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On Inverse Sturm- Liouville Problems With Symmetric Potentials

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Abstract

In this paper an inverse problem

$$-y'' + p(x)y = \lambda y$$
$$y'(0) - hy(0) = y'(1) + hy(1) = 0$$

where p are integrable on $[0,1]$ and satisfy the symmetry conditions $p(x) = p(1-x)$ almost everywhere in the interval $0 \le x \le 1$. We reconstruct the potential of the inverse Sturm Liouville problem's solution is obtain for finite interval with symmetric potential.

Keywords: Inverse problem symmetric potential, fixed point theorem,second-order differential equation.

Introducton

In general, an inverse eigenproblem measures the frequencies of a vibrating system and tries to obtain some physical properties of the system. Difficulties in achieving higher eigenvalues in the field of practice exist, but only in a limited amount of data. However, the solution may have been predicted beforehand. The chronological problem, for example, indicates whether the first guess is consistent with the data and how it can be changed if it does not. The results to be found are very suitable to answer such a question.
The reverse Sturm-Liouville problem is examined in at least four different ways. The most common is the problem investigated by Gel'fand and Levitan [6], in which the potential and boundary conditions are uniquely determined by spectral function. This situation has also been investigated by [13,16,22]. Second, the potential and boundary conditions are uniquely determined by two spectra. This situation can be reduced to the first case by examining [5,15,16,22]. The third is uniquely determined by potential, boundary conditions and two possible reduced spectra. This is also discussed in [3,8,14]. Knowing the boundary conditions implies that the lowest eigenvalue in one of the spectra is neglected. The latter has been shown to be potentially specific, provided that boundary conditions and a possible reduced spectrum are given, as long as it is an equally functioning function in the middle of the potential. The study [3,8,14] In this article we will introduce a constructive method by dealing with an inverse sturm liouville problem for the fourth stage
The main situation is similar to the formula obtained by Hochstadt for two potential

differences and is based on the technique developed by Hochstadt [8]. This formula leads to uniqueness theories, as shown in [3,7,8,14].
However, taking into account the relationship between the lowest eigenvalue and the boundary conditions, it is clearly stated that if the boundary conditions are given and mixed, the lowest eigenvalue is neglected. In addition, the inverse Sturm-Liouville problem leads to a natural generalization of the Borg original formula and establishes a connection between the four above mentioned conditions.

1. Statement of the main results

In this section we will consider two Sturm-Liouville problems with different potentials

Theorem 1. 1 Consider the eigenvalue problems

$$-y'' + p(x)y = \lambda y$$
$$y'(0) - hy(0) = y'(1) + hy(1) = 0 \qquad (1)$$

$$-y'' + \tilde{p}(x)y = \lambda y$$
$$y'(0) - \tilde{h}\, y(0) = y'(1) + \tilde{h}\, y(1) = 0 \qquad (2)$$

where p and $\tilde{p}$ are integrable on [0,1] and satisfy the symmetry conditions $p(x) = p(1-x)$ and $\tilde{p}(x) = \tilde{p}(1-x)$ almost everywhere in the interval $0 \le x \le 1$

Let λ_j and $\tilde{\lambda}_j$, be the eigenvalues of (1) and (2). Let $\tilde{u}_j$ and $\tilde{v}_j$ be the solution of

$$y'' + (\lambda - \tilde{p})y = 0 \qquad (3)$$
$$y(0) = 1 \quad , y'(0) = \tilde{h} \qquad (4)$$
$$v(1) = 1 \quad , v'(1) = -\tilde{h} \qquad (5)$$

with $\lambda = \lambda_j$,. Define the functions $\tilde{y}$ by

$$\tilde{z}_j = 2\frac{\tilde{v}_j - k_j \tilde{y}_j}{\omega'(\lambda_j)} \qquad (6)$$

Here $\frac{k_j}{\omega'(\lambda_j)} = \frac{1}{\int_0^1 y_j^2 dx}$ where $k_j = (-1)^j$ and $y_j(x)$ are the eigenfunctions of (1)

normalized such that $y_j(x) = 1$. If

$$\sum_j \left|\lambda_j - \tilde{\lambda}_j\right| \le \infty \qquad (7)$$

then

$$p - \tilde{p} = \sum_j \left(\tilde{z}_{,j} y_{,j}\right)' \qquad (8)$$

Proof: Then y satisfies the Volterra integral equation

$$y(x) = \cos\sqrt{\lambda}x + \frac{h}{\sqrt{\lambda}}\sin\sqrt{\lambda}x + \frac{1}{\sqrt{\lambda}}\int_0^x \sin\sqrt{\lambda}(x-t)p(t)y(t)dt \qquad (9)$$

