

Some Novel Exponential and Complex Structural Properties of the Fisher Equation Arising in Mathematical Bioscience

Hasan Bulut^{1,*}, Sibel Sehriban Atas^{1,**}, and Haci Mehmet Baskonus^{2,***}

¹Faculty of Science, Firat University, Elazig, Turkey

²Faculty of Engineering, Munzur University, Tunceli, Turkey

Abstract. In this manuscript, we consider the Bernoulli sub-equation function method for obtaining new exponential and complex prototype structures to the Fisher Equation arising in Mathematical biosciences. We obtain new results by using the technique for new properties of model and for more understanding of properties of model. We plot two- and three-dimensional surfaces of the results by using Wolfram Mathematica 9. At the end of this manuscript, we submit a conclusion in the comprehensive manner.

1 Introduction

The studies of the solutions to the nonlinear partial differential equation have become of significance important in the fields of science, engineering, mathematical physics and biosciences, such as chemistry, plasma physics, fluid mechanics, etc.

Many studies have been presented to the literature in terms of Mathematical modal of humankind problem. For example, E.C. Alvord et al have investigated the density or concentration of cancer cells [1, 2]. M. Beren et al one characteristic of glioblastoma cells which has gained considerable attention is the 'go or grow'- hypothesis, which states that proliferation and migration are mutually exclusive phenotypes of glioblastoma cells [3]. Many different models of glioblastoma growth have been proposed by D.Basanta et al have searched for game theoretical models [4], and K.R.Swanson et al systems of partial differential equations [5], E. Khain et al to individual-based models [6]. The effects of the density driven switching have observed by Pham et al [7].

In recent years, some researchers have focused on individual-based model. For example, Nelander et al have concerned with the analysis of an individual-based model [8]. Moreover, some important results have been submitted to the literature by various scientists [9, 10].

The paper is organized as follows: In section 2, we describe the Bernoulli Sub-Equation function method (BSEFM). In section 3, we consider the following Fisher equation defined by [11]

$$\omega u_{xx} - u_t + \rho u - \rho u^2 = 0, \quad (1)$$

Moreover, in section 4 we give the physical interpretations and remarks of the solutions obtained by BSEFM. Also a comprehensive conclusion is given in section 5.

*e-mail: hbulut@firat.edu.tr

**e-mail: suzundag90@gmail.com

***e-mail: hmbaskonus@gmail.com

2 Fundamental Properties of Bernoulli Sub-Equation Function Method

An approach to the mathematical models including partial differential equations will be presented in this sub-section of paper. The steps of the this technique which is partially new modified can be taken as follows [12].

Step-1. We consider the partial differential equation in two variables such as x, t and a dependent variable u ;

$$P(u_x, u_t, u_{xx}, u_{xt}, \dots), \tag{2}$$

and take the wave transformation

$$u(x, t) = U(\eta), \quad \eta = x - ct, \tag{3}$$

where $c \neq 0$. Substituting Eq. (3) into Eq. (2), it gives us the following nonlinear ordinary differential equation;

$$N(U, U', U'', U''', \dots). \tag{4}$$

where $U = U(\eta), U' = \frac{dU}{d\eta}, U'' = \frac{d^2U}{d\eta^2}, \dots$

Step-2. Taking the trial solution for Eq. (4) as:

$$U(\eta) = \sum_{i=0}^n a_i F^i(\eta) = a_0 + a_1 F(\eta) + a_2 F^2(\eta) + \dots + a_n F^n(\eta), \tag{5}$$

where

$$F' = bF(\eta) + dF^M(\eta), \tag{6}$$

where $b \neq 0, d \neq 0, M \in \mathbb{R} \setminus \{0, 1, 2\}$, and $F(\eta)$ is Bernoulli differential polynomial. Substituting above relations into Eq. (4), we obtain an equation of polynomial $\Omega(F(\eta))$ of $F(\eta)$;

$$\Omega(F(\eta)) = \rho_s F^s(\eta) + \dots + \rho_1 F(\eta) + \rho_0 = 0. \tag{7}$$

According to the balance principle, we can get values of s and M and n .

Step-3. Let us consider the coefficients of $\Omega(F(\eta))$ all be zero, we will obtain an algebraic equations system:

$$\rho_i = 0, \quad i = 0, \dots, s. \tag{8}$$

Solving this system, we will determine the values of a_0, \dots, a_n .

