Recent results on weakly factorial domains

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Abstract. In this paper, we will survey recent results on weakly factorial domains based on the results of [11, 13, 14]. Let $D$ be an integral domain, $X$ be an indeterminate over $D$, $d \in D$, $R = D[X, \frac{d}{X}]$ be a subring of the Laurent polynomial ring $D[X, \frac{1}{X}]$, $\Gamma$ be a nonzero torsionless commutative cancellative monoid with quotient group $G$, and $D[\Gamma]$ be the semigroup ring of $\Gamma$ over $D$. Among other things, we show that $R$ is a weakly factorial domain if and only if $D$ is a weakly factorial GCD-domain and $d = 0$, $d$ is a unit of $D$ or $d$ is a prime element of $D$. We also show that if $\text{char}(D) = 0$ (resp., $\text{char}(D) = p > 0$), then $D[\Gamma]$ is a weakly factorial domain if and only if $D$ is a weakly factorial GCD-domain, $\Gamma$ is a weakly factorial GCD semigroup, and $G$ is of type $(0, 0, 0, \ldots)$ (resp., $(0, 0, 0, \ldots)$ except $p$).

Introduction

Let $D$ be an integral domain with quotient field $K$, $X$ be an indeterminate over $D$, $\Gamma$ be a nonzero torsionless commutative cancellative monoid (written additively) with quotient group $G$, and $D[\Gamma]$ be the semigroup ring of $\Gamma$ over $D$. A nonzero nonunit element $p \in D$ is called a primary element if $p | ab$ for each $a, b \in D$ implies that $p | a$ or $p | b^n$ for some positive integer $n$; equivalently, $pD$ is a primary ideal. We say that $D$ is a unique factorization domain (UFD) (resp., weakly factorial domain (WFD)) if each nonzero nonunit of $D$ can be written as a finite product of prime (resp. primary) elements. Clearly, a prime element is primary, and hence a UFD is a WFD. The concept of a WFD was first introduced by Anderson and Mahaney [3]. Chang introduced the notion of weakly factorial semigroups [11] in order to study when $D[\Gamma]$ is a WFD. A nonzero nonunit element $s \in \Gamma$ is primary if, for each $a, b \in \Gamma$, $a + b \in s + \Gamma$ implies that $a \in s + \Gamma$ or $nb \in s + \Gamma$ for some positive integer $n$. It is clear that $s \in \Gamma$ is primary if and only if $s + \Gamma$ is a primary ideal. We say that $\Gamma$ is a weakly factorial semigroup if each nonunit of $\Gamma$ is a finite sum of primary elements.

Chang has worked on WFDs [6–8, 11, 13, 14] and its generalization to rings with zero divisors [10, 12]. In this paper, among them, we survey the recent results of [11, 13, 14]. In Section 1, for easy reference, we review some definitions and preliminary results on $t$-operations, monoinds, semigroup rings, and weakly Krull domains. Let $d \in D$ and $R = D[X, \frac{d}{X}]$; so $R$ is a subring of the Laurent polynomial ring $D[X, \frac{1}{X}]$ containing $D[X]$. In Section 2, we show that $R$ is a weakly factorial domain if and only if $D$ is a weakly factorial GCD-domain and $d = 0$, $d$ is a unit of $D$ or $d$ is a prime element of $D$. We also show that if $D$ is a weakly factorial GCD-domain, $p$ is a prime element of $D$, and $n \geq 2$ is an integer.

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then $D[X, \frac{\partial}{\partial X}]$ is an almost weakly factorial domain with $Cl(D[X, \frac{\partial}{\partial X}]) = \mathbb{Z}_n$, the cyclic group of order $n$. In Section 3, we completely characterize when $D[\Gamma]$ is a WFD. We first study the notion of weakly factorial GCD-semigroups. Then, among other things, we prove that if $\text{char}(D) = 0$, then $D[\Gamma]$ is a WFD if and only if $D$ is a weakly factorial GCD domain, $\Gamma$ is a weakly factorial GCD semigroup, and $G$ is of type $(0,0,0,\ldots)$. We also show that if $\text{char}(D) = p > 0$, then $D[\Gamma]$ is a WFD if and only if $D$ is a weakly factorial GCD domain, $\Gamma$ is a weakly factorial GCD semigroup, and $G$ is of type $(0,0,0,\ldots)$ except $p$. However, we omit the proofs of the results; so the reader who is interested in the proofs can refer to [11, 13, 14].