Let $y(x)$ and $v(x)$ be the solutions of eq. (3) with initial conditions (4) and (5), where $\tilde{h}$ and $\tilde{p}$ are replaced by h and p.
Also $\lambda = s^2$ where $s = \sigma + i\tau$. From eq. (1.9) follows that for each $x, y(x)$ is an entire function of λ of order $\frac{1}{2}$ and asymptotically we have

$$y(x,\lambda) = \cos sx + O\left(\frac{e^{|\tau|x}}{|s|}\right) \qquad (10)$$

$$y'(x,\lambda) = -s\sin sx + O\left(e^{|\tau|x}\right) \qquad (11)$$

Titchmarsh [19]. Since $p(x) = p(1-x)$ a.e. we find that $v(x) = y(1-x)$ and thus e (10) and (11) provide the asymptotic expansions for v and v' as well. We introduce now the Wronskian

$$\omega(\lambda) = -hy(1,\lambda) + y'(1,\lambda) \qquad (12)$$

and note that λ is an eigenvalue of (1) iff $\omega(\lambda) = 0$. From (10) and (11) follows that the asymptotic expansion for $\omega(\lambda)$ is

$$\omega(\lambda) = s\sin s\pi + O\left(e^{|\tau|\pi}\right). \qquad (13)$$

f is continuous and that the square of f is integrable and assume that f is meromorphic

$$\Omega(x,\lambda) = \frac{\tilde{v}\int_0^x yfdz + \tilde{y}\int_x^1 vfdz}{\omega(\lambda)}$$

Here $\tilde{y}$ and $\tilde{v}$ are the solutions of (3) with initial conditions (4) and (5), and have the same asymptotic expansions as y and v. We will integrate Ω along a large contour in the λ plane.
In the s plane we let R be the rectangle with vertices at $\pm d + iO$ and $\pm d + id$ where $d = n + \frac{1}{2}$ and we let Γ be the contour in the λ plane which corresponds to the points of R for which $\tau > 0$. By using the asymptotic estimates for y, $\tilde{v}$ and ω we find that

$$\frac{\tilde{v}(x,\lambda)y(z,\lambda)}{\omega(\lambda)} = \frac{\cos s(1-x)\cos z}{s\sin s\pi} + O\left(\frac{e^{\tau(z-x)}}{|s|^2}\right)$$

This is precisely what is needed to make Titchmarsh's [19], and we conclude that

$$\frac{1}{2\pi i}\int_{\Gamma}\Omega(x,\lambda)d\lambda-\frac{1}{2\pi i}\int_{\Gamma}\int_{0}^{x}\frac{\cos s(1-x)\cos z}{s\sin s\pi}f(z)dzd\lambda-\frac{1}{2\pi i}\int_{\Gamma}\int_{x}^{1}\frac{\cos sx\cos s(1-z)}{s\sin s\pi}f(z)dzd\lambda$$

converges uniformly to zero on the interval $0 \le x \le 1$ as n $\to \infty$ .It follows from the residue theorem that the sum of the last two terms is the first $n+1$ terms of the Fourier cosine expansion of the function f. It is therefore natural to extend f as an even, 2π periodic function and since f is square integrable and $f(-\pi) = f(\pi)$ we know that the Fourier series for f converges uniformly, Zygmund [22]. By using the residue theorem to evaluate the first term in the above expression and letting $n \to \infty$ we obtain

$$f(x) = \sum_{j=0}^{\infty}\frac{\tilde{v}_j\int_0^x y_j f dz+\tilde{y}_j\int_x^1 v_j f dz}{\omega'(\lambda_j)}$$

We note that y_j and v_j represent the same eigenfunction, whereas fit y_j and v_j are just solutions of (3) with $\lambda = \lambda_j$ Since $p(x) = p(1-x)$ we see that $v_j = k_j y_j$ where $k = (-1)^j$.

