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# Multipoint Numerical Criterion for Manifolds to Guarantee Attractor Properties in the Problem of Synergetic Control Design

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**Abstract.** The paper presents the results of our research on the application of symbolic regression methods for the numerical solution of the problem of synthesis of synergetic control. Synergetic control is characterized by the presence of manifolds in the state space that must have the properties of an attractor, in particular terminal manifolds, as well as the existence of regions of state space through which system solutions do not pass, for example, phase constraints, that must have the properties of repellers. In the present paper we formulate a numerical criterion for ensuring the property of an attractor for a terminal manifold of arbitrary dimension, taking into account the dynamic behavior of the system in a neighborhood of a given manifold. The results of computational experiments on the effective application of the proposed criterion are presented. It is shown that the proposed approach makes it possible to automate the synthesis of synergistic control using the methods of symbolic regression.

## 1 Introduction

Herman Haken [1] defines synergetic as the field of science which study self-organization in physical systems and also other related phenomena in wider class of systems.

Synergetic control as an approach to solve the problems of control synthesis was proposed in the works of A.A. Kolesnikov. [2-4]. The main idea of the approach is following. When control is synthesized that is control function that depends on the vector of coordinates of state space of control object is found, we are able to receive such system of differential equations describing dynamics of the control object together with the control system which does not contain a free vector of control in the right parts of the equations because we replace the control by the synthesized function. So all solutions of the received system of the differential equations have the properties set by the developer. Properties that must be satisfied by solutions of the differential equations describing the received closed control system are determined by the presence of manifolds in the state space, in particular terminal manifolds must possess properties of an attractor, and phase constraints must point to regions of the state space through which solutions of the system do not pass or possess the properties of repellers.

Unfortunately, the most qualitative examples of control synthesis on the basis of the synergetic theory consider objects of a low order, in particular not above the second order where properties of solutions of systems of the differential equations are obviously displayed on the plane. Practically all synergetic approaches [2], despite general conclusions, are limited to consideration of examples of second-order systems. Such low order of systems as the condition of application of the synergetic theory cannot satisfy experts in the field of control because modern control objects, such as robots or groups of robots, are described by the systems of the differential equations of much higher order. The main mathematical apparatus for synergetic control is a method of analytical construction of aggregated controllers [3-6]. This method uses the solution of a system of algebraic equations, and if the dimension of control and terminal manifold do not match, a subjective approach is required to choose the solution. The method works only for the systems of a low order too.

In this of work we consider application of modern numerical methods of symbolic regression [7-9] for the solution of tasks of control on the basis of the synergetic theory. We formulate numerical criteria to check properties of an attractor for manifolds in the state space. The presented numerical criteria take into account not only hit of the solution into the neighbourhood of the manifold but also the very behaviour of the solutions in the neighbourhood of this manifold. The proposed criterion turns out to be especially important for one-dimensional manifolds, for which it is necessary not only to get into the neighbourhood of a manifold, but also to provide a definite form of solution changing in time in the neighbourhood of the manifold.

We provide the results of numerical experiments which show efficiency of the proposed numerical criterion to check attractor properties.

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## 2 Method of Analytical Construction of Aggregated Controllers

Consider the method of analytical construction of aggregated controllers.

Given a model of control object

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}). \quad (1)$$

We want to receive such control that a terminal manifold became an attractor

$$\psi_i(\mathbf{x}) = 0, \ i = 1, \dots, r. \quad (2)$$

Let

$$\boldsymbol{\psi}(\mathbf{x}) = [\psi_1(\mathbf{x}) \ \dots \ \psi_r(\mathbf{x})]^T. \quad (3)$$

Notice

$$(Tp + 1)\boldsymbol{\psi}(\mathbf{x}) = 0, \quad (4)$$

where $T$ is a positive constant, $p$ is a differentiation operator.