1 Definitions and preliminary results

In this section, we review some definitions and preliminary results on the $t$-operations, semigroups, semigroup rings, and weakly Krull domains.

1.1 The $t$-operations

Let $D$ be an integral domain with quotient field $K$ and $F(D)$ be the set of nonzero fractional ideals of $D$. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$; then $I^{-1} \in F(D)$. Hence, we can define the $v$- and $t$-operations as follows: $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup \{J_v \mid J \subseteq I \land J \in F(D) \text{ is finitely generated}\}$. An $I \in F(D)$ is called a $t$-ideal if $I_t = I$. A $t$-ideal of $D$ is a maximal $t$-ideal if it is maximal among proper integral $t$-ideals of $D$. Let $t\text{-Max}(D)$ be the set of maximal $t$-ideals of $D$. It is well known that $t\text{-Max}(D) \neq \emptyset$ if $D$ is not a field; each ideal in $t\text{-Max}(D)$ is a prime ideal; each prime ideal minimal over a $t$-ideal is a $t$-ideal (which implies that every height-one prime ideal is a $t$-ideal); and $D = \bigcap_{t\text{-Max}(D)} D_P$. We say that $D$ is of finite $t$-character if every nonzero nonunit of $D$ is contained in only finitely many maximal $t$-ideals of $D$.

An $I \in F(D)$ is said to be $t$-invertible if $(II^{-1})_t = D$. Let $T(D)$ be the group of $t$-invertible fractional $t$-ideals of $D$ under $I \ast J = (IJ)_t$, and $\text{Prin}(D)$ be its subgroup of principal fractional ideals. Then $Cl(D) = T(D)/\text{Prin}(D)$, called the $t$-class group of $D$, is an abelian group. We say that $D$ is a Prüfer $v$-multiplication domain (PrMD) if each nonzero finitely generated ideal of $D$ is $t$-invertible. It is known that $D$ is a PrMD if and only if $D[X]$ is a PrMD [26, Corollary 4], and if and only if $D[X, \frac{\partial}{\partial X}]$ is a PrMD [24, Theorem 3.10]. Also, $D$ is a GCD-domain if and only if $D$ is a PrMD with $Cl(D) = \{0\}$ [9, Proposition 2].

1.2 Semigroups

Let $\Gamma$ be a nonzero torsionless grading monoid, i.e., a nonzero torsionless commutative cancellative monoid (written additively), and $\langle \Gamma \rangle = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of $\Gamma$; so $\langle \Gamma \rangle$ is a torsionfree abelian group and $\Gamma$ is given a total order compatible with the monoid operation [22, page 123]. Let $G$ be a torsionfree abelian group. We say that $G$ is of type $(0,0,0,\ldots)$ except $p$ if $G$ satisfies the following two conditions: for each nonzero element $g \in G$, (i) the number of prime numbers dividing $g$ is finite and (ii) for each prime number $q \neq p$, $q^n$ does not divide $g$ for some positive integer $n$. The notion of type $(0,0,0,\ldots)$ except $p$ was introduced by Matsuda [23, 25] in order to study when $K[G]$, where $K$ is a field with $\text{char}(K) = p$, is a generalized Krull domain. Clearly, a torsionfree abelian group of type $(0,0,0,\ldots)$ is of type $(0,0,0,\ldots)$ except $p$ for all prime numbers $p$. As in the case of integral domains, we can define the $t$-operation on $\Gamma$. For more on definitions and basic results (e.g., maximal $t$-ideals, class groups, weakly factorial semigroups), see [20, Chapter 11], [15], or [11].
1.3 Semigroup rings

For a nonzero torsionless commutative cancellative monoid \( \Gamma \) and an integral domain \( D \), let \( D[\Gamma] \) be the semigroup ring of \( \Gamma \) over \( D \). Then \( D[\Gamma] \) is an integral domain [17, Theorem 8.1] and each nonzero element \( f \in D[\Gamma] \) can be written uniquely as \( f = a_1X^{s_1} + \cdots + a_nX^{s_n} \) for \( 0 \neq a_i \in D \) and \( s_i \in \Gamma \) with \( s_1 < \cdots < s_n \) [17, Corollary 3.4]. Each nonzero element of the form \( aX^s \in D[\Gamma] \) is said to be homogeneous with \( \deg(aX^s) = s \). Let \( H \) be the set of nonzero homogeneous elements of \( D[\Gamma] \). Then \( D[\Gamma]_H = K[\langle \Gamma \rangle] \), and hence \( D[\Gamma]_H \) is a completely integrally closed GCD-domain [18, Theorem 6.4]. For more on semigroup rings, the reader can refer to [17].

