A study on the improved $\tan(\phi(\xi)/2)$ –expansion method

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Abstract. In this study, the improved $\tan(\phi(\xi)/2)$ -expansion method (ITEM), one of the improved expansion methods, has been applied to (3+1)- dimensional Jimbo Miwa and Sharma-Tasso-Olver equations using symbolic computation. With the aid of the method, many new and abundant analytical solutions have been obtained. The newly obtained results show that ITEM is a new and significant technique for solving nonlinear differential equations which plays an important role on fluids mechanics, engineering and many physics fields.

1 Introduction

Scientists have used mathematics to describe the physical properties of the universe for many years by modelling. Many fundamental phenomena observed in ecology, physics, finance, data science, mechanical engineering etc. can be formulated using differential equations. Therefore obtain analytical and numerical solutions for differential equations is a significant part of scientific studies. So, over the years a wide range of efficient methods have been suggested and improved for solving those equations such as Refs. [1–6] and many other techniques in their references. The aim of the current study is to derive new analytical solutions of (3+1)-dimensional Jimbo Miwa (JM) and Sharma-Tasso-Olver (STO) equations which are physical model by using an improved expansion method called as improved $\tan(\phi(\xi)/2)$ -expansion method. The rest of the study is organized as follows, in the second section is a description of improved $\tan(\phi(\xi)/2)$ -expansion method (ITEM) and third section consists of an application of the method to the STO and JM equations and newly obtained results. At the end of the study, conclusion and references are laid out, respectively.

2 Algorithm of the improved $\tan(\phi(\xi)/2)$ -expansion method

In this section of the study, we are going to present the algorithm of ITEM for constructing analytical solutions of nonlinear differential equations successfully. The algorithm has four steps given as follows:

Step 1: The main idea of this step is to reduce the nonlinear partial differential equation to the nonlinear ordinary differential equation. First of all, let us symbolize the NPDEs with polynomial R of $u(x, y, z, t)$ as

$$R(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, \dots) = 0 \quad (1)$$

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where x, y, z and t are independent variables and u is an unknown function. R consist of partial derivatives, the highest order derivative and nonlinear term. For reduction process, combining four independent variables x, y, z and t into one parameter ξ , i. e, using traveling wave transformation given as

$$u(x, t) = u(\xi) \quad \xi = \alpha x + \beta y + \gamma z - \mu t. \quad (2)$$

Eq. (1) reduces to nonlinear ODE follows

$$R'(u, \alpha u', \beta u', \gamma u', -\mu u', \alpha^2 u'', \beta^2 u'', \dots) = 0 \quad (3)$$

where α, β, γ and μ are real parameters.

Step 2: Assuming that the wave solution of Eq. (3) can be expressed as

$$u(\xi) = \sum_{k=0}^m A_k [p + \tan(\phi(\xi)/2)]^k + \sum_{k=1}^m B_k [p + \tan(\phi(\xi)/2)]^{-k} \quad (4)$$

where $A_0, A_k, B_k, (1 \leq k \leq m)$ and p are constants to be determined by symbolic computation, m is a positive integer to be calculated using homogenous balance between the highest order derivative and nonlinear term. Also $\phi(\xi)$ is the general solution of certain Riccati equation with constant coefficients given as

$$\phi'(\xi) = a \sin(\phi(\xi)) + b \cos(\phi(\xi)) + c. \quad (5)$$

Manafian, obtained special solutions of Eq. (5) and expressed in the following nineteen solution families, one can see in Ref. [7]

Step 3: In this step, substitution of derivatives for $u(\xi)$ respect to ξ into (3) is going to be carried out. After this proceeding, a polynomial yields in term of $\tan(\phi(\xi)/2)^k$ and $\cot(\phi(\xi)/2)^k (k = 1, 2, \dots)$ function. Collecting coefficients of the newly polynomial according to the power of trigonometric functions and setting to zero, we can get an algebraic equation system in terms of A_0, A_k, B_k, a, b, c and p .

Step 4: At the final step, after solving the algebraic equation systems which are obtained in Step 3 for A_0, A_k, B_k and μ and using these values in (4) with Families 1 ~ 19 seen in Ref.[7], the desired solutions are obtained.

3 Applications

We are going to now exert ITEM described in Section 2 to solve Jimbo Miwa and Sharma-Tasso-Olver equations.

