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On Some Spectral Problems for Diffusion Operator

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Abstract. In this study, we attain several spectral results for Diffusion operator. In particular, the solution functions belong to Paley-Wiener space:

$$PW_{\pi}=\left\{f\ entire,\ \left|f(\mu)\right|\leq Ce^{\pi|\operatorname{Im}\mu|},\ \int_R\left|f(\mu)\right|^2 d\mu<\infty\right\}.$$

so that required theorems are proved.

1 Introduction

The aim of this paper is to solve the following diffusion equation

$$-y''+\left[2\lambda p(x)+q(x)\right]y=\lambda^2 y,\quad x\in[0,\pi] \tag{1}$$

where the functions $p(x)$ and $q(x)$ are real-valued and $p(x)\in W_2^{m+1}[0,\pi]$, $q(x)\in W_2^{m}[0,\pi]$ for $m\geq 1$. Some spectral problems were extensively solved for the diffusion operator in references [1-4].

Consider the problem

$$-y''+\left[2\lambda p(x)+q(x)\right]y=\lambda^2 y, \tag{2}$$

$$y(0)=1,\quad y'(0)=h, \tag{3}$$

where h is a finite number. Let us denote by $\varphi(x,\lambda)$ the solution of (2) satisfying the initial conditions (3). Let [2]

$$\varphi(x,\lambda)=\cos\left[\lambda x-\alpha(x)\right]+\int_0^x A(x,t)\cos\lambda t\,dt+\int_0^x B(x,t)\sin\lambda t\,dt, \tag{4}$$

where

$$\alpha(x)=x.p(0)+2\int_0^x\left\{A(\xi,\xi)\sin\alpha(\xi)-B(\xi,\xi)\cos\alpha(\xi)\right\}d\xi$$

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$$q(x)=-p^2(x)+2\frac{d}{dx}\{A(x,x)\cos\alpha(x)+B(x,x)\sin\alpha(x)\}$$
$$A(0,0)=h,\quad B(x,0)=0,\quad \left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0}=0,\quad \alpha(x)=\int_0^x p(t)\,dt \tag{5}$$

and

$$\lambda_n=n+c_0+\frac{c_1}{n}+\frac{c_{1,n}}{n}, \tag{6}$$

be the nth eigenvalue where

$$c_0=\frac{1}{\pi}\int_0^\pi p(x)\,dx,\qquad \sum_n\left|c_{1,n}\right|^2<\infty,$$
$$c_1=\frac{1}{\pi}\left(h+H+\frac{1}{2}\int_0^\pi\left[q(x)+p^2(x)\right]dx\right), \tag{7}$$

and H is a finite number.

Consider the Diffusion equation with more general separable boundary conditions

$$-y''+\left[2\lambda p(x)+q(x)\right]y=\lambda^2 y,\quad x\in[0,\pi],$$
$$a_{11}y(0,\lambda)-a_{12}y'(0,\lambda)=0, \tag{8}$$
$$a_{21}y(\pi,\lambda)+a_{22}y'(\pi,\lambda)=0.$$

where $a_{11}^2+a_{12}^2\neq 0$, $a_{21}^2+a_{22}^2\neq 0$. So, let $\lambda=\mu^2$ and $y(x,\mu^2)$ denote the solution of the initial value problem

$$-y''+\left[2\mu^2 p(x)+q(x)\right]y=\mu^4 y,$$
$$y(0,\mu^2)=a_{12},\quad y'(0,\mu^2)=a_{11}.$$

The eigenvalues of (8) are the square of the zeroes of the boundary function $B(\mu)$,

$$B(\mu):=a_{21}y(\pi,\mu^2)+a_{22}y'(\pi,\mu^2).$$

In the Dirichlet case, this boundary function is an entire function of μ of order 1 and type π and is square integrable on the real line. Therefore, it belongs to the Paley-Wiener space.

2 Main Results

Let

$$v_1(x,\mu)=y(x,\mu^2)-\cos\left[\mu^2 x-\alpha(x)\right]$$
$$=\int_0^x A(x,t)\cos(\mu^2 t)\,dt+\int_0^x B(x,t)\sin(\mu^2 t)\,dt \tag{1}$$

In the following, we shall make use of the estimates [5],

$$|\cos u|\le e^{|\mathrm{Im}\,u|},\quad |\sin u|\le c_0 e^{|\mathrm{Im}\,u|}, \tag{2}$$

where c_0 is some constant (we may take $c_0=1.72$ for numerical purposes).

Define the constants

$$c_1=\int_0^{\pi}\max_{0\le x\le\pi}\left|A(x,t)\right|dt,\quad c_2=\int_0^{\pi}\max_{0\le x\le\pi}\left|B(x,t)\right|dt,$$
$$c_3=\left(c_1+c_0c_2\right)\left|a_{21}\right|+c_0c_2\left|a_{22}\right|. \tag{3}$$

we claim the subsequent results.

