On Some Spectral Problems for Diffusion Operator

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Abstract. In this study, we attain several spectral results for Diffusion operator. In particular, the solution functions belong to Paley-Wiener space:

\[ PW_2 = \left\{ f \text{ entire}, \left| f(\mu) \right| \leq C e^{\pi \Im \mu}, \int_{\mathbb{R}} |f(\mu)|^2 d\mu < \infty \right\} , \]

so that required theorems are proved.

1 Introduction

The aim of this paper is to solve the following diffusion equation

\[-y'' + \left[ 2\lambda p(x) + q(x) \right] y = \lambda^2 y, \quad x \in [0, \pi] \quad (1)\]

where the functions \( p(x) \) and \( q(x) \) are real-valued and \( p(x) \in W_2^{m+1}[0, \pi] \), \( q(x) \in W_2^m[0, \pi] \) for \( m \geq 1 \). Some spectral problems were extensively solved for the diffusion operator in references [1-4].

Consider the problem

\[-y'' + \left[ 2\lambda p(x) + q(x) \right] y = \lambda^2 y, \quad (2)\]

\[ y(0) = 1, \quad y'(0) = h, \quad (3)\]

where \( h \) is a finite number. Let us denote by \( \varphi(x, \lambda) \) the solution of (2) satisfying the initial conditions (3). Let [2]

\[ \varphi(x, \lambda) = \cos \left[ \lambda x - \alpha(x) \right] + \int_0^x A(x, t) \cos \lambda t \, dt + \int_0^x B(x, t) \sin \lambda t \, dt , \quad (4)\]

where

\[ \alpha(x) = x \cdot p(0) + 2 \int_0^x \left\{ A(\xi, \xi) \sin \alpha(\xi) - B(\xi, \xi) \cos \alpha(\xi) \right\} d\xi \]

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$$q(x) = -p^2(x) + 2 \frac{d}{dx} \{ A(x,x) \cos \alpha(x) + B(x,x) \sin \alpha(x) \}$$

$$A(0,0) = h, \quad B(x,0) = 0, \quad \frac{\partial A(x,t)}{\partial t} \bigg|_{t=0} = 0, \quad \alpha(x) = \int_0^x p(t) dt$$

and

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n},$$

be the $n$th eigenvalue where

$$c_0 = \frac{1}{\pi} \int_0^\pi p(x) dx, \quad \sum_n |c_{1,n}|^2 < \infty, \quad c_1 = \frac{1}{\pi} \left( h + H + \frac{1}{2} \int_0^\pi \left[ q(x) + p^2(x) \right] dx \right),$$

and $H$ is a finite number.

Consider the Diffusion equation with more general separable boundary conditions

$$-y'' + \left[ 2 \lambda p(x) + q(x) \right] y = \lambda^2 y, \quad x \in [0,\pi],$$

$$a_{11}y(0,\lambda) - a_{12}y'(0,\lambda) = 0,$$

$$a_{21}y(\pi,\lambda) + a_{22}y'(\pi,\lambda) = 0. \quad (8)$$

where $a_{11}^2 + a_{12}^2 \neq 0, \quad a_{21}^2 + a_{22}^2 \neq 0.$ So, let $\lambda = \mu^2$ and $y(x,\mu^2)$ denote the solution of the initial value problem

$$-y'' + \left[ 2 \mu^2 p(x) + q(x) \right] y = \mu^4 y,$$

$$y(0,\mu^2) = a_{12}, \quad y'(0,\mu^2) = a_{11}. \quad$$

The eigenvalues of (8) are the square of the zeroes of the boundary function $B(\mu),$

$$B(\mu) := a_{21}y(\pi,\mu^2) + a_{22}y'(\pi,\mu^2).$$

In the Dirichlet case, this boundary function is an entire function of $\mu$ of order 1 and type $\pi$ and is square integrable on the real line. Therefore, it belongs to the Paley-Wiener space.

### 2 Main Results

Let

$$v_1(x,\mu) = y(x,\mu^2) - \cos \left[ \mu^2 x - \alpha(x) \right]$$

$$= \int_0^x A(x,t) \cos (\mu^2 t) dt + \int_0^x B(x,t) \sin (\mu^2 t) dt$$

In the following, we shall make use of the estimates [5],

$$|\cos u| \leq e^{\Im u}, \quad |\sin u| \leq c_0 e^{\Im u}, \quad (2)$$

where $c_0$ is some constant (we may take $c_0 = 1.72$ for numerical purposes).

Define the constants
Define the constants
\[ c_1 = \int_0^\pi \max_{0 \leq x \leq \pi} |A(x,t)| dt, \quad c_2 = \int_0^\pi \max_{0 \leq x \leq \pi} |B(x,t)| dt, \]
\[ c_3 = (c_1 + c_0c_2) |a_{21}| + c_0c_2 |a_{22}|. \]
we claim the subsequent results.