1.4 Weakly Krull domains

Let \( X^1(D) \) be the set of height-one prime ideals of an integral domain \( D \). We say that \( D \) is a weakly Krull domain if \( D = \bigcap_{P \in X^1(D)} D_P \) and this intersection has finite character, i.e., each nonzero nonunit of \( D \) is a unit in \( D_P \) except finitely many primes in \( X^1(D) \).

**Lemma 1.1.** \( D \) is a weakly Krull domain if and only if \( D = \bigcap_{P \in X^1(D)} D_P \) and \( D \) is of finite \( t \)-character. In this case, \( t\)-dim\( (D) = 1 \), i.e., \( t\)-Max\( (D) = X^1(D) \).

**Proof.** [4, Lemma 2.1].

A nonzero prime ideal \( Q \) of \( D[X] \) is called an upper to zero in \( D[X] \) if \( Q \cap D = (0) \), and we say that \( D \) is a UMT-domain if each upper to zero in \( D[X] \) is a maximal \( t \)-ideal of \( D[X] \). It is known that \( D \) is a PrMD if and only if \( D \) is an integrally closed UMT-domain [21, Proposition 3.2].

**Proposition 1.2.** The following statements are equivalent for \( D \).

1. \( D[X] \) is a weakly Krull domain.
2. \( D[X, X^{-1}] \) is a weakly Krull domain.
3. \( D \) is a weakly Krull UMT-domain.

**Proof.** [2, Propositions 4.7 and 4.11].

Clearly, Krull domains are weakly Krull domains. Recall that \( D \) is a UFD if and only if \( D \) is a Krull domain and \( Cl(D) = \{0\} \) [16, Proposition 6.1].

**Theorem 1.3.** \( D \) is a WFD if and only if \( D \) is a weakly Krull domain and \( Cl(D) = \{0\} \).

**Proof.** [5, Theorem].

An almost weakly factorial domain (AWFD) is an integral domain \( D \) in which for each \( 0 \neq d \in D \), there is an integer \( n = n(d) \geq 1 \) such that \( d^n \) can be written as a finite product of primary elements of \( D \). It is known that \( D \) is an AWFD if and only if \( D \) is a weakly Krull domain with \( Cl(D) \) torsion [4, Theorem 3.4]; hence

\[
\text{UFD} \Rightarrow \text{WFD} \Rightarrow \text{AWFD} \Rightarrow \text{weakly Krull domain}.
\]

It is easy to see that if \( N \) is a multiplicative subset of a weakly Krull domain (resp., WFD, an AWFD), then \( D_N \) satisfies the corresponding property.
2 Weakly factorial generalized Rees rings

Let $D$ be an integral domain, $I$ be a proper ideal of $D$, and $t$ be an indeterminate over $D$. Then $R = D[t, t^{-1}]$ is a subring of $D[t, t^{-1}]$, called the generalized Rees ring of $D$ with respect to $I$. In [27], Whithman proved that if $I$ is finitely generated, then $R$ is a UFD if and only if $D$ is a UFD and $t^{-1}$ is a prime element of $R$. Let $I = dD$ for $d \in D$ and $t^{-1} = X$; so $R = D[X, \frac{d}{X}]$. In [1], the authors studied several kinds of divisibility properties of $R$ including Krull domains, UFDs, and GCD-domains. In this section we study when $R = D[X, \frac{d}{X}]$ is a WFD.

**Proposition 2.1.** Let $d \in D$ be a nonzero element and $R = D[X, \frac{d}{X}]$. Then the following statements are equivalent.

1. $X$ is irreducible (resp., prime, primary) in $R$.
2. $\frac{d}{X}$ is irreducible (resp., prime, primary) in $R$.
3. $d$ is a nonunit (resp., prime, primary) in $D$.

