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Recent results on weakly factorial domains

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Abstract. In this paper, we will survey recent results on weakly factorial domains base on the results of [11, 13, 14]. Let D be an integral domain, X be an indeterminate over D, $d \in D$, $R = D[X, \frac{d}{X}]$ be a subring of the Laurent polynomial ring $D[X, \frac{1}{X}]$, Γ be a nonzero torsionless commutative cancellative monoid with quotient group G, and $D[\Gamma]$ be the semigroup ring of Γ over D. Among other things, we show that R is a weakly factorial domain if and only if D is a weakly factorial GCD-domain and $d = 0$, d is a unit of D or d is a prime element of D. We also show that if char$(D) = 0$ (resp., char$(D) = p > 0$), then $D[\Gamma]$ is a weakly factorial domain if and only if D is a weakly factorial GCD domain, Γ is a weakly factorial GCD semigroup, and G is of type $(0, 0, 0, \dots)$ (resp., $(0, 0, 0, \dots)$ except p).

Introduction

Let D be an integral domain with quotient field K, X be an indeterminate over D, Γ be a nonzero torsionless commutative cancellative monoid (written additively) with quotient group G, and $D[\Gamma]$ be the semigroup ring of Γ over D. A nonzero nonunit element $p \in D$ is called a *primary element* if $p|ab$ for each $a, b \in D$ implies that $p|a$ or $p|b^n$ for some positive integer n; equivalently, pD is a primary ideal. We say that D is a *unique factorization domain* (UFD) (resp., *weakly factorial domain* (WFD)) if each nonzero nonunit of D can be written as a finite product of prime (resp. primary) elements. Clearly, a prime element is primary, and hence a UFD is a WFD. The concept of a WFD was first introduced by Anderson and Mahaney [3]. Chang introduced the notion of weakly factorial semigroups [11] in order to study when $D[\Gamma]$ is a WFD. A nonzero nonunit element $s \in \Gamma$ is *primary* if, for each $a, b \in \Gamma$, $a + b \in s + \Gamma$ implies that $a \in s + \Gamma$ or $nb \in s + \Gamma$ for some positive integer n. It is clear that $s \in \Gamma$ is primary if and only if $s + \Gamma$ is a primary ideal. We say that Γ is a *weakly factorial semigroup* if each nonunit of Γ is a finite sum of primary elements.

Chang has worked on WFDs [6–8, 11, 13, 14] and its generalization to rings with zero divisors [10, 12]. In this paper, among them, we survey the recent results of [11, 13, 14]. In Section 1, for easy reference, we review some definitions and preliminary results on t-operations, monoinds, semigroup rings, and weakly Krull domains. Let $d \in D$ and $R = D[X, \frac{d}{X}]$; so R is a subring of the Laurent polynomial ring $D[X, \frac{1}{X}]$ containing $D[X]$. In Section 2, we show that R is a weakly factorial domain if and only if D is a weakly factorial GCD-domain and $d = 0$, d is a unit of D or d is a prime element of D. We also show that if D is a weakly factorial GCD-domain, p is a prime element of D, and $n \geq 2$ is an integer,

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then $D[X, \frac{p^n}{X}]$ is an almost weakly factorial domain with $Cl(D[X, \frac{p^n}{X}]) = \mathbb{Z}_n$, the cyclic group of order n. In Section 3, we completely characterize when $D[\Gamma]$ is a WFD. We first study the notion of weakly factorial GCD-semigroups. Then, among other things, we prove that if char$(D) = 0$, then $D[\Gamma]$ is a WFD if and only if D is a weakly factorial GCD domain, Γ is a weakly factorial GCD semigroup, and G is of type $(0, 0, 0, \dots)$. We also show that if char$(D) = p > 0$, then $D[\Gamma]$ is a WFD if and only if D is a weakly factorial GCD domain, Γ is a weakly factorial GCD semigroup, and G is of type $(0, 0, 0, \dots)$ except p. However, we omit the proofs of the results; so the reader who is interested in the proofs can refer to [11, 13, 14].

1 Definitions and preliminary results

In this section, we review some definitions and preliminary results on the t-operations, semigroups, semigroup rings, and weakly Krull domains.