Step-4. When we solve nonlinear Bernoulli differential equation that is Eq. (6), we obtain the following two situations according to b and d ;

$$F(\eta) = \left[\frac{-d}{b} + \frac{E}{e^{b(M-1)\eta}} \right]^{\frac{1}{1-M}}, \quad b \neq d \tag{9}$$

$$F(\eta) = \left[\frac{(E-1) + (E+1)\tanh\left(\frac{b(1-M)\eta}{2}\right)}{1 - \tanh\left(\frac{b(1-M)\eta}{2}\right)} \right]^{\frac{1}{1-M}}, \quad b = d, E \in \mathbb{R}. \tag{10}$$

Using a complete discrimination system for polynomial, we obtain the solutions to Eq.(4) with the help of Wolfram Mathematica 9 programming and classify the exact solutions to Eq.(4). For a better understanding of results obtained in this way, we can plot two- and three-dimensional surfaces of solutions by taking into consideration suitable values of parameters.

3 Application

In this section, we have successfully applied the BSEFM to the Fisher equation for getting some new complex travelling wave solutions.

When we consider the travelling wave transformation and perform the transformation $u = U(\eta), \eta = x - ct$, where c is constant non-zero in Eq. (1), we obtain NLODE as following;

$$\omega U' + cU' + \rho U - \rho U^2 = 0. \tag{11}$$

Applying the homogeneous balancing technique on Eq. (11) by using the highest derivative U'' and the highest power nonlinear term U^2 , yields the following relation between n and M ;

$$2M = n + 2 \quad n, M \in \mathbb{Z}^+. \tag{12}$$

If we take as $n = 4$ and $M = 3$ in Eq. (5), we can write following equation;

$$U(\eta) = a_0 + a_1 F + a_2 F^2 + a_3 F^3 + a_4 F^4, \tag{13}$$

differentiating Eq. (13) twice, gives the following equations;

$$U' = a_1 b F + a_1 d F^3 + 2a_2 b F^2 + 2a_2 d F^4 + 3a_3 F^3 + 3a_3 d F^5 + 4a_4 b F^4 + 4a_4 d F^6, \tag{14}$$

$$U'' = a_1 b F' + a_1 d F^2 F' + 4a_2 b F F' + 16a_4 d F^3 F' + 8a_2 d F^3 F' + 9a_3 b F^2 F' + 15a_3 F^4 F' + 24a_4 d F^5 F', \tag{15}$$

where $a_2 \neq 0, b \neq 0, d \neq 0$. When we put Eqs. (12-14) in Eq.(10), we obtain a system of algebraic equations for Eq. (13). Therefore, we obtain a system of algebraic equations from these coefficients of polynomial of F . Solving this system with the help of wolfram Mathematica 9, we find the following coefficients and solutions;

Case 1a: For $b \neq d$, it can be considered the following coefficients;

$$a_0 = 1, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = \frac{24d^2\omega}{\rho}, b = -\frac{i\sqrt{\rho}}{2\sqrt{6\omega}}, c = \frac{5i\sqrt{\rho\omega}}{\sqrt{6}}.$$

Substituting these coefficients in Eq. (13) along with Eq. (9), we obtain the following new complex exponential function solution for Eq. (1);

$$u_1(x, t) = 1 + 24d^2\omega \left(E \sqrt{\rho} e^{\frac{5\rho t}{6} + \frac{\rho}{\sqrt{6\omega}} ix} - 2\sqrt{6\omega} di \right)^{-2}, \tag{16}$$

where d, ω, ρ are real constants.

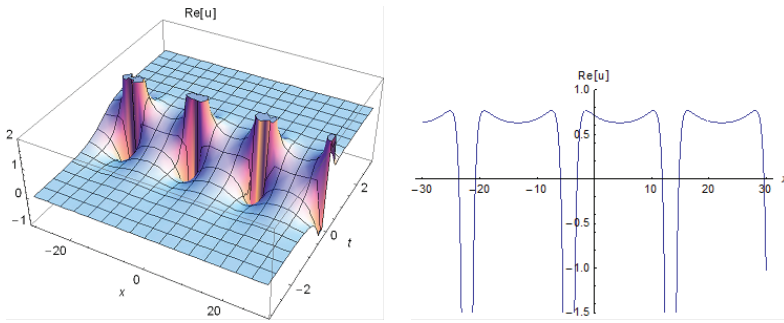


Figure 1. The The 2D and 3D surfaces of real part of Eq. (16) under the values of $\omega = 4$, $\rho = 3$, $E = -0.7$, $d = 0.2$, $-30 < x < 30$, $-3 < t < 3$ and $t = 0.02$ for the 2D surface.