**Proof.** For the properties of irreducible and prime, see [1, Proposition 1]. For the property of primary, see [13, Proposition 2].

Let $0 \neq a \in D$. It is easy to see that $a$ is primary if and only if $\sqrt{aD}$ is a maximal $t$-ideal [7, Lemma 2.1]. Hence, by Proposition 2.1, we have

**Corollary 2.2.** [13, Corollary 3] Let $d \in D$ be a nonzero nonunit and $R = D[X, \frac{d}{X}]$. If $dD$ is primary in $D$, then $\sqrt{XR}$ is a maximal $t$-ideal of $R$ and $(X, \frac{d}{X})_v = R$.

The next lemma can be easily proved by Proposition 1.2 which is necessary for the proof of Theorem 2.4.

**Lemma 2.3.** [13, Proposition 4] Let $d \in D$ be a nonzero nonunit and $R = D[X, \frac{d}{X}]$. Then $R$ is a weakly Krull domain if and only if $D$ is a weakly Krull UMT-domain.

By using the results of Corollary 2.2 and Lemma 2.3, we can completely characterize when $R = D[X, \frac{d}{X}]$ is a WFD.

**Theorem 2.4.** [13, Theorem 6] Let $d \in D$ and $R = D[X, \frac{d}{X}]$. Then the following statements are equivalent.

1. $R$ is a WFD.
2. $R$ is a weakly factorial GCD-domain.
3. $D$ is a weakly factorial GCD-domain and $d = 0$, $d$ is a unit of $D$, or $d$ is a prime element of $D$.

A ring of Krull type is a PrMD of finite $t$-character [19, Theorem 7]. It is known that $D$ is a ring of Krull type if and only if $D[X]$ is a ring of Krull type, or equivalently, $D[X, \frac{1}{X}]$ is a ring of Krull type (cf. [19, Propositions 9 and 12]).

**Theorem 2.5.** [13, Theorem 7] Let $d \in D$ and $R = D[X, \frac{d}{X}]$. Then $R$ is a ring of Krull type if and only if $D$ is a ring of Krull type.

A generalized Krull domain is an integral domain $D$ such that (i) $D_P$ is a valuation domain for all $P \in X^1(D)$, (ii) $D = \bigcap_{P \in X^1(D)} D_P$, and (iii) this intersection has finite character. Clearly, $D$ is a generalized Krull domain if and only if $D$ is a weakly Krull PrMD, if and only if $D$ is a ring of Krull type with $t$-$\dim(D) = 1$. Hence, by Lemma 2.3 and Theorem 2.5, we have
Lemma 2.1. Hence, by Proposition 2.1, we have a ring of Krull type with $t_D$ primary in $D$, then when primary, see [13, Proposition 2]. For the properties of irreducible and prime, see [1, Proposition 1]. For the property of $R$, let $1$, the authors studied several kinds of divisibility properties of $D$.

By using the results of Corollary 2.2 and Lemma 2.3, we can completely characterize $A$ ring of Krull type $D$ and $R$.

Corollary 2.5. Let $d_D$ be a generalized Krull domain if and only if $D$ is a ring of Krull type.

A ring of Krull type $D$ and $R$.

Corollary 2.6. [13, Corollary 8] Let $d_D$ be a ring of Krull type with $D(X, \frac{d_D}{X})$. Then $R$ is a generalized Krull domain if and only if $D$ is a generalized Krull domain.

It is known that if $D$ is a UFD, $p$ is a prime element of $D$, and $n \geq 1$ is an integer, then $D(X, \frac{p_D}{X})$ is a Krull domain with $Cl(D(X, \frac{p_D}{X})) = \mathbb{Z}_n$ [1, Theorems 8 and 16]. The next result is an AWFD analog.

Corollary 2.7. [13, Corollary 9] Let $D$ be a weakly factorial GCD-domain, $p$ be a prime element of $D$, and $n \geq 2$ be an integer. Then $R = D(X, \frac{p_D}{X})$ is an AWFD with $Cl(R) = \mathbb{Z}_n$.

Let $V$ be a rank-one nondiscrete valuation domain, $y$ be an indeterminate over $V$, and $D = V[y]$. Then $D$ is a weakly factorial GCD-domain and $y$ is a prime of $D$. Thus, by Theorem 2.4 and Corollary 2.7, $R = D(X, \frac{p_D}{X})$ is a WFD and $D(X, \frac{p_D}{X})$ is an AWFD with $Cl(D(X, \frac{p_D}{X})) = \mathbb{Z}_n$ for all integers $n \geq 2$.