1.1 The t-operations

Let D be an integral doman with quotient field K and $F(D)$ be the set of nonzero fractional ideals of D. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$; then $I^{-1} \in F(D)$. Hence, we can define the v- and t-operations as follows: $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup\{J_v \mid J \subseteq I$ and $J \in F(D)$ is finitely generated$\}$. An $I \in F(D)$ is called a t-ideal if $I_t = I$. A t-ideal of D is a maximal t-ideal if it is maximal among proper integral t-ideals of D. Let t-Max(D) be the set of maximal t-ideals of D. It is well known that t-Max$(D) \neq \emptyset$ if D is not a field; each ideal in t-Max(D) is a prime ideal; each prime ideal minimal over a t-ideal is a t-ideal (which implies that every height-one prime ideal is a t-ideal); and $D = \bigcap_{P \in t\text{-Max}(D)} D_P$. We say that D is *of finite t-character* if every nonzero nonuint of D is contained in only finitely many maximal t-ideals of D.

An $I \in F(D)$ is said to be t-invertible if $(II^{-1})_t = D$. Let $T(D)$ be the group of t-invertible fractional t-ideals of D under $I * J = (IJ)_t$, and $Prin(D)$ be its subgroup of principal fractional ideals. Then $Cl(D) = T(D)/Prin(D)$, called the t-class group of D, is an abelian group. We say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. It is known that D is a PvMD if and only if $D[X]$ is a PvMD [26, Corollary 4], if and only if $D[X, \frac{1}{X}]$ is a PvMD [24, Theorem 3.10]. Also, D is a GCD-domain if and only if D is a PvMD with $Cl(D) = \{0\}$ [9, Proposition 2].

1.2 Semigroups

Let Γ be a nonzero torsionless grading monoid, i.e., a nonzero torsionless commutative cancellative monoid (written additively), and $\langle\Gamma\rangle = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of Γ; so $\langle\Gamma\rangle$ is a torsionfree abelian group and Γ is given a total order compatible with the monoid operation [22, page 123]. Let G be a torsionfree abelian group. We say that G is *of type* $(0, 0, 0, \dots)$ if G satisfies the ascending chain condition on its cyclic subgroups (equivalently, for each nonzero element $g \in G$, there exists a largest positive integer n_g such that $n_g x = g$ is solvable in G) [17, Theorem 14.10]. For a prime number p, G is said to be of type $(0, 0, 0, \dots)$ except p if G satisfies the following two conditions; for each nonzero element $g \in G$, (i) the number of prime numbers dividing g is finite and (ii) for each prime number $q \neq p$, q^n does not divide g for some positive integer n. The notion of type $(0, 0, 0, \dots)$ except p was introduced by Matsuda [23, 25] in order to study when $K[G]$, where K is a field with char$(K) = p$, is a generalized Krull domain. Clearly, a torsionfree abelian group of type $(0, 0, 0, \dots)$ is of type $(0, 0, 0, \dots)$ except p for all prime numbers p. As in the case of integral domains, we can define the t-operation on Γ. For more on definitions and basic results (e.g., maximal t-ideals, class groups, weakly factorial semigroups), see [20, Chapter 11], [15], or [11].

1.3 Semigroup rings

For a nonzero torsionless commutative cancellative monoid Γ and an integral domain D, let $D[\Gamma]$ be the semigroup ring of Γ over D. Then $D[\Gamma]$ is an integral domain [17, Theorem 8.1] and each nonzero element $f \in D[\Gamma]$ can be written uniquely as $f = a_1X^{s_1} + \cdots + a_nX^{s_n}$ for $0 \neq a_i \in D$ and $s_i \in \Gamma$ with $s_1 < \cdots < s_n$ [17, Corollary 3.4]. Each nonzero element of the form $aX^s \in D[\Gamma]$ is said to be homogeneous with $\deg(aX^s) = s$. Let H be the set of nonzero homogeneous elements of $D[\Gamma]$. Then $D[\Gamma]_H = K[\langle\Gamma\rangle]$, and hence $D[\Gamma]_H$ is a completely integrally closed GCD-domain [18, Theorem 6.4]. For more on semigroup rings, the reader can refer to [17].

1.4 Weakly Krull domains

Let $X^1(D)$ be the set of height-one prime ideals of an integral domain D. We say that D is a weakly Krull domain if $D = \bigcap_{P\in X^1(D)} D_P$ and this intersection has finite character, i.e., each nonzero nonunit of D is a unit in D_P except finitely many primes in $X^1(D)$.