Differentiate

$$T \frac{d}{dt}\boldsymbol{\psi}(\mathbf{x}) + \boldsymbol{\psi}(\mathbf{x}) = 0, \quad (5)$$

$$T \frac{\partial \boldsymbol{\psi}(\mathbf{x})}{d\mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) + \boldsymbol{\psi}(\mathbf{x}) = 0. \quad (6)$$

We have obtained a system of $r$ algebraic equations with $m$ unknown components of control vector. We need to solve these equations concerning the vector of control. If $r = m$, then the system (6) has the only one solution. If $r < m$, the system (6) has many solutions, and if $r > m$, the system has no solution.

## 3 Problem Statement of Numerical Control Synthesis

Consider the formulation of the problem of control synthesis, taking into account the possibility of solving it numerically.

Given a model of control object

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad (7)$$

where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{u} \in \mathrm{U} \subseteq \mathbf{R}^m$, U - is a compact set, $\mathbf{x}$ is a vector of state space, $\mathbf{x} = [x_1 \ \dots \ x_n]^T$, $\mathbf{u}$ is a vector of control, $\mathbf{u} = [u_1 \ \dots \ u_m]^T$, $\mathbf{f}(\mathbf{x}, \mathbf{u}) : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $\mathbf{f}(\mathbf{x}, \mathbf{u}) = [f_1(\mathbf{x}, \mathbf{u}) \ \dots \ f_n(\mathbf{x}, \mathbf{u})]^T$, $m \le n$.

Given a region of initial conditions

$$\mathrm{X}_0 \subseteq \mathbf{R}^n. \quad (8)$$

Given a terminal manifold

$$g_i(\mathbf{x}) = 0, \ i = 1, \dots, r, \ r \le n. \quad (9)$$

Given a quality criterion

$$J = \int_0^{t_f} f(\mathbf{x}, \mathbf{u}) dt \to \min, \quad (10)$$

where $t_f$ is a given time of the control process.

It is necessary to find control in the form of function of coordinates of state space

$$\mathbf{u} = \mathbf{h}(\mathbf{x}), \quad (11)$$

where $\mathbf{h}(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}^m$.

Function $\mathbf{h}(\mathbf{x})$ satisfies the following conditions: the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})),$$

for any initial condition from the region (8)

$$\forall \mathbf{x}(0) \in \mathrm{X}_0 \subseteq \mathbf{R}^n,$$

has a solution

$$\mathbf{x} = \mathbf{v}(\mathbf{x}(0), t),$$

such that it attains in a given time $t_f$ the terminal manifold (9)

$$g_i(\mathbf{v}(\mathbf{x}(0), t_f)) = 0, \ i = 1, \dots, r,$$

and does not violate constraints of control

$$\mathbf{h}(\mathbf{v}(\mathbf{x}(0), t)) \in \mathrm{U}, \ 0 \le t \le t_f,$$

and provides minimum to the quality criterion (10)

$$\int_0^{t_f} f(\mathbf{v}(\mathbf{x}(0), t), \mathbf{h}(\mathbf{v}(\mathbf{x}(0), t))) dt = \min_{\mathbf{u}(\cdot) \in \mathbf{U}^*} \int_0^{t_f} f(\mathbf{x}, \mathbf{u}) dt,$$

where $\mathrm{U}^*$ is a set of admissible controls that satisfy the control constraints and ensure the achievement of the terminal manifold (9) for the same initial conditions of the object $\mathbf{x}(0) \in \mathrm{X}_0$.

For synthesis of control we use methods of symbolic regression [7-9] which allow to find the coded mathematical expression of the required synthesizing function by means of an evolutionary algorithm.

On the basis of the synergetic theory the terminal manifold as a result of synthesis of control must possess the property of an attractor. If the terminal manifold has dimension zero, $n - r = 0$, i.e. it is a point in state space, then in numerical synthesis, the fulfilment of terminal conditions in the form

$$\| \mathbf{g}(\mathbf{x}) \| \le \varepsilon, \quad (12)$$

where $\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}) \ \dots \ g_n(\mathbf{x})]^T$, $\varepsilon$ is a set positive small value, ensures the obtaining of properties of an attractor for the terminal manifold. Here in (12) $r = n$ because the manifold is a point and it has zero dimension.