3 Weakly factorial semigroup rings

Let $D$ be an integral domain with quotient field $K$, $\Gamma$ be a torsionless grading monoid with quotient group $G$, $D[\Gamma]$ be the semigroup ring of $\Gamma$ over $D$, and $H$ be the set of nonzero homogeneous elements of $D[\Gamma]$.

In order to study when $D[\Gamma]$ is a UFD, we first need the notion of factorial semigroups. As in the domain case, an $\alpha \in \Gamma$ is called a prime element if $\alpha + \Gamma$ is a prime ideal of $\Gamma$, and we say that $\Gamma$ is a factorial semigroup if each nonunit of $\Gamma$ can be written as a finite sum of prime elements of $\Gamma$.

Theorem 3.1. $D[\Gamma]$ is a UFD if and only if $D$ is a UFD, $\Gamma$ is a factorial semigroup, and the quotient group of $\Gamma$ is of type $(0, 0, 0, \ldots)$.

Proof. [18, Theorem 7.17 and Lemma 7.15].

Theorem 3.1 is the motivation of the results in this section, i.e., we completely characterize when $D[\Gamma]$ is a WFD.

3.1 Weakly factorial semigroups

Lemma 3.2. [11, Lemma 1] Let $\alpha \in \Gamma$ be a nonunit. Then the following statements are equivalent.

1. $X^\alpha$ is primary in $D[\Gamma]$.
2. $\sqrt{X^\alpha D[\Gamma]}$ is a maximal $t$-ideal of $D[\Gamma]$.
3. $D[\sqrt{\alpha^* + \Gamma}]$ is a maximal $t$-ideal of $D[\Gamma]$.
4. $\sqrt{\alpha + \Gamma}$ is a maximal $t$-ideal of $\Gamma$.
5. $\alpha$ is primary in $\Gamma$.

It is well known that $D$ is a GCD-domain if and only if $aD \cap bD$ is principal, if and only if $(a, b)_D$ is principal for each pair $0 \neq a, b \in D$. Similarly, note that if $\Gamma$ is a GCD-semigroup, then $gcd(\alpha, \beta) = a \iff (\alpha + \Gamma) \cap (\beta + \Gamma) = \alpha + \beta - a + \Gamma, \iff ((\alpha + \Gamma) \cup (\beta + \Gamma))_\alpha = a + \Gamma$ for each pair $\alpha, \beta \in \Gamma$. Thus, $\Gamma$ is a GCD-semigroup if and only if $\bigcap_{i=1}^{k} (\alpha_i + \Gamma)$ is principal, if and only if $(\bigcup_{i=1}^{k} (\alpha_i + \Gamma))_\alpha$ is principal for any $\alpha_1, \ldots, \alpha_k \in \Gamma$ (cf. [20, Exercise 1 (page 117) and Theorem 11.5]). The next result is the semigroup analog of [3, Theorem 18] that a WFD $D$ is a GCD-domain if and only if $D_p$ is a valuation domain for each height-one prime ideal $P$ of $D$.
Proposition 3.3. [11, Proposition 5] Suppose that $\Gamma$ is a weakly factorial semigroup. Then $\Gamma$ is a GCD-semigroup if and only if $\alpha + \Gamma \subseteq \beta + \Gamma$ or $\beta + \Gamma \subseteq \alpha + \Gamma$ for all primary elements $\alpha$ and $\beta$ of $\Gamma$ with $\sqrt{\alpha + \Gamma} = \sqrt{\beta + \Gamma}$.

It is clear that a factorial semigroup is a GCD-semigroup. Also, prime elements are primary, and hence a factorial semigroup is a weakly factorial GCD-semigroup. However, the next example shows that a weakly factorial GCD-semigroup whose quotient group is of type $(0, 0, 0, \ldots)$ need not be a factorial semigroup.

Example 3.4. [11, Example 12] Let $G = \{m + n \sqrt{2} \mid m, n \in \mathbb{Z}\}$ be the additive group and $\Gamma = \{\alpha \in G \mid \alpha \geq 0\}$ be a subsemigroup of $G$.