If p= p and $h = \tilde{h}$ then (13) reduces to the Sturm-Liouville expansion and consequently $\frac{k_j}{\omega'(\lambda_j)} = \frac{1}{\int_0^1 y_j^2 dx}$. Let now f be equal to the first eigenfunction y_0 of (1). From (13) and definition (6) follows that

$$y_0 = \tilde{y}_0 + \frac{1}{2}\sum_j \tilde{z}_j\int_0^x y_j y_0 dt \qquad (14)$$

We can now obtain the results stated in the theorem by differentiating eq. (14) formally. To realize that let $f_j = \tilde{z}_j \int_0^x y_j y_0 dt$ dır. Thus f(0)=0 and f(0)= $\tilde{y}_j(0)$. Since y_j and y_0 are eigenfunctions of (1) and $\tilde{z}_j$ is a solution of (3) with $\lambda = \lambda_j$ we find by differentiating f_j twice and using integration by parts that

$$f_j'' + (\lambda_0 - \tilde{q})f_j = 2(\tilde{z}_j y_j)' y_{0.} \qquad (15)$$

If the expressions for f_j, f_j', f_j'' are

$$f_j = \tilde{z}_j\int_0^x y_j y_0 dt$$

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$$f_j' = \tilde{z}_j y_j y_0 + \tilde{z}_j' \int_0^x y_j y_0 dt$$

$$f_j' = 2\,\tilde{z}_j' y_j y_0 + \tilde{z}_j y_j' y_0 + \tilde{z}_j y_j y_0' + \tilde{z}_j'' \int_0^x y_j y_0 dt.$$

If you write instead of the above expressions to find the left side of equation (15)

$$f_j'' + (\lambda_0 - \tilde{q}) f_j = 2\left(\tilde{z}_j' y_j\right)' y_0 + (\lambda_0 - \lambda j)\tilde{z}_j \int_0^x y_j y_0 + \left[y_j y_0' - y_j' y_0\right]\tilde{z}_j dt$$

If we consider the integral part

$$I = (\lambda_0 - \lambda j) \int_0^x y_j y_0 dt$$

$$= \int_0^x y_j \lambda_0 y_0 dt - \int_0^x \lambda_j y_j y_0 dt$$

$$= \int_0^x y_j [\tilde{q} y_0 - y_0''] dt - \int_0^x [\tilde{q} y_0 - y_0''] y_0 dt$$

$$= \int_0^x \left(y_j'' y_0 - y_0'' y_j\right) dt$$

$$= \int_0^x \left(y_j' y_0 - y_0' y_j\right)' dt$$

We can now derive (7) and (8). By differentiating (14) and using (15) we obtain

$$y_0' - \tilde{y}_0' = \frac{1}{2} \sum_j f_j'$$

$$y_0'' - \tilde{y}_0'' = (\tilde{q} - \lambda_0)\left(y_0 - \tilde{y}_0\right) + \sum (\tilde{z}_j y_j)' y_0.$$

Thus eq. (7) follows by setting $x = 0$ in the first equation. To derive (8) from the second equation we use that $y_0'' = (q - \lambda_0) y_0$ and $\tilde{y}_0'' = (\tilde{q} - \lambda_0)\tilde{y}_0$ and note that the eigenfunction y_0 is positive in the whole interval.

We will show that $\sqrt{\lambda_j}\tilde{z}_j$, and $\tilde{z}_j$ are $O(\lambda_j - \tilde{\lambda}_j)$. Let z_j, be the eigenfunction of (1.2) corresponding to the eigenvalue $\tilde{\lambda}_j$. We will compare $\tilde{y}_j$ with z_j and let $w_j = (\tilde{y}_j - u_j)/\ (\lambda_j - \tilde{\lambda}_j)$.

Thus $w = w_j$ satisfies the differential equation

$$w'' + (\lambda - \tilde{p})w = -u_j$$

$$w(0) = w'(0) = 0$$

with $\lambda = \lambda_j$. Let φ_1 and φ_2 be solutions of the homogeneous equation, i.e. (3), with initial conditions $\varphi_{i+1}^{(j)} = \delta_{ij}$ at $x = 0$. The solution of the inhomogeneous equation is then given by

$$w(x) = \int_0^x [\varphi_1(x)\varphi_2(z) - \varphi_2(x)\varphi_1(z)]u_j(z)dz \tag{16}$$

To estimate w we must estimate φ_1 , φ_2 and z_j. We first observe that φ_1 is the solution

of the Volterra integral equation (9) with $h = 0$ and p replaced by $\tilde{p}$ and $\lambda = \lambda_j$