Lemma 1.1. *D is a weakly Krull domain if and only if $D = \bigcap_{P\in X^1(D)} D_P$ and D is of finite t-character. In this case, t-$\dim(D) = 1$, i.e., t-$Max(D) = X^1(D)$.*

Proof. [4, Lemma 2.1]. □

A nonzero prime ideal Q of $D[X]$ is called an upper to zero in $D[X]$ if $Q \cap D = (0)$, and we say that D is a UMT-domain if each upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$. It is known that D is a PvMD if and only if D is an integrally closed UMT-domain [21, Proposition 3.2].

Proposition 1.2. *The following statements are equivalent for D.*

1. *$D[X]$ is a weakly Krull domain.*
2. *$D[X, X^{-1}]$ is a weakly Krull domain.*
3. *D is a weakly Krull UMT-domain.*

Proof. [2, Propositions 4.7 and 4.11]. □

Clearly, Krull domains are weakly Krull domains. Recall that D is a UFD if and only if D is a Krull domain and $Cl(D) = \{0\}$ [16, Proposition 6.1].

Theorem 1.3. *D is a WFD if and only if D is a weakly Krull domain and $Cl(D) = \{0\}$.*

Proof. [5, Theorem]. □

An *almost weakly factorial domain* (AWFD) is an integral domain D in which for each $0 \neq d \in D$, there is an integer $n = n(d) \geq 1$ such that d^n can be written as a finite product of primary elements of D. It is known that D is an AWFD if and only if D is a weakly Krull domain with $Cl(D)$ torsion [4, Theorem 3.4]; hence

$$\text{UFD} \Rightarrow \text{WFD} \Rightarrow \text{AWFD} \Rightarrow \text{weakly Krull domain.}$$

It is easy to see that if N is a multiplicative subset of a weakly Krull domain (resp., WFD, an AWFD), then D_N satisfies the corresponding property.

2 Weakly factorial generalized Rees rings

Let D be an integral domain, I be a proper ideal of D, and t be an indeterminate over D. Then $R = D[tI, t^{-1}]$ is a subring of $D[t, t^{-1}]$, called the *generalized Rees ring of D with respect to I*. In [27], Whithman proved that if I is finitely generated, then R is a UFD if and only if D is a UFD and t^{-1} is a prime element of R. Let $I = dD$ for $d \in D$ and $t^{-1} = X$; so $R = D[X, \frac{d}{X}]$. In [1], the authors studied several kinds of divisibility properties of R including Krull domains, UFDs, and GCD-domains. In this section we study when $R = D[X, \frac{d}{X}]$ is a WFD.

Proposition 2.1. *Let $d \in D$ be a nonzero element and $R = D[X, \frac{d}{X}]$. Then the following statements are equivalent.*

1. *X is irreducible (resp., prime, primary) in R.*
2. *$\frac{d}{X}$ is irreducible (resp., prime, primary) in R.*
3. *d is a nonunit (resp., prime, primary) in D.*

Proof. For the properties of irreducible and prime, see [1, Proposition 1]. For the property of primary, see [13, Proposition 2]. □

Let $0 \neq a \in D$. It is easy to see that a is primary if and only if $\sqrt{aD}$ is a maximal t-ideal [7, Lemma 2.1]. Hence, by Proposition 2.1, we have

Corollary 2.2. *[13, Corollary 3] Let $d \in D$ be a nonzero nonunit and $R = D[X, \frac{d}{X}]$. If dD is primary in D, then $\sqrt{XR}$ is a maximal t-ideal of R and $(X, \frac{d}{X})_v = R$.*

The next lemma can be easily proved by Proposition 1.2 which is necessary for the proof of Thorem 2.4.

Lemma 2.3. *[13, Proposition 4] Let $d \in D$ be a nonzero nonunit and $R = D[X, \frac{d}{X}]$. Then R is a weakly Krull domain if and only if D is a weakly Krull UMT-domain.*

By using the results of Corollary 2.2 and Lemma 2.3, we can completely characterize when $R = D[X, \frac{d}{X}]$ is a WFD.

Theorem 2.4. *[13, Theorem 6] Let $d \in D$ and $R = D[X, \frac{d}{X}]$. Then the following statements are equivalent.*

1. *R is a WFD.*
2. *R is a weakly factorial GCD-domain.*
3. *D is a weakly factorial GCD-domain and $d = 0$, d is a unit of D, or d is a prime element of D.*

A *ring of Krull type* is a PvMD of finite t-character [19, Theorem 7]. It is known that D is a ring of Krull type if and only if $D[X]$ is a ring of Krull type, or equivalently, $D[X, \frac{1}{X}]$ is a ring of Krull type (cf. [19, Propositions 9 and 12]).

