ITM Web of Conferences **13**, 01007 (2017) DOI: 10.1051/itmconf/20171301007
CMES2017

Generalized Cesàro Summable Difference Sequence Spaces

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Abstract. Et (*M. Et, On some generalized Cesàro difference sequence spaces, İstanbul Üniv. Fen Fak. Mat. Derg. 55/56 (1996/97), 221–229*) introduced Cesàro difference sequence spaces $C_p(\Delta^m)\,(1 \leq p < \infty)\,, C_\infty(\Delta^m)$ and studied some topological properties. In this paper we continue to examine others relations with related the sequence spaces $C_1(\Delta^m)$ and $C_\infty(\Delta^m)$.

1 Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\|_\infty = \sup|x_k|$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers. Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely summable and $p-$absolutely summable series, respectively.

A sequence space X with a linear topology is called a $K-$space provided each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for each $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A $K-$space X is called an $FK-$space provided X is a complete linear metric space. An $FK-$space whose topology is normable is called a $BK-$space. We say that an $FK-$space X has AK (or has the AK property), if (e_k) (the sequence of unit vectors) is a Schauder bases for X.

Study of difference sequence spaces is a recent development in the summability theory. The approach of using differences of terms does not help for all the problems involving sequences. However, for many sequences, the use of differences can be very profitable and, moreover, the differences themselves prove extremely valuable in many other contexts. Sometimes a situation may arise that we have a sequence at hand and we are interested in sequences formed by its successive differences and in the structure of these new sequences.

The notion of difference sequence spaces was introduced by Kızmaz [16] and the notion was generalized by Et and Çolak [11]. Later on Et and Nuray [12] generalized these sequence spaces to the following sequence spaces. Let X be any sequence space and let m be a non-negative integer. Then,

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

$\Delta^0 x = (x_k)\,, \Delta^m x = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$. The sequence spaces $\Delta^m(X)$ are Banach spaces normed by

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$$\|x\|_{\Delta} = \sum_{i=1}^{m} |x_i| + \|\Delta^m x_k\|_{\infty} .$$

Let X be any sequence space, if $x \in X(\Delta^m)$ then there exists one and only one $y = (y_k) \in X$ such that

$$x_k = \sum_{i=1}^{k-m} (-1)^m \binom{k-i-1}{m-1} y_i = \sum_{i=1}^{k} (-1)^m \binom{k+m-i-1}{m-1} y_{i-m},$$
$$y_{1-m} = y_{2-m} = \cdots = y_0 = 0 \tag{1}$$

for sufficiently large k, for instance $k > 2m$. Recently, a large amount of work has been carried out by many mathematicians regarding various generalizations of difference sequence spaces. For a detailed account of difference sequence spaces one may refer to ([1-12],[14-15]).

The Cesàro sequence spaces Ces_p and Ces_∞ have been introduced by Shiue [19]. Jagers [13] has determined the Köthe duals of the sequence space Ces_p $(1 < p < \infty)$. It can be shown that the inclusion $\ell_p \subset Ces_p$ is strict for $1 < p < \infty$. Later on the Cesàro sequence spaces X_p and X_∞ of non-absolute type are defined by Ng and Lee ([17],[18]).

2 Topological Properties of $C_1(\Delta^m)$ and $C_\infty(\Delta^m)$

In this section we prove some results involving the sequence spaces $C_1(\Delta^m)$ and $C_\infty(\Delta^m)$.

Definition 1 *Let m be a non-negative integer. We define the following sequence spaces:*

$$C_1(\Delta^m) = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} (\Delta^m x_k - L) = 0 \right\},$$
$$C_\infty(\Delta^m) = \left\{ x = (x_k) : \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right) < \infty \right\}.$$

The above sequence spaces contain some unbounded sequences for $m \geq 1$, for example let $x = (k^m)$, then $x \in C_\infty(\Delta^m)$, but $x \notin \ell_\infty$.