**Theorem 1** \( v_1(x,\mu) \in PW_x \) is a function of \( \mu \) for each \( x \) and the following estimate hold:
\[ |v_1(x,\mu)| \leq (c_1 + c_0c_2) e^{x|\text{Im}\mu^2|} \] 

**Proof** In the first instance
\[ |v_1(x,\mu)| \leq \int_0^x |A(x,t)||\cos(\mu^2 t)| dt + \int_0^x |B(x,t)||\sin(\mu^2 t)| dt \]
Thus,
\[ |v_1(x,\mu)| \leq \int_0^x |A(x,t)| e^{x|\text{Im}(\mu^2)|} dt + c_0 \int_0^x |B(x,t)| e^{x|\text{Im}(\mu^2)|} dt \]
\[ \leq e^{x|\text{Im}\mu^2|} \left( \int_0^x |A(x,t)| dt + c_0 \int_0^x |B(x,t)| dt \right) \]
\[ \leq e^{x|\text{Im}\mu^2|} \left( \int_0^\pi \max_{0 \leq x \leq \pi} (|A(x,t)|) dt + c_0 \int_0^\pi \max_{0 \leq x \leq \pi} (|B(x,t)|) dt \right) \]
from which we obtain
\[ |v_1(x,\mu)| \leq e^{x|\text{Im}\mu^2|} (c_1 + c_0c_2) \]
So, we have proved the estimate. Therefore, \( v_1(x,\mu) \) is entirely of type \( x \) order 1 and square integrable on the real line as a function of \( \mu \) for each \( x \).

**Theorem 2** \( v_2(x,\mu) \in PW_x \) is a function of \( \mu \) for each \( x \) and the following estimate hold:
\[ |v_2(x,\mu)| \leq c_0c_2 e^{x|\text{Im}\mu^2|} \] 

**Proof**
\[ v_2(x,\mu) = \int_0^x \left\{ A(x,t)\cos(\mu^2 t) dt + B(x,t)\sin(\mu^2 t) \right\} dt - \int_0^x \left\{ A(x,t)\cos(\mu^2 t) \right\} dt \]
\[ |v_2(x,\mu)| \leq \int_0^x |B(x,t)||\sin(\mu^2 t)| dt \]
\[ \leq c_0 \int_0^x |B(x,t)| e^{x|\text{Im}\mu^2|} dt \]
\[ \leq c_0e^{x|\text{Im}\mu^2|} \left( \int_0^\pi \max_{0 \leq x \leq \pi} |B(x,t)| dt \right) \]
from which we get
Hence, function $v_2(x, \mu)$ is entirely of type $x$ order 1 and square integrable on the real line as a function of $\mu$ for each $x$.

The boundary function (characteristic equation) $B(\mu)$ is not necessarily in $PW_\pi$ as in the Dirichlet-Dirichlet case. However, we have the following theorem.

**Theorem 3** $\tilde{B}(\pi, \mu) = a_{21}v_1(\pi, \mu) + a_{22}v_2(\pi, \mu) \in PW_\pi$ is a function of $\mu$ and the following estimate holds:

$$|\tilde{B}(\pi, \mu)| \leq c_3 e^{\pi|\text{Im} \mu^2|} \quad (7)$$

**Proof** Primarily, we have

$$|\tilde{B}(x, \mu)| \leq |a_{21}| |v_1(x, \mu)| + |a_{22}| |v_2(x, \mu)|$$

from which we acquire

$$|\tilde{B}(x, \mu)| \leq |a_{21}| (c_1 + c_0c_2) e^{\pi|\text{Im} \mu^2|} + |a_{22}| c_0c_2 e^{\pi|\text{Im} \mu^2|}$$

$$\leq e^{\pi|\text{Im} \mu^2|} \left( (c_1 + c_0c_2 |a_{21}| + c_0c_2 |a_{22}|) \right)$$

so

$$|\tilde{B}(\pi, \mu)| \leq c_3 e^{\pi|\text{Im} \mu^2|}$$

$|\tilde{B}(\pi, \mu)|$ is easily seen from above mentioned inequality. Hence, the following theorem is applicable.

**Theorem 4** Let $f \in PW_\pi$, then

$$f(\mu) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi (\mu - k)}{\pi (\mu - k)}, \quad (8)$$

where the series converges uniformly on compact set of $\mathbb{R}$ and also in $L^2_{d\mu}$ [6].

Let $\tilde{B}_N(\pi, \mu)$ denote the truncation of $\tilde{B}(\pi, \mu)$

$$\tilde{B}_N(\pi, \mu) = \sum_{k=-N}^{N} \tilde{B}(\pi, k) \frac{\sin \pi (\mu - k)}{\pi (\mu - k)}, \quad (9)$$

and $B_N(\pi, \mu)$ the corresponding approximation to $B(\pi, \mu)$.

**Conclusion**

In this paper, we examined Diffusion operator and succeeded in performing our approach for Diffusion equation. The approach is based on the well established technique: Shannon’s sampling theorem. Thus, we obtained satisfactory results by using the Paley-Wiener spaces.
Hence, function \( f(x, \mu) \) is entirely of type \( x \) order 1 and square integrable on the real line as a function of \( \mu \) for each \( x \).

The boundary function (characteristic equation) \( b_{\mu} \) is not necessarily in \( PW \( \pi \) as in the Dirichlet-Dirichlet case. However, we have the following theorem.

**Theorem 3**

\[
\sin k_{\mu} f(k_{\mu}) = \sum_{k=-\infty}^{\infty} \sin k_{\mu} f(k_{\mu})
\]

where the series converges uniformly on compact set of \( \mathbb{R} \) and also in \( L^2 \).

Let \( b_{\mu} \) denote the truncation of \( b_{\mu} \) the corresponding approximation to \( b_{\mu} \).

**Conclusion**

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**References**