Theorem 2.5. *[13, Theorem 7] Let $d \in D$ and $R = D[X, \frac{d}{X}]$. Then R is a ring of Krull type if and only if D is a ring of Krull type.*

A generalized Krull domain is an integral domain D such that (i) D_P is a valuation domain for all $P \in X^1(D)$, (ii) $D = \bigcap_{P \in X^1(D)} D_P$, and (iii) this intersection has finite character. Clearly, D is a generalized Krull domain if and only if D is a weakly Krull PvMD, if and only if D is a ring of Krull type with t-dim$(D) = 1$. Hence, by Lemma 2.3 andTheorem 2.5, we have

Corollary 2.6. *[13, Corollary 8] Let $d \in D$ and $R = D[X, \frac{d}{X}]$. Then R is a generalized Krull domain if and only if D is a generalized Krull domain.*

It is known that if D is a UFD, p is a prime element of D, and $n \geq 1$ is an integer, then $D[X, \frac{p^n}{X}]$ is a Krull domain with $Cl(D[X, \frac{p^n}{X}]) = \mathbb{Z}_n$ [1, Theorems 8 and 16]. The next result is an AWFD analog.

Corollary 2.7. *[13, Corollary 9] Let D be a weakly factorial GCD-domain, p be a prime element of D, and $n \geq 2$ be an integer. Then $R = D[X, \frac{p^n}{X}]$ is an AWFD with $Cl(R) = \mathbb{Z}_n$.*

Let V be a rank-one nondiscrete valuation domain, y be an indeterminate over V, and $D = V[y]$. Then D is a weakly factorial GCD-domain and y is a prime of D. Thus, by Theorem 2.4 and Corollary 2.7, $R = D[X, \frac{y}{X}]$ is a WFD and $D[X, \frac{y^n}{X}]$ is an AWFD with $Cl(D[X, \frac{y^n}{X}]) = \mathbb{Z}_n$ for all integers $n \geq 2$.

3 Weakly factorial semigroup rings

Let D be an integral domain with quotient field K, Γ be a torsionless grading monoid with quotient group G, $D[\Gamma]$ be the semigroup ring of Γ over D, and H be the set of nonzero homgeneous elements of $D[\Gamma]$.

In order to study when $D[\Gamma]$ is a UFD, we first need the notion of factorial semigroups. As in the domain case, an $\alpha \in \Gamma$ is called a *prime element* if $\alpha + \Gamma$ is a prime ideal of Γ, and we say that Γ is a *factorial semigroup* if each nonunit of Γ can be written as a finite sum of prime elements of Γ.

Theorem 3.1. *$D[\Gamma]$ is a UFD if and only if D is a UFD, Γ is a factorial semigroup, and the quotient group of Γ is of type $(0, 0, 0, \dots)$.*

Proof. [18, Theorem 7.17 and Lemma 7.15]. □

Theorem 3.1 is the motivation of the results in this section, i.e., we completely characterize when $D[\Gamma]$ is a WFD.

3.1 Weakly factorial semigroups

Lemma 3.2. *[11, Lemma 1] Let $\alpha \in \Gamma$ be a nonunit. Then the following statements are equivalent.*

1. *X^α is primary in $D[\Gamma]$.*
2. *$\sqrt{X^\alpha D[\Gamma]}$ is a maximal t-ideal of $D[\Gamma]$.*
3. *$D[\sqrt{\alpha + \Gamma}]$ is a maximal t-ideal of $D[\Gamma]$.*
4. *$\sqrt{\alpha + \Gamma}$ is a maximal t-ideal of Γ.*
5. *α is primary in Γ.*

It is well known that D is a GCD-domain if and only if $aD \cap bD$ is principal, if and only if $(a, b)_v$ is principal for each pair $0 \neq a, b \in D$. Similarly, note that if Γ is a GCD-semigrup, then $gcd(\alpha,\beta) = a \Leftrightarrow (\alpha + \Gamma) \cap (\beta + \Gamma) = \alpha + \beta - a + \Gamma$, $\Leftrightarrow ((\alpha + \Gamma) \cup (\beta + \Gamma))_v = a + \Gamma$ for each pair $\alpha, \beta \in \Gamma$. Thus, Γ is a GCD-semigroup if and only if $\bigcap_{i=1}^k (\alpha_i + \Gamma)$ is principal, if and only if $(\bigcup_{i=1}^k (\alpha_i + \Gamma))_v$ is principal for any $\alpha_1, \dots, \alpha_k \in \Gamma$ (cf. [20, Exercise 1 (page 117) and Theorem 11.5]). The next result is the semigroup analog of [3, Theorem 18] that a WFD D is a GCD-domain if and only if D_P is a valuation domain for each height-one prime ideal P of D.