If dimension of terminal manifold is bigger than zero, $n - r > 0$, than the condition (12) is not enough for obtaining properties of attractor. It is necessary to describe behaviour of the control object after achievement of terminal manifold.

For example, the system of equations (9) at $r < n$ has many solutions. Let $\mathbf{x}^*$ is one of these solutions, $g_i(\mathbf{x}^*) = 0$, $i = 1,\ldots,r$. Then condition

$$\|\mathbf{x} - \mathbf{x}^*\| \le \varepsilon , \qquad (13)$$

is equivalent to the condition (12) and, in the synthesis, the point $\mathbf{x}^*$ can be found, which belongs to terminal manifold (9). In this case we at synthesis will provide stability of system concerning a point $\mathbf{x}^*$ of state space on terminal manifold, but not the movement of the control object on a set of points of terminal manifold.

## 4 Multipoint Criterion to Provide Attractor Properties

A multipoint criterion is appropriate to apply in the numerical synthesis of synergetic control for terminal manifolds whose dimension is greater than zero.

A multipoint criterion is in establishing additional requirements which consider peculiarities of the motion of an object after hitting to the neighborhood of a terminal manifold. According to the additional requirements, it is needed to reach a certain number of points on the terminal manifold, remaining in the neighbourhood of this manifold.

To determine the requirements to achieve a set of points on a terminal manifold, we introduce an integer function

$$\varphi(K^*, t) = K^* - \min\{K^*, k(t)\} , \qquad (14)$$

where $K^*$ is a given number of different points on the terminal manifold that must be achieved, $k(t)$ is the number of different points on the terminal manifold attained by the moment $t$

$$k(t) \leftarrow \begin{cases} 1, \text{ if } t \le t_f,\ k(t) = 0,\ \|\mathbf{g}(\mathbf{x}(t))\| \le \varepsilon, \\ \qquad \mathrm{X}_f = \{\mathbf{x}^1 = \mathbf{x}(t)\} \\ k(t)+1, \text{ if } t \le t_f,\ \|\mathbf{g}(\mathbf{x}(t))\| \le \varepsilon, \text{and} \\ \qquad \forall \mathbf{x}^j \in \mathrm{X}_f,\ \|\mathbf{x}(t) - \mathbf{x}^j\| > \varepsilon, \\ \qquad j = 1,\ldots,k(t), \\ \qquad \mathrm{X}_f \leftarrow \mathrm{X}_f \cup \{\mathbf{x}^{k(t)+1} = \mathbf{x}(t)\} \\ 0, \text{ if } t \le t_f,\ \|\mathbf{g}(\mathbf{x}(t))\| > \varepsilon \end{cases} . \qquad (15)$$

The function $\varphi(K^*, t_f)$ is either zero, if the solution succeeds in reaching $K^*$ different points on the terminal manifold by time $t_f$, or is equal to the difference between the given $K^*$ and the reached by time $t_f$ number of points $k(t_f)$ on the terminal manifold.

## 5 Examples of Numerical Synthesis of Synergetic Control

Consider a task of synthesis of control for nonlinear control object of the second order

$$\dot{x}_1 = x_2 + u_1 ,$$

$$\dot{x}_2 = -x_2 - x_1 - x_1^3 + u_2 ,$$

where

$$-1 \le u_i \le 1 ,\ i = 1,2 .$$

In the first example we consider the following terminal manifold

$$x_2 = -x_1 ,\ x_1^- \le x_1 \le x_1^+ .$$

It is necessary to rich the given terminal manifold from the region of initial conditions