Theorem 2 *The sequence spaces $C_1(\Delta^m)$ and $C_\infty(\Delta^m)$ are Banach spaces normed by*

$$\|x\|_{\Delta} = \sum_{i=1}^{m} |x_i| + \sup_n \frac{1}{n} \left| \sum_{k=1}^{n} \Delta^m x_k \right| .$$

Proof. Prof follows from Theorem 2.2 of Et and Nuray [12]. ■

Theorem 3 $\ell_\infty(\Delta^{m-1}) \subset C_1(\Delta^m)$ *and the inclusion is strict.*

Proof. Let $x \in \ell_\infty(\Delta^{m-1})$, then there exists $M > 0$ such that $\left|\Delta^{m-1} x_k\right| \le M$, for all $k \in \mathbb{N}$. On the other hand, we can write

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} \Delta^m x_k\right| &= \frac{1}{n}\left|(\Delta^m x_1 + \Delta^m x_2 + \ldots + \Delta^m x_n)\right| \\
&= \frac{1}{n}\left|\left\{\left(\Delta^{m-1} x_1 - \Delta^{m-1} x_2\right) + \left(\Delta^{m-1} x_2 - \Delta^{m-1} x_3\right) + \ldots + \left(\Delta^{m-1} x_n - \Delta^{m-1} x_{n+1}\right)\right\}\right| \\
&= \frac{1}{n}\left|\left\{\Delta^{m-1} x_1 - \Delta^{m-1} x_{n+1}\right\}\right| \\
&\le \frac{1}{n}\left|\Delta^{m-1} x_1\right| + \frac{1}{n}\left|\Delta^{m-1} x_{n+1}\right| \\
&\le \frac{2M}{n} \to 0, \;\; (n \to \infty)\,.
\end{aligned}$$

So $x \in C_1(\Delta^m)$. For strict inclusion, observe that $(k^m) \in C_1(\Delta^m)$ but $(k^m) \notin \ell_\infty(\Delta^{m-1})$ (If $x_k = k^m$, then $\Delta^m x_k = (-1)^m\, m!$ and $\Delta^{m-1} x_k = (-1)^{m+1}\, m! \left[k + \frac{m-1}{2}\right]$ for all $k \in \mathbb{N}$). ■

Theorem 4 *The sequence spaces $C_1(\Delta^{m-1})$ and $c(\Delta^m)$ overlap without containing each other, but $c(\Delta^m) \subset C_1(\Delta^m)$ and the inclusion is strict.*

Proof. Proof is similar to that of Theorem 3. ■

The proof of the following result is easy, so we state without proof.

Theorem 5 *$C_1(\Delta^m)$ is a closed subspace of $C_\infty(\Delta^m)$.*

Theorem 6 *$C_1(\Delta^m)$ is a nowhere dense subset of $C_\infty(\Delta^m)$.*

Proof. Proof follows from the fact that $C_1(\Delta^m)$ is a proper and complete subspace of $C_\infty(\Delta^m)$. ■

Theorem 7 *$C_\infty(\Delta^m)$ is not separable, in general.*

Proof. Proof follows from Theorem 5 of Bhardwaj et al. [5]. ■

Theorem 8 *$C_\infty(\Delta^m)$ does not have Schauder basis.*

Proof. Proof follows from the fact that if a normed space has a Schauder basis, then it is separable. ■

Theorem 9 *$C_1(\Delta^m)$ is separable.*

Proof. Proof follows from Theorem 2.5 of Et and Nuray [12]. ■

Theorem 10 *$C_1(\Delta^m)$ does not have the AK property.*

Proof. Let $x = (x_k) = (k^m) = (1^m, 2^m, 3^m, 4^m, ...) \in C_1(\Delta^m)$. Consider the n^{th} section of the sequence (x_k) as $x^{[n]} = (1^m, 2^m, 3^m, 4^m, ...n^m, 0, 0, ...)$. Then

$$\begin{aligned} \left\| x - x^{[n]} \right\|_\Delta &= \left\| \left(0, 0, 0, 0, ...0, n^{m+1}, n^{m+2}, ...\right) \right\|_\Delta \\ &= \frac{1}{n} \left[(-1)^{m+1}\, m! \left((n+1) + \frac{m-1}{2} \right) \right] \end{aligned}$$

which does not tend to 0 as $n \to \infty$. ■

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