Proposition 3.3. *[11, Proposition 5] Suppose that* Γ *is a weakly factorial semigroup. Then* Γ *is a GCD-semigroup if and only if* $\alpha + \Gamma \subseteq \beta + \Gamma$ *or* $\beta + \Gamma \subseteq \alpha + \Gamma$ *for all primary elements* α *and* β *of* Γ *with* $\sqrt{\alpha + \Gamma} = \sqrt{\beta + \Gamma}$.

It is clear that a factorial semigroup is a GCD-semigroup. Also, prime elements are primary, and hence a factorial semigroup is a weakly factorial GCD-semigroup. However, the next example shows that a weakly factorial GCD-semigroup whose quotient group is of type $(0, 0, 0, \dots)$ need not be a factorial semigroup.

Example 3.4. *[11, Example 12]* Let $G = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$ be the additive group and $\Gamma = \{\alpha \in G \mid \alpha \geq 0\}$ be a subsemigroup of G.

1. Γ is a torsionless grading monoid with quotient group G.
2. G is of type $(0, 0, 0, \dots)$.
3. Γ is a weakly factorial GCD-semigroup.
4. Γ is not a factorial semigroup.

Proof. (1), (2) and (3). These are easy exercises. (4) This can be proved by the fact that $\sqrt{2}$ is irrational. □

3.2 When G is of type $(0, 0, 0, \dots)$

Let $D[X]$ be the polynomial ring over D and $\mathbb{N}_0$ be the semigroup of nonnegative integers. Then (i) $D[X] = D[\mathbb{N}_0]$ and (ii) $D[X]$ is a WFD if and only if D is a weakly factorial GCD-domain (Theorem 2.4). Note that $\mathbb{N}_0$ is a torsionless grading monoid whose quotient group $\mathbb{Z}$ is of type $(0, 0, 0, \dots)$; hence the next result is a natural generalization of the result (ii) to $D[\Gamma]$ with G of type $(0, 0, 0, \dots)$.

Theorem 3.5. *[11, Theorem 9] Assume that G is of type* $(0, 0, 0, \dots)$. *Then* $D[\Gamma]$ *is a WFD if and only if D is a weakly factorial GCD-domain and* Γ *is a weakly factorial GCD-semigroup.*

Since a factorial semigroup is a weakly factorial GCD semigroup, by Theorem 3.5, we have

Corollary 3.6. *[11, Corollary 10] Assume that G is of type* $(0, 0, 0, \dots)$ *and* Γ *is a factorial semigroup. Then* $D[\Gamma]$ *is a WFD if and only if D is a weakly factorial GCD-domain.*

Set $H_\alpha = \mathbb{N}_0$ for each α, and let $\Gamma = \sum_\alpha H_\alpha$. Then $K[\Gamma] = K[\{X_\alpha\}]$ is a factorial domain, and hence Γ is a factorial semigroup whose quotient group is of type $(0, 0, 0, \dots)$. Thus, by Corollary 3.6, we have

Corollary 3.7. *[11, Corollary 11] Let* $\{X_\alpha\}$ *be a nonempty set of indeterminates over D. Then* $D[\{X_\alpha\}]$ *is a WFD if and only if D is a weakly factorial GCD-domain.*

Example 3.8. Let $\mathbb{Z}$ be the ring of integers and $\Gamma = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}$ and $m + n\sqrt{2} \geq 0\}$. Then $\mathbb{Z}[\Gamma]$ is a WFD but not a UFD by Theorems 3.1, 3.5 and Example 3.4.

The next example shows that Theorem 3.5 is not true when G is not of type $(0, 0, 0, \dots)$.

Example 3.9. *[11, Example 13]* Let $\mathbb{R}$ be the field of real numbers and Γ be the additive semigroup of nonnegative rational numbers.

1. Γ is a torsion-free grading monoid.

2. Γ is a weakly factorial GCD-semigroup.

3. $\mathbb{R}[\Gamma]$ has the (Krull) dimension one, and hence t-dim$(\mathbb{R}[\Gamma]) = 1$.

4. $\mathbb{R}[\Gamma]$ is not a WFD.