$$\mathrm{X}_0 = \{[x_1(0)\ x_2(0)]^T : -1.1 \le x_i(0) \le 1, i = 1,2\} .$$

As criteria of optimality of synthesis we consider the following functional

$$J_1 = \alpha\varphi(K^*, t_f) + \int_0^{t_f} \Delta_f(x_2(t), x_1(t))dt ,$$

$$J_2 = \alpha\varphi(K^*, t_f) + \Delta_f(x_2(t_f), x_1(t_f)) ,$$

where $\Delta_f(x_2(t), x_1(t))$ is a distance to terminal manifold

$$\Delta_f(x_2(t), x_1(t)) = \begin{cases} \min\{\Delta^-(t), \Delta^+(t)\}, \\ \quad \text{if } (\beta^-(t) < 0) \vee (\beta^+(t) < 0) \\ 2S(t)/l_f \ -\text{otherwise} \end{cases} ,$$

$$\Delta^-(t) = \sqrt{(x_1(t) - x_1^-)^2 + (x_2(t) - x_2^-)^2} ,$$

$$\Delta^+(t) = \sqrt{(x_1(t) - x_1^+)^2 + (x_2(t) - x_2^+)^2} ,$$

$$\beta^-(t) = \sum_{i=1}^{2} (x_i^+ - x_i^-)(x_i(t) - x_i^-) ,$$

$$\beta^+(t) = \sum_{i=1}^{2} (x_i^- - x_i^+)(x_i(t) - x_i^+) ,$$

$$l_f = \sqrt{(x_1^+ - x_1^-)^2 + (x_2^+ - x_2^-)^2} ,$$

$$S(t) = \sqrt{p(t)(p(t) - \Delta^-(t))(p(t) - \Delta^+(t))(p(t) - l_f)} ,$$

$$p(t) = \frac{\Delta^-(t) + \Delta^+(t) + l_f}{2} ,\ x_2^- = -x_1^- ,\ x_2^+ = -x_1^+ ,$$

$\alpha$ is a weight coefficient.

Synthesis was carried out by method of binary variational genetic programming [6, 9]. In a computing experiment the following values of constants has been used: $x_1^- = -0.5$, $x_1^+ = 0.5$, $\varepsilon = 0.005$, $\alpha = 0.1$, $K^* = 8$, $t_f = 8$.

The following control has been synthesized

$$u_i = \begin{cases} 1, \text{ if } \tilde{u}_i > 1 \\ -1, \text{ if } \tilde{u}_i < -1 \\ \tilde{u}_i - \text{otherwise} \end{cases}, \quad i = 1,2,$$

where

$$\tilde{u}_1 = \sin((q_2+1)(\cos(q_1)+x_1)+q_3q_2\sin(q_1))+$$
$$+\operatorname{sgn}(q_2x_1\cos(q_2)+(q_2+1)x_2\sin(q_4)),$$
$$\tilde{u}_2 = -(q_1+1)(q_4+2)\arctan(x_2/q_2)+$$
$$(2q_4+1)\vartheta(-q_1-1+\arctan(q_3/x_1)),$$

$q_1 = 0.5439$, $q_2 = 2.7022$, $q_3 = 0.2197$, $q_4 = 3.69531$,

$\vartheta(a)$ is a Heaviside function

$$\vartheta(a) = \begin{cases} 1, \text{ if } \ a \ge 0 \\ 0 - if \end{cases},$$

Fig.1 shows the received solutions for four initial values. The red line notes the terminal manifold.

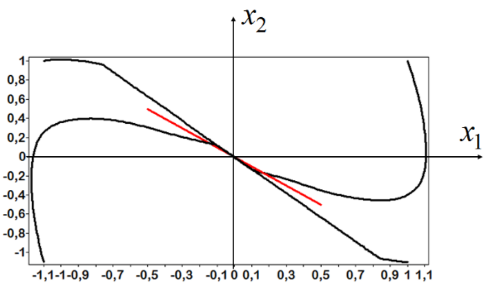


**Fig. 1.** Results of the 1st experiment.

In the second experiment we consider the following terminal manifold

$$x_2 = x_1, \quad x_1^- \le x_1 \le x_1^+,$$

where $x_1^- = -0.5$, $x_2^- = -0.5$, $x_1^+ = 0.5$, $x_2^+ = 0.5$.