1. $\Gamma$ is a torsionless grading monoid with quotient group $G$.
2. $G$ is of type $(0, 0, 0, \ldots)$.
3. $\Gamma$ is a weakly factorial GCD-semigroup.
4. $\Gamma$ is not a factorial semigroup.

Proof. (1), (2) and (3). These are easy exercises. (4) This can be proved by the fact that $\sqrt{2}$ is irrational.

3.2 When $G$ is of type $(0, 0, 0, \ldots)$

Let $D[X]$ be the polynomial ring over $D$ and $\mathbb{N}_0$ be the semigroup of nonnegative integers. Then (i) $D[X] = D[\mathbb{N}_0]$ and (ii) $D[X]$ is a WFD if and only if $D$ is a weakly factorial GCD-domain (Theorem 2.4). Note that $\mathbb{N}_0$ is a torsionless grading monoid whose quotient group $\mathbb{Z}$ is of type $(0, 0, 0, \ldots)$; hence the next result is a natural generalization of the result (ii) to $D[\Gamma]$ with $G$ of type $(0, 0, 0, \ldots)$.

Theorem 3.5. [11, Theorem 9] Assume that $G$ is of type $(0, 0, 0, \ldots)$. Then $D[\Gamma]$ is a WFD if and only if $D$ is a weakly factorial GCD-domain and $\Gamma$ is a weakly factorial GCD-semigroup.

Since a factorial semigroup is a weakly factorial GCD semigroup, by Theorem 3.5, we have

Corollary 3.6. [11, Corollary 10] Assume that $G$ is of type $(0, 0, 0, \ldots)$ and $\Gamma$ is a factorial semigroup. Then $D[\Gamma]$ is a WFD if and only if $D$ is a weakly factorial GCD-domain.

Set $H_\alpha = \mathbb{N}_0$ for each $\alpha$, and let $\Gamma = \sum_\alpha H_\alpha$. Then $K[\Gamma] = K[\{X_\alpha\}]$ is a factorial domain, and hence $\Gamma$ is a factorial semigroup whose quotient group is of type $(0, 0, 0, \ldots)$. Thus, by Corollary 3.6, we have

Corollary 3.7. [11, Corollary 11] Let $\{X_\alpha\}$ be a nonempty set of indeterminates over $D$. Then $D[\{X_\alpha\}]$ is a WFD if and only if $D$ is a weakly factorial GCD-domain.

Example 3.8. Let $\mathbb{Z}$ be the ring of integers and $\Gamma = \{m + n \sqrt{2} \mid m, n \in \mathbb{Z}$ and $m + n \sqrt{2} \geq 0\}$. Then $\mathbb{Z}[\Gamma]$ is a WFD but not a UFD by Theorems 3.1, 3.5 and Example 3.4.

The next example shows that Theorem 3.5 is not true when $G$ is not of type $(0, 0, 0, \ldots)$.

Example 3.9. [11, Example 13] Let $\mathbb{R}$ be the field of real numbers and $\Gamma$ be the additive semigroup of nonnegative rational numbers.

1. $\Gamma$ is a torsion-free grading monoid.
2. \( \Gamma \) is a weakly factorial GCD-semigroup.

3. \( \mathbb{R}[\Gamma] \) has the (Krull) dimension one, and hence \( t\)-dim(\( \mathbb{R}[\Gamma] \)) = 1.

4. \( \mathbb{R}[\Gamma] \) is not a WFD.

Proof. (1) Clear. (2) This is an easy exercise. (3) Let \( \mathbb{N}_0 \) be the semigroup of nonnegative integers. Then \( \mathbb{R}[\Gamma] \) is integral over \( \mathbb{R}[\mathbb{N}_0] \) and \( \dim(\mathbb{R}[\mathbb{N}_0]) = 1 \). Thus, \( \dim(\mathbb{R}[\Gamma]) = t\)-dim(\( \mathbb{R}[\Gamma] \)) = 1. (4) It can be shown that \( 1 - X \in \mathbb{R}[\Gamma] \) is contained in infinitely many height-one prime ideals of \( \mathbb{R}[\Gamma] \).

3.3 \( \text{char}(D) = 0 \)

In this section, we study the WFD property of \( D[\Gamma] \) under the assumption that \( \text{char}(D) = 0 \). It is known that if \( \text{char}(D) = 0 \), then \( D[G] \) is a ring of Krull type if and only if \( D \) is a ring of Krull type and \( G \) is of type \((0,0,0,\ldots)\) [25, Theorem 3].