Let $\|\tilde{p}\|_1 = \int_0^1 |\tilde{p}|dx$ Since the potential $\tilde{p}$ is symmetric we find that $u_j(1-u) = k_j u_j(x)$ and $\tilde{v}_j(x) = \tilde{y}_j(1-x)$.It therefore follows from the definition of w, and formula (6) that

$$\tilde{z}_j(x) = 2\frac{\lambda_j - \tilde{\lambda}_j}{\omega'(\lambda_j)}[w_j(1-x) - k_j w_j(x)] \tag{17}$$

To complete our study of $\tilde{z}_j$ we must find a lower bound for $|\omega'(\lambda_j)| = \|y_j\|_2$

Here y_j is the eigenfunction of (1) normalized such that $y_j(0) = 1$.
We can now prove the validity of the formal arguments leading to (7) and (8). From the solution of equation (15) follows that

$$\frac{1}{2}\sum f_j - \frac{1}{2}\sum \tilde{z}_j(0)\varphi_2(x) - \int_0^x G(x,y)\sum(\tilde{z}_j y_j)' y_0 dz \tag{18}$$

Here $G(x,y) = \varphi_1(x)\varphi_2(z) - \varphi_2(x)\varphi_1(z)$, where φ_1 and φ_2 are solutions of the homogeneous (3) with $\lambda = \lambda_j$ and satisfy the initial conditions $\varphi_{i+1}^{(j)} = \delta_{ij}$ at $x = 0.$ To interchange the order of integration and summation we have used that $\sum(\tilde{z}_j y_j)'$ converges in L^∞, On the other hand, $y = y_0 - \tilde{y}_0$ satisfies the differential equation $y'' + (\lambda_0 - \tilde{p})y = (p - \tilde{p})y_0$ with the initial conditions $y(0) = 0$ and $y'(0) = h - \tilde{h}$ and consequently

$$y_0 - y_0 = (h - \tilde{h})\varphi_2(x) - \int_0^x G(x,y)(p - \tilde{p})y_0 dz.$$

The theorem now follows from (14) by comparing the last two equations and using the uniqueness theorem for ordinary differential equations with summable coefficients, Neumark [18]. This completes the proof.

2. Uniqueness results

The explicit formula (1.8) for the difference between two potentials is well suited for deriving some well-known uniqueness results. We will show that the potential and the boundary conditions are uniquely determined by the full spectrum. Moreover, we will prove that if the boundary conditions are known then the potential is uniquely determined by the reduced spectrum. Here the reduced spectrum is the full spectrum with the lowest eigenvalue omitted. Finally, we will show that the assumption $\sum|\lambda_j - \tilde{\lambda}_j|$ in Theorem 1 is quite restrictive. In particular, it implies that if the comparison spectrum $\{\tilde{\lambda}_j\}$ corresponds to a Sturm-Liouville problem with constant coefficients, then the Fourier series for the potential $p(x)$ will be absolutely convergent.

Corollary 2.1. Consider the eigenvalue problem (1) where p is integrable in $[0,1]$. If $p(x) = p(1-x)$ almost everywhere in $0 < x < 1$ then $p(x)$ and h are uniquely determined by the spectrum.

Proof. We have two Sturm-Liouville problems, let's have the eigenvalues $\lambda_j = \tilde{\lambda}_j$. From equations (3) and (4) follows that $\tilde{y}_j$ is an eigenfunction, and since the potential $\tilde{p}$ is symmetric we conclude that $v_j = k_j\tilde{y}_j$. This shows that all $\tilde{z}_j$ vanish identically and the right hand sides of equations (7) and (8) are zero. This completes the proof.
The symmetric property of the potential indicates that the eigenfunctions have single or double $\frac{1}{2}$ functions. For this reason, we can divide the eigenvalue problem over $[0,1]$ into two problems with the boundary conditions of Dirichlet or Neumann at $x = 1/2$. It has been shown by [12,15,16], that the potential and the boundary conditions are uniquely determined by two spectra, and can be reconstructed from this data. The result is sometimes credited to [5,15]. However, Borg proved a different, but equally precise result, namely that if the boundary conditions are given then two-possible reduced-spectra determine the potential uniquely.

Corollary 2.2. Consider the eigenvalue problem (1) where p is integrable on $[0,1]$ and satisfies $p(x) = p(1-x)$ a.e. If $p(x)$ is replaced by another symmetric potential $\tilde{p}(x)$ and the two problem8 have the same reduced spectrum then $p = \tilde{p}$.