The following control has been received

$$\tilde{u}_1 = (q_2+q_3)x_1\cos(q_1)+(q_3+1)x_2\sin(1)+$$
$$(q_3^2+1)x_2\sin(q_1)+q_2x_1\cos(q_2)+$$
$$q_2x_1\cos(q_2)+(q_3+1)x_2\sin(1),$$
$$\tilde{u}_2 = \exp((q_3+\sin(q_4)+q_1+\cos(1))\times$$
$$\times(\arctan(q_3\operatorname{sgn}(x_1)\sqrt{|x_1|})-(\ln|q_1|+1))+$$
$$+\operatorname{sgn}((q_1+q_2+1-(q_2+1)^3)\times$$
$$\times(\arctan(x_2/x_1)-1-\arctan(x_1)))\times$$
$$\times\sqrt{|(q_1+q_2+1-(q_2+1)^3)|}\times$$
$$\times\sqrt{|(\arctan(x_2/x_1)-1-\arctan(x_1))|}),$$

$q_1 = 2.979$, $q_2 = 2.786$, $q_3 = 3.392$, $q_4 = 1.505$.

Fig.2 shows the received solutions for four initial values and the terminal manifold.

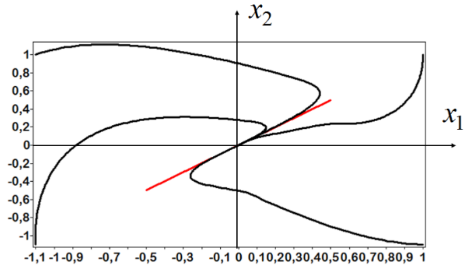


**Fig. 2.** Results of the 2nd experiment.

In the third experiment the following manifold was considered

$$x_2 = 0, \quad x_1^- \le x_1 \le x_1^+,$$

where $x_1^- = -0.5$, $x_1^+ = 0.5$,

As a result the following control has been received

$$\tilde{u}_1 = (q_2+q_2^2)x_1\cos(q_2)+(q_3+1)\sqrt[3]{\operatorname{sgn}(x_2)\sqrt{|x_2|}},$$
$$\tilde{u}_2 = (\vartheta(x_2)+q_3+1)^3(\pi/4-(x_2+\sin(q_1)))^3,$$
$$q_1 = 1.472, \quad q_2 = 2.530, \quad q_3 = 2.157, \quad q_4 = 1.213.$$

Fig.3 shows the received solutions for four initial values and the terminal manifold.

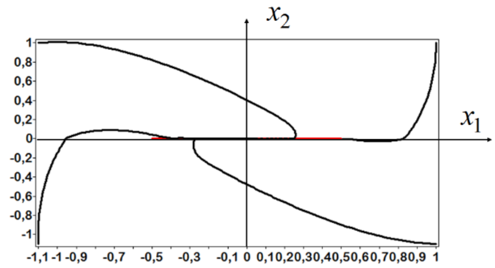


**Fig. 3.** Results of the 3rd experiment.

In the fourth experiment we considered the following terminal manifold

$$x_1 = 0, \quad x_2^- \le x_2 \le x_2^+,$$

where $x_2^- = -0.5$, $x_2^+ = 0.5$.

As a result the following control has been received

$$\tilde{u}_1 = -q_1x_1(q_2+q_1)+(q_3+1)\sqrt[3]{x_2}\sin(q_1)+$$
$$+\operatorname{sgn}((q_2+\exp(q_4))\cos(q_2)\arctan(x_1)-$$

$$-q_2 x_2 \arctan(q_3)) \times$$

$$\times(|(q_2+\exp(q_4))\cos(q_2)\arctan(x_1)-$$

$$-q_2 x_2 \arctan(q_3)|)^{0.5},$$

$$\tilde{u}_2=(\arctan(x_2/x_1)-(q_1+1))(x_2+\sqrt[3]{q_4}+2)-$$

$$-q_1(\sqrt{q_4}+3)\arctan(q_2/x_1),$$

$$q_1=3.049,\ q_2=3.395,\ q_3=1.156,\ q_4=0.002.$$

Fig.4 shows the received solutions for four initial values and the terminal manifold.