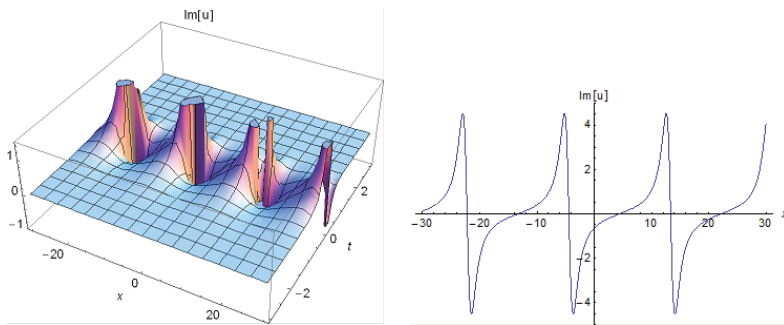


Figure 2. The The 2D and 3D surfaces of imaginary part of Eq. (16) under the values of $\omega = 4$, $\rho = 3$, $E = -0.7$, $d = 0.2$, $-30 < x < 30$, $-3 < t < 3$ and $t = 0.02$ for the 2D surface.

Case 1b: Another coefficients for Eq. (1) and for $b \neq d$, it can be considered follows;

$$a_0 = 1, a_1 = 0, a_2 = \frac{4d\sqrt{6\omega}}{\sqrt{\rho}}, a_3 = 0, a_4 = \frac{24d^2\omega}{\rho}, b = \frac{\sqrt{\rho}}{2\sqrt{6\omega}}, c = \frac{5\sqrt{\rho\omega}}{\sqrt{6}}.$$

Substituting these coefficients in Eq. (13) along with Eq. (9), we obtain the new exponential function solution for Eq. (1);

$$u_2(x, t) = E^2 \sqrt{\rho} \left(E \sqrt{\rho} - 2 \sqrt{6\omega} d e^{-\frac{5\rho}{6}t + \frac{\sqrt{\rho}}{\sqrt{6\omega}}x} \right)^{-2}, \tag{17}$$

where d, ω, ρ are real constants.

Case 1c: Another coefficients for Eq. (1) and for $b \neq d$, it can be considered follows;

$$a_0 = 0, a_1 = 0, a_2 = \frac{4di\sqrt{6\omega}}{\sqrt{\rho}}, a_3 = 0, a_4 = \frac{24d^2\omega}{\rho}, b = \frac{i\sqrt{\rho}}{2\sqrt{6\omega}}, c = \frac{5i\sqrt{\rho\omega}}{\sqrt{6}}.$$

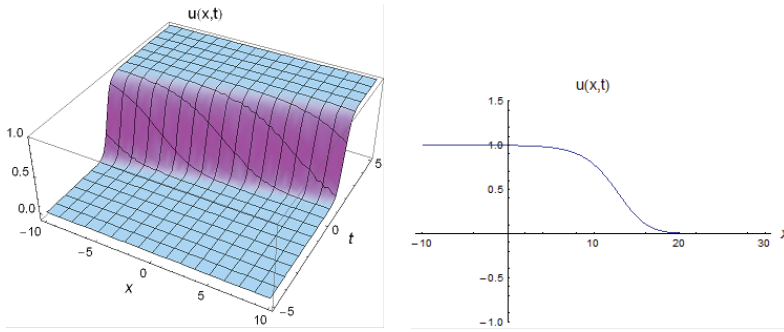


Figure 3. The The 2D and 3D surfaces of Eq. (17) under the values of $\omega = 4, \rho = 5, E = -0.7, d = 0.9,$ $-10 < x < 10, -5 < t < 5$ and $t = 2$ for the 2D surface.

Substituting these coefficients in Eq. (13) along with Eq. (9), we obtain the following new complex exponential function solution for Eq. (1);

$$u_3(x, t) = 1 - E^2 \sqrt{\rho} \left(E \sqrt{\rho} - 2i \sqrt{6\omega} de^{\frac{5\rho}{6}t + \frac{i\sqrt{\rho}}{\sqrt{6\omega}}x} \right)^{-2}, \tag{18}$$

where d, ω, ρ are real constants.