Proof. Let λ_j and $\tilde{\lambda}_j$ be the eigenvalues corresponding to p and $\tilde{p}$. If $h = \tilde{h}$ and $\lambda_j = \tilde{\lambda}_j$ for $j = 1,2,..$ then we from Theorem-1 that $\tilde{z}_0(0) = 0$ and

$p-\tilde{p} = (\tilde{z}_0 y_0)'$. We will Show that $\tilde{z}_0$ vanishes identically. From (6) follows that $\tilde{z}_0(1-x) = \tilde{z}_0(x)$ and thus $\tilde{z}_0$ vanishes at $x=0,\frac{1}{2}$ and 1.Since $\lambda_0 < \lambda_1 = \tilde{\lambda}_1$ we conclude from Sturm comparison theorem that if $\tilde{z}_0 \neq 0$ then the eigenfunction $\tilde{y}(x,\lambda_1)$ will have at least two zeros in $(0,1)$. This is contrary to Sturm's theory of oscillation and the result follows contraception.

We can think that the problems discussed in the 1st Chamber and the 2 nd Criminal can emerge in practice. Thus, the inverse eigen problem of a cylinder can be reduced

To solve the two inverse Sturm-Liouville problem, in this case, the boundary conditions in the second eigenproblem, the first property spectrum

The eigenvalue problem using Corollary I. In Corollary 2, the lowest eigenvalue plays a special role.

References

1.V. Ambarzumian, Uber eine Frage der Eigenwerttheorie. Z. Physik., **53**, 690-695 (1929)

2.V. Barcilon, Iterative solution of the inverse Sturm-Liouville problem. J. Mathematical Phys., **15,** 287-298 (1974)

3.G.Borg, Eine Umkehrung der Sturm-Liouvillesehen Eigenwertaufgabe. Acta Math., **78,** 1-96 (1946)

4.E.Coddinoton, N.Levinson, Theory of ordinary differential equations. McGraw-Hill, New York, (1955)

5. G.M.Gasymov, B.M.Levitan, On Sturm-Liouville differential operators with discrete spectra. Amer. Math. Soc. Transl. Series 2, **68,** 21-33 (1968).

6. I. M. Gel'fand , B.M.Levitan, On the determination of a differential equation from its spectral function. Amer. Math. Soe. Transl. Series 2, **1**, 253-304 (1955)

7. O.H.Hald, Inverse eigenvalue problems for layered media. Comm. Pure Appl. Math., **30**, 69-94 (1977)

8.H. Hochstadt, The inverse Sturm-Liouville problem. Comm. Pure Appl. Math., **26**, 715-729 (1973.

9. F.W.J. Olver, Introdiction to Asymptotics and Spesial Functions,New-York and London,Academic Pres.(1974)

10. C.J.Chysan, J.Henderson, Positive solutions for singular higher order nonlinear equations,Differential Equations Dynam.Systems 2 ,153-160 (1994)

11.K. Jörgens, Spectral theory of second-order ordinary differential operators, Lectmyes delivered at Aarhus Universitet, 1962/63, Aarhus, (1964)

12. M. G. Krein, Solution of the inverse Sturm-Liouville problem. Dokl. Akad. Vaulc SSSR (N.S.), **76**, 21-24 (1941)

13. On the transfer function of a one-dimensional boundary problem of second order. Dokl. Akat. 1Vaulc SSSR (N.S.), **88**, 405-408 (1953)

14.N. Levinson, The inverse Sturm-Liouville problem. Mat. Tidsskr. B., 25-30 (1949)

15. B.M.Levitan, On the determination of a Sturm-Liouville equation by two spectra. Amer. Math. Soc. Transl. Series 2, **68**, 1-20 (1968)

16. V.A. Marcenko, Concerning the theory of a differential operator of the second order. Dokl. Akad. Nauk SSSR (N.S.), **72**, 457-460 (1950)

17. C. Willis, Inverse Sturm- liouville problems with two discontinuites. Inverse problems 2, 111-130 (1986)

18. M. A Neumark, Lineare Differential Operatoren. Akademie-Verlag, Berlin, (1963)

19. E. C.Titchmarsh, The theory of functions. Oxford University Press, London, (1939)
20. M.Koboyashi, A uniqueness proof for discontinuous inverse Sturm- liouville problems with symmetric potentials, 767-781(1989)
21.V.V.Zrkov, On inverse Sturm-Liouville problems on a finite segment. Math. USSR--Izv. 1 923-934 (1967)
22. A.Zygmund, Trigonometric series,vol.I,2nd ed.Cambridge University Press,London, (1959)