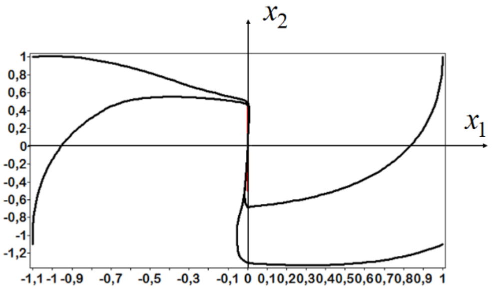


**Fig. 4.** Results of the 4th experiment.

In the fifth experiment we considered the following terminal manifold

$$x_2=x_1^2-1,\ x_1^- \le x_1 \le x_1^+,$$

where $x_1^-=-1$, $x_1^+=1$.

The following control has been received

$$u_1=\mathrm{sgn}(q_4q_3(x_2+\sin(q_1)))\sqrt{|q_4q_3(x_2+\sin(q_1)|}+$$

$$\vartheta(q_1+1+\exp(q_2^3\sin(q_1)))+$$

$$+(q_2+q_3)\cos(q_2)\sin(x_1)+$$

$$\left(\ln(q_3+1)\arctan\left(\frac{\sin(q_4)}{q_1}\right)\right)^2,$$

$$u_2=\mathrm{sgn}(1-\exp(-\mathrm{sgn}(\vartheta(x_1)-$$

$$-\sin(q_4))\sqrt{|\vartheta(x_1)+\sin(q_4)|}))\times$$

$$\times\mathrm{sgn}(x_2-\sqrt{q_4\exp(q_2)}-\sqrt{q_1}-x_2)\times$$

$$\mathrm{sgn}(1+\exp(-\mathrm{sgn}(\vartheta(x_1)-\sin(q_4))\sqrt{|\vartheta(x_1)+\sin(q_4)|}))\times$$

$$\times\mathrm{sgn}\left(\ln\left|x_2^2+q_1^2+1+q_2^3\frac{1-\exp(-x_1)}{1+\exp(-x_1)}-q_4-\sqrt{q_3}\right|\right),$$

$$q_1=1.812,\ q_2=2.815,\ q_3=3.623,\ .\ q_3=2.251$$

Fig.5 shows the received solutions for four initial values and the terminal manifold.

As can be seen from the results of the experiments, the multipoint criterion (14) provides the possibility of obtaining an attractor and moving along it for a terminal linear manifold.

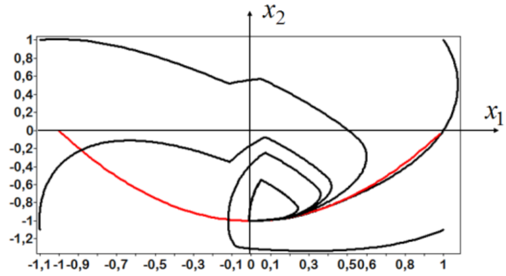


**Fig. 5.** Results of the 5th experiment.

## 6 Conclusion

For synthesis of system of synergetic control it is offered to use methods of symbolic regression. These methods allow to find needed mathematical expressions in the coded form by means of evolutionary algorithms. Criterion of search of the mathematical expression is the criterion of control system design. It is shown that for synergetic control it is insufficient to describe terminal manifold in the form of the system of algebraic equations. The multipoint criterion to check the properties of an attractor for terminal manifold is offered. By means of numerical examples it is shown that the offered multipoint criterion allows to receive controls with the necessary attractor properties of terminal manifold.

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