Theorem 1 $v_1(x,\mu)\in PW_x$ is a function of μ for each x and the following estimate hold:

$$\left|v_1(x,\mu)\right|\le\left(c_1+c_0c_2\right)e^{x\left|\operatorname{Im}\mu^2\right|} \tag{4}$$

Proof In the first instance

$$\left|v_1(x,\mu)\right|\le\int_0^x\left|A(x,t)\right|\left|\cos\left(\mu^2t\right)\right|dt+\int_0^x\left|B(x,t)\right|\left|\sin\left(\mu^2t\right)\right|dt \tag{5}$$

Thus,

$$\left|v_1(x,\mu)\right|\le\int_0^x\left|A(x,t)\right|e^{\left|\operatorname{Im}\left(\mu^2t\right)\right|}dt+c_0\int_0^x\left|B(x,t)\right|e^{\left|\operatorname{Im}\left(\mu^2t\right)\right|}dt$$
$$\le e^{x\left|\operatorname{Im}\mu^2\right|}\left(\int_0^x\left|A(x,t)\right|dt+c_0\int_0^x\left|B(x,t)\right|dt\right)$$
$$\le e^{x\left|\operatorname{Im}\mu^2\right|}\left(\int_0^{\pi}\max_{0\le x\le\pi}\left(\left|A(x,t)\right|\right)dt+c_0\int_0^{\pi}\max_{0\le x\le\pi}\left(\left|B(x,t)\right|\right)dt\right)$$

from which we obtain

$$\left|v_1(x,\mu)\right|\le e^{x\left|\operatorname{Im}\mu^2\right|}\left(c_1+c_0c_2\right)$$

So, we have proved the estimate. Therefore, $v_1(x,\mu)$ is entirely of type x order 1 and square integrable on the real line as a function of μ for each x.

Theorem 2 $v_2(x,\mu)\in PW_x$ is a function of μ for each x and the following estimate hold:

$$\left|v_2(x,\mu)\right|\le c_0c_2e^{x\left|\operatorname{Im}\mu^2\right|} \tag{6}$$

Proof

$$v_2(x,\mu)=\int_0^x\left\{A(x,t)\cos\left(\mu^2t\right)dt+B(x,t)\sin\left(\mu^2t\right)\right\}dt-\int_0^x\left\{A(x,t)\cos\left(\mu^2t\right)\right\}dt$$

$$\left|v_2(x,\mu)\right|\le\int_0^x\left|B(x,t)\right|\left|\sin\left(\mu^2t\right)\right|dt$$
$$\le\int_0^x c_0\left|B(x,t)\right|e^{\left|\operatorname{Im}\left(\mu^2t\right)\right|}dt$$
$$\le c_0e^{x\left|\operatorname{Im}\mu^2\right|}\left(\int_0^{\pi}\max_{0\le x\le\pi}\left|B(x,t)\right|dt\right)$$

from which we get

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$$\left|v_2(x,\mu)\right| \le (c_0 c_2) e^{x\left|\operatorname{Im}\mu^2\right|}$$

Hence, function $v_2(x,\mu)$ is entirely of type x order 1 and square integrable on the real line as a function of μ for each x.

The boundary function(characteristic equation) $B(\mu)$ is not necessarily in PW_π as in the Dirichlet-Dirichlet case. However, we have the following theorem.

Theorem 3 $\tilde{B}(\pi,\mu) = a_{21}v_1(\pi,\mu) + a_{22}v_2(\pi,\mu) \in PW_\pi$ is a function of μ and the following estimate holds:

$$\left|\tilde{B}(\pi,\mu)\right| \le c_3 e^{\pi\left|\operatorname{Im}\mu^2\right|} \tag{7}$$

Proof Primarily, we have

$$\left|\tilde{B}(x,\mu)\right| \le \left|a_{21}\right|\left|v_1(x,\mu)\right| + \left|a_{22}\right|\left|v_2(x,\mu)\right|$$

from which we acquire

$$\begin{aligned}\left|\tilde{B}(x,\mu)\right| &\le \left|a_{21}\right|(c_1 + c_0 c_2) e^{x\left|\operatorname{Im}\mu^2\right|} + \left|a_{22}\right| c_0 c_2 e^{x\left|\operatorname{Im}\mu^2\right|} \\ &\le e^{x\left|\operatorname{Im}\mu^2\right|}\left((c_1 + c_0 c_2)\left|a_{21}\right| + c_0 c_2\left|a_{22}\right|\right)\end{aligned}$$

so

$$\left|\tilde{B}(\pi,\mu)\right| \le c_3 e^{\pi\left|\operatorname{Im}\mu^2\right|}$$

$\left|\tilde{B}(\pi,\mu)\right|$ is easily seen from above mentioned inequality. Hence, the following theorem is applicable.

Theorem 4 Let $f \in PW_\pi$, then

$$f(\mu) = \sum_{k=-\infty}^{\infty} f(k)\frac{\sin\pi(\mu-k)}{\pi(\mu-k)}, \tag{8}$$

where the series converges uniformly on compact set of $\mathbb{R}$ and also in $L^2_{d\mu}$ [6].

Let $\tilde{B}_N(\pi,\mu)$ denote the truncation of $\tilde{B}(\pi,\mu)$

$$\tilde{B}_N(\pi,\mu) = \sum_{k=-N}^{N} \tilde{B}(\pi,k)\frac{\sin\pi(\mu-k)}{\pi(\mu-k)}, \tag{9}$$

and $B_N(\pi,\mu)$ the corresponding approximation to $B(\pi,\mu)$.

Conclusion

In this paper, we examined Diffusion operator and succeeded in performing our approach for Diffusion equation. The approach is based on the well established technique: Shannon's sampling theorem. Thus, we obtained satisfactory results by using the Paley-Wiener spaces.

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