Lemma 3.10. [14, Theorem 3.1] Let \( G \) be a torsionfree abelian group, \( D[G] \) be the group ring of \( G \) over \( D \), and assume that \( \text{char}(D) = 0 \). Then \( D[G] \) is of finite \( t \)-character if and only if \( D \) is of finite \( t \)-character and \( G \) is of type \((0,0,0,\ldots)\).

Corollary 3.11. [14, Corollary 3.2] Let \( K \) be a field, \( G \) be a torsionfree abelian group, and assume that \( \text{char}(K) = 0 \). Then the following statements are equivalent.

1. \( K[G] \) is a WFD.

2. \( K[G] \) is a weakly Krull domain.

3. \( K[G] \) is of finite \( t \)-character.

4. \( G \) is of type \((0,0,0,\ldots)\).

5. \( K[G] \) is a UFD.

6. \( K[G] \) is a Krull domain.

7. \( K[G] \) is a generalized Krull domain.

Proof. (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) Clear. (3) \( \Rightarrow \) (4) Lemma 3.10. (4) \( \Rightarrow \) (5) Theorem 3.1. (5) \( \Rightarrow \) (1) Clear. (5) \( \Rightarrow \) (6) \( \Rightarrow \) (7) \( \Rightarrow \) (3) Clear.

Theorem 3.12. [14, Theorem 3.4] Assume that \( \text{char}(D) = 0 \). Then \( D[\Gamma] \) is a WFD if and only if \( D \) is a weakly factorial GCD-domain, \( \Gamma \) is a weakly factorial GCD-semigroup, and \( G \) is of type \((0,0,0,\ldots)\).

Proof. If \( D[\Gamma] \) is a WFD, then \( D[\Gamma]_H \) is a WFD. Note that \( D[\Gamma]_H = K[G] \); hence \( G \) is of type \((0,0,0,\ldots)\) by Corollary 3.11. Thus, the result follows directly from Theorem 3.5.

3.4 \( \text{char}(D) = p > 0 \)

In this section, we study when \( D[\Gamma] \) is a WFD under the assumption that \( \text{char}(D) = p > 0 \). Recall that if \( \text{char}(D) = p > 0 \), then \( D[G] \) is a ring of Krull type if and only if \( D \) is a ring of Krull type and \( G \) is of type \((0,0,0,\ldots)\) except \( p \) [25, Theorem 5].
Lemma 3.13. [14, Theorem 4.2] Let G be a torsionfree abelian group, D[G] be the group ring of G over D, and assume that char(D) = p > 0. Then D[G] is of finite t-character if and only if D is of finite t-character and G is of type (0, 0, 0, ...) except p.

Corollary 3.14. [14, Corollary 4.3] Let K be a field, G be a torsionfree abelian group, and assume that char(K) = p > 0. Then the following statements are equivalent.

1. K[G] is a WFD.
2. K[G] is a weakly Krull domain.
4. G is of type (0, 0, 0, ...) except p.
5. K[G] is a generalized Krull domain.

We next give a complete characterization of a WFD D[Γ] with char(D) = p > 0.

Theorem 3.15. [14, Theorem 4.5] Assume that char(D) = p > 0. Then D[Γ] is a WFD if and only if D is a weakly factorial GCD-domain, Γ is a weakly factorial GCD-semigroup, and G is of type (0, 0, 0, ...) except p.

Example 3.16. [14, Example 4.6] Let K be a field, p be a prime number, G = \bigcup_{n=1}^{\infty} (1/p^n)\mathbb{Z}, and K[G] be the group ring of G over K.

1. G is of type (0, 0, 0, ...) except p but not of type (0, 0, 0, ...).
2. If char(K) = p > 0, then K[G] is a weakly factorial domain.
3. If char(K) ≠ p (e.g., char(K) = 0), then K[G] is not a WFD.

Hence, \mathbb{R}[G] is not a WFD, while \mathbb{Z}_p[G] is a WFD, where \mathbb{Z}_p is a field of p elements.

Proof. For (1), see [25, Proposition 12]. For (2) and (3), see (1) and Theorems 3.5 and 3.15.

References