3.1 Jimbo-Miwa equation

Consider the following nonlinear partial differential equation of which name is Jimbo Miwa equation

$$u_{xxx}y + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0. \quad (6)$$

Here we apply traveling wave transformation to Eq.(6) given by (2) with $\alpha = \beta = \gamma = 1$ and as a result of the first step, we obtain the following nonlinear ODE:

$$u^{iv} + 6u' u'' - (2\mu + 3) u'' = 0 \quad (7)$$

During the second step, using homogeneous balance between the highest order derivative u^{iv} and nonlinear term $u'u''$, solution of Eq. (7) expressed for $n = 1$ is obtained by

$$u(\xi) = A_0 + A_1 [p + \tan(\phi(\xi)/2)] + B_1 [p + \tan(\phi(\xi)/2)]^{-1} \quad (8)$$

At the last for steps 3 and 4, when we substitute the first, second and fourth order derivatives of $u(\xi)$ given in (8) into (7), we hold an over determined algebraic polynomial in terms of $A_0, A_k, B_k, p, a, b, c$ and μ . Then collecting coefficients of the newly obtained polynomial as trigonometric functions and equate to zero, and solving the algebraic system for A_0, A_k, B_k and p by symbolic computation software Maple, the following solution set is obtained.

$$\begin{aligned} \text{set 1: } \{ & A_0 = A_0, \quad A_1 = b - c, \quad B_1 = 0, \quad \mu = \frac{a^2 + b^2 - c^2 - 3}{2} \} \\ \text{set 2: } \{ & A_0 = A_0, \quad A_1 = b - c, \quad B_1 = \frac{b^2 + a^2 - c^2}{b - c}, \quad \mu = \frac{2(b^2 + a^2 - c^2) - 3}{2}, \quad p = \frac{a}{c - b} \} \end{aligned}$$

To show power and effect of the improved method, using the solution set 2 with Families 1, 2, 3, 4 and 8 in Ref. [7], we get the following solutions;

- For $a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$,

$$u_1 = A_0 + (c^2 - a^2 - b^2)^{1/2} \left(\cot \left(\frac{\sqrt{c^2 - a^2 - b^2}}{2} (\xi + C) \right) - \tan \left(\frac{\sqrt{c^2 - a^2 - b^2}}{2} (\xi + C) \right) \right)$$

- For $a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$,

$$u_2 = A_0 + (a^2 + b^2 - c^2)^{1/2} \left(\tanh \left(\frac{\sqrt{a^2 + b^2 - c^2}}{2} (\xi + C) \right) + \coth \left(\frac{\sqrt{a^2 + b^2 - c^2}}{2} (\xi + C) \right) \right)$$

- For $a^2 + b^2 - c^2 > 0$ and $b \neq 0$ and $c = 0$,

$$u_3 = A_0 + (a^2 + b^2)^{1/2} \left(\tanh \left(\frac{\sqrt{a^2 + b^2}}{2} (\xi + C) \right) + \coth \left(\frac{\sqrt{a^2 + b^2}}{2} (\xi + C) \right) \right)$$

- For $a^2 + b^2 - c^2 < 0$ and $c \neq 0$ and $b = 0$, $u_4 = A_0 - (c^2 - a^2)^{1/2} \left(\tan \left(\frac{\sqrt{c^2 - a^2}}{2} (\xi + C) \right) - \cot \left(\frac{\sqrt{c^2 - a^2}}{2} (\xi + C) \right) \right)$

- For $a^2 + b^2 = c^2$, $u_8 = A_0 + \frac{2}{\xi + C}$

In the rest of the subsection, we depict graphical simulation of the considered equation. In Fig 1, we plot 3D graphs of the solutions given above. plotting graphics, we have chosen $x, y \in [-15, 15]$, $t = 1$, $C = 1$ for all figures and $a = 2, 2, 1, 1$ $b = 3, 3, 3, 0, 1$ and $c = 4, 2, 0, 3, \sqrt{2}$ for u_1, u_2, u_3, u_4 and u_8 , respectively.

