-Valued Random Variables, Kluwer Academic Publishers, 2002.