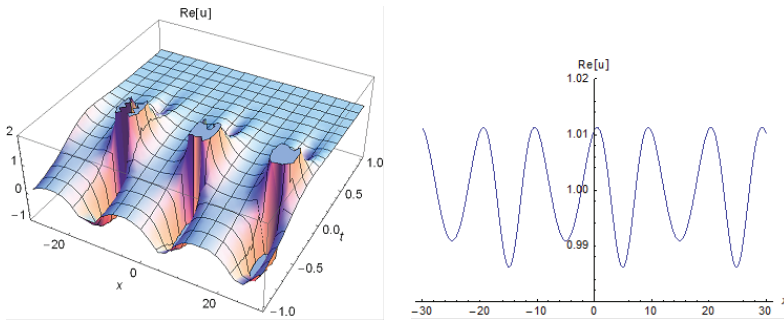


Figure 4. The The 2D and 3D surfaces of real part of Eq. (18) under the values of $\omega = 5, \rho = 3, E = 27, d = 10,$ $-30 < x < 30, -1 < t < 1$ and $t = 0.02$ for the 2D surface.

Case 1d: Another coefficients for Eq. (1) and for $b \neq d$, it can be considered follows;

$$a_0 = 0, a_1 = 0, a_2 = -\frac{4di\sqrt{6\omega}}{\sqrt{\rho}}, a_3 = 0, a_4 = \frac{24d^2\omega}{\rho}, b = -\frac{i\sqrt{\rho}}{2\sqrt{6\omega}}, c = -\frac{5i\sqrt{\rho\omega}}{\sqrt{6}}.$$

Substituting these coefficients in Eq. (13) along with Eq. (9), we obtain the following new complex exponential function solution for Eq. (1);

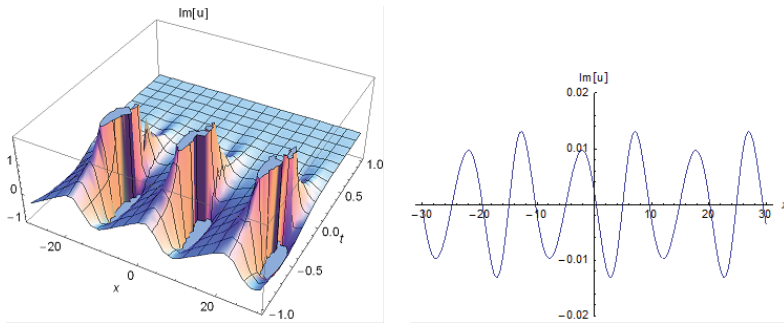


Figure 5. The The 2D and 3D surfaces of imaginary part of Eq. (18) under the values of $\omega = 5, \rho = 3, E = 27, d = 10, -30 < x < 30, -1 < t < 1$ and $t = 0.02$ for the 2D surface.

$$u_4(x, t) = \frac{4E \sqrt{6\rho\omega d} i e^{-\frac{5\rho}{6}t + \frac{i\sqrt{\rho}}{\sqrt{6\omega}}x} + 24d^2\omega}{\left(E \sqrt{\rho} i e^{-\frac{5\rho}{6}t + \frac{i\sqrt{\rho}}{\sqrt{6\omega}}x} - 2 \sqrt{6\omega d}\right)^2}, \quad (19)$$

where d, ω, ρ are real constants.

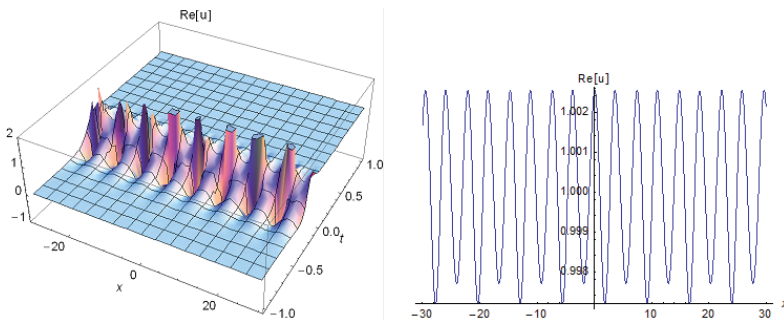


Figure 6. The The 2D and 3D surfaces of real part of Eq. (19) under the values of $\omega = 3, \rho = 13, E = 7, d = 20, -30 < x < 30, -1 < t < 1$ and $t = 0.1$ for the 2D surface.

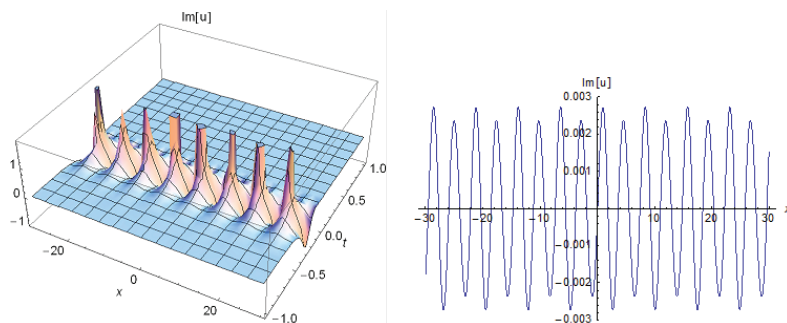


Figure 7. The 2D and 3D surfaces of imaginary part of Eq. (19) under the values of $\omega = 3, \rho = 13, E = 7, d = 20, -30 < x < 30, -1 < t < 1$ and $t = 0.1$ for the 2D surface.