3.2 Sharma-Tasso-Olver equation

In this subsection, we will next obtain analytical solutions of the following Sharma-Tasso-Olver equation through the ITEM

$$u_t + \alpha (u^3)_x + \frac{3}{2} \alpha (u^2)_{xx} + \alpha u_{xxx} = 0 \quad (9)$$

To start, let us transform Eq.(9) into nonlinear ODE by $u(x, t) = u(\xi)$, $\xi = x - \mu t$ where with, $\beta = \gamma = 0$ and α, μ is an arbitrary parameters and μ symbolizes wave speed. Thus, we get the following equation

$$-\mu u' + 3\alpha u u' + 3\alpha (u')^2 + 3\alpha u u'' + \alpha u''' = 0. \quad (10)$$

Figure 1: JM: The simulations of u_1, u_2, u_3, u_4 and u_8 for *set2*, respectively.

Considering the balance between u''' and $(u')^2$, we find $n + 3 = 2n + 2$ so that $n = 1$. Now we can build solution structure of Eq. (10) as

$$u(\xi) = A_0 + A_1 [p + \tan(\phi(\xi)/2)] + B_1 [p + \tan(\phi(\xi)/2)]^{-1}. \quad (11)$$

Substituting solution (11) into (10) and setting each coefficients of $\tan(\phi(\xi)/2)$ and $\cot(\phi(\xi)/2)$ functions to zero, we obtain an algebraic equation system for $A_0, A_k, B_k, p, \alpha, a, b, c$ and μ . After solving these system, we attain the values of A_0, A_k, B_k and μ as follow:

$$\begin{aligned} \text{Set1} : A_0 &= \frac{p(c-b)-a+1}{3} & A_1 &= \frac{b-c}{3} & B_1 &= 0 & \mu &= \alpha \\ \text{Set2} : A_0 &= \frac{1}{3} & A_1 &= \frac{b-c}{3} & B_1 &= \frac{b^2+a^2-c^2}{3(b-c)} & p &= \frac{-a}{b-c} \end{aligned}$$

Finally, substituting obtained parameters in set 2 to the solution families 1, 2, 8 and 11 into (11), we get new analytical solutions of STO equation as follow

- For $a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$, $u_1 = \frac{1}{3} + \frac{\sqrt{c^2 - a^2 - b^2}}{3} \left(\cot\left(\frac{\sqrt{c^2 - a^2 - b^2}}{2}(\xi + C)\right) - \tan\left(\frac{\sqrt{c^2 - a^2 - b^2}}{2}(\xi + C)\right) \right)$
- For $a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$, $u_2 = \frac{1}{3} + \frac{\sqrt{a^2 + b^2 - c^2}}{3} \left(\tanh\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}(\xi + C)\right) + \coth\left(\frac{\sqrt{a^2 + b^2 - c^2}}{2}(\xi + C)\right) \right)$
- For $a^2 + b^2 = c^2$, $u_8 = \frac{1}{3} + \frac{2}{(\xi + C)}$
- For $c = a$, $u_{11} = \frac{1}{3} + \frac{b-a}{3} \left(-\frac{a}{b-a} - \frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1} \right) + \frac{b^2+a^2-c^2}{3(b-c)} \left(-\frac{a}{b-a} - \frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1} \right)^{-1}$

Then, 3D graphical simulations of u_1, u_2, u_8 and u_{11} are depicted in Figure 3 using Set 2 in Fig. 2. For plotting, the range of x and t values are chosen the same with previous subsection. In Figures, for u_1, u_2, u_8 and u_{11} , we chose $a = 1, 2, 1, 2$, $b = 1, 3, 1, 1$ and $c = 2, 3, \sqrt{2}, 2$, respectively. In the literature, there are many studies on obtaining analytical solutions of Jimbo Miwa equation [8, 9] and Sharma-Tasso-Olver equation [10, 11]. To the best knowledge of the authors of this article there is not any study in the literature about to obtain analytical solutions of the mentioned equations using ITEM and these new results.

Figure 2: STO: The simulations of u_1 , u_2 , u_8 and u_{11} for *set2*, respectively.

4 Conclusions

In summary, the aim of this study is to obtain new analytical solutions of Jimbo Miwa and Sharma-Tasso-Olver equations with the aid of ITEM and symbolic programming Maple. The newly obtained results are expressed as hyperbolic, trigonometric, exponential and rational functions. Evidently, it can be concluded that ITEM is an effective, powerful and concise mathematical tool for finding analytical solutions of other nonlinear evolution equations in science and physical nature.

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