4 Conclusions

The solutions such as Eq. (16, 17, 18, 19) obtained by using BSEFM are the new complex and exponential function structures solutions for Eq. (1) when we compare with the paper submitted to the literature by Nelander et al [11]. It has been observed that all solutions have verified the Eq. (1) by using Wolfram Mathematica 9. To the best of our knowledge, the application of BSEFM to the Eq. (1) has not been previously submitted to the literature. The method proposed in this paper can be used to seek more travelling wave solutions of NLEEs because the method has some advantages such as easily calculations, writing programme for obtaining coefficients and so many. Eq (17) can be re-written as hyperbolic functions by using the fundamental properties of hyperbolic functions as follows;

$$u_2(x, t) = E^2 \sqrt{\rho} \left(E \sqrt{\rho} - 2 \sqrt{6\omega} d e^{-\frac{5\rho}{6}t + \frac{\sqrt{\rho}}{\sqrt{6\omega}}x} \right)^{-2}, \tag{20}$$

where d, ω, ρ are real constants. When we consider the general properties of hyperbolic and exponential functions, we can re-write $u_2(x, t)$ solution as follows;

$$u_2(x, t) = E^2 \sqrt{\rho} \left(E \sqrt{\rho} - 2 \sqrt{6\omega} d \left(\frac{1 \pm \sqrt{1 - \operatorname{sech}^2(T)}}{\operatorname{sech}(T)} \right) \right)^{-2}, \tag{21}$$

where $T = -\frac{5\rho}{6}t + \frac{\sqrt{\rho}}{\sqrt{6\omega}}x$.

These travelling wave solutions have been introduced to the literature some important physical information about Eq. (1). As mentioned in section 1, it is estimated that this solution is related to physical features of hyperbolic function [13, 14].

When we compare these structures found in this paper with the paper which has been published by P. Gerlee and S. Nelander [11], we can say that the complex solutions is new results for the considered model. We think that they will give more properties of model considered. Therefore, the method can be applied to the another models in terms of giving new features of models.

References

- [1] A Giese, R Bjerkving and M Westphal, *J. Clin. Oncol* **21(8)**, 1624 (2003)
- [2] A Farin, SO Suzuki, M Weiker, JE Goldman, JN Bruce and P Canoll, *A dynamic analysis, Glia* **53(8)**, 799-808 (2006)
- [3] H Hatzikirou, D Basanta, M Basanta, M Simon, K Schaller and A Deutsch, *Math, Med, Biol* **29**, 49-65 (2012)
- [4] D Basanta, S JG, R Rockne, K Swanson and A Alexander, *Phys. Biol* **8**, 015016 (2011)
- [5] KR Swanson, RC Rockne, J Claridge, MA Chaplain, EC Alvord and ARA Anderson, *Cancer Res.* **71(24)**, 7366-7375 (2011)
- [6] E Khain, M Katakowski, N Charteries, F Jiang and M. Chopp, *Phys. Rev. E* **86(1)**, 011904 (2012)
- [7] K Pham, A Chauviere, H Hatzikirou, X Li, HM Byrne, V Cristini and J Lowengrub, *J. Biol, Dynam* **6**, 1-18 (2011)
- [8] P Gerlee and S Nelander, *Plos Comput, Biol* **8**, e1002556 (2012)
- [9] HM Baskonus, H Bulut, FM Belgacem, *J. Comput. Appl. Math* **312(1)**, 257-266 (2017)
- [10] FM Belgacem, H Bulut, HM Baskonus, *Adv. Math. Stat. Sci*, 445-453 (2015)
- [11] P Gerlee and S Nelander, *Mathematical Biosciences* **276**, 75-81 (2016)
- [12] B Zheng, *WSEAS Transactions on Mathematics* **7(11)**, 618-626 (2012)
- [13] EW Weisstein, *Concise Encyclopedia of Mathematics* 2nd edition, Newyork: CRC (2002)
- [14] WH Beyer, *CRC Standard Mathematical Tables* 28th edition, Boca Raton, FL: CRC (1987)