Proof. (1) Clear. (2) This is an easy exercise. (3) Let $\mathbb{N}_0$ be the semigroup of nonnegative integers. Then $\mathbb{R}[\Gamma]$ is integral over $\mathbb{R}[\mathbb{N}_0]$ and $\dim(\mathbb{R}[\mathbb{N}_0]) = 1$. Thus, $\dim(\mathbb{R}[\Gamma]) = t$-$\dim(\mathbb{R}[\Gamma]) = 1$. (4) It can be shown that $1 - X \in \mathbb{R}[\Gamma]$ is contained in infinitely many height-one prime ideals of $\mathbb{R}[\Gamma]$. □

3.3 char$(D) = 0$

In this section, we study the WFD property of $D[\Gamma]$ under the assumption that char$(D) = 0$. It is known that if char$(D) = 0$, then $D[G]$ is a ring of Krull type if and only if D is a ring of Krull type and G is of type $(0, 0, 0, \dots)$ [25, Theorem 3].

Lemma 3.10. *[14, Theorem 3.1] Let G be a torsionfree abelian group, $D[G]$ be the group ring of G over D, and assume that char$(D) = 0$. Then $D[G]$ is of finite t-character if and only if D is of finite t-character and G is of type $(0, 0, 0, \dots)$.*

Corollary 3.11. *[14, Corollary 3.2] Let K be a field, G be a torsionfree abelian group, and assume that char$(K) = 0$. Then the following statements are equivalent.*

1. *$K[G]$ is a WFD.*

2. *$K[G]$ is a weakly Krull domain.*

3. *$K[G]$ is of finite t-character.*

4. *G is of type $(0, 0, 0 \dots)$.*

5. *$K[G]$ is a UFD.*

6. *$K[G]$ is a Krull domain.*

7. *$K[G]$ is a generalized Krull domain.*

Proof. (1) ⇒ (2) ⇒ (3) Clear. (3) ⇒ (4) Lemma 3.10. (4) ⇒ (5) Theorem 3.1. (5) ⇒ (1) Clear. (5) ⇒ (6) ⇒ (7) ⇒ (3) Clear. □

Theorem 3.12. *[14, Theorem 3.4] Assume that char$(D) = 0$. Then $D[\Gamma]$ is a WFD if and only if D is a weakly factorial GCD-domain, Γ is a weakly factorial GCD-semigroup, and G is of type $(0, 0, 0, ...)$.*

Proof. If $D[\Gamma]$ is a WFD, then $D[\Gamma]_H$ is a WFD. Note that $D[\Gamma]_H = K[G]$; hence G is of type $(0, 0, 0 \dots)$ by Corollary 3.11. Thus, the result follows directly from Theorem 3.5. □

3.4 char$(D) = p > 0$

In this section, we study when $D[\Gamma]$ is a WFD under the assumption that char$(D) = p > 0$. Recall that if char$(D) = p > 0$, then $D[G]$ is a ring of Krull type if and only if D is a ring of Krull type and G is of type $(0, 0, 0, \dots)$ except p [25, Theorem 5].

Lemma 3.13. *[14, Theorem 4.2] Let G be a torsionfree abelian group, $D[G]$ be the group ring of G over D, and assume that char$(D) = p > 0$. Then $D[G]$ is of finite t-character if and only if D is of finite t-character and G is of type $(0, 0, 0, \ldots)$ except p.*

Corollary 3.14. *[14, Corollary 4.3] Let K be a field, G be a torsionfree abelian group, and assume that char$(K) = p > 0$. Then the following statements are equivalent.*

1. *$K[G]$ is a WFD.*
2. *$K[G]$ is a weakly Krull domain.*
3. *$K[G]$ is of finite t-character.*
4. *G is of type $(0, 0, 0 \ldots)$ except p.*
5. *$K[G]$ is a generalized Krull domain.*

We next give a complete characterization of a WFD $D[\Gamma]$ with char$(D) = p > 0$.

Theorem 3.15. *[14, Theorem 4.5] Assume that char$(D) = p > 0$. Then $D[\Gamma]$ is a WFD if and only if D is a weakly factorial GCD-domain, Γ is a weakly factorial GCD-semigroup, and G is of type $(0, 0, 0, ...)$ except p.*

Example 3.16. *[14, Example 4.6]* Let K be a field, p be a prime number, $G = \bigcup_{n=1}^{\infty}(1/p^n)\mathbb{Z}$, and $K[G]$ be the group ring of G over K.

1. G is of type $(0, 0, 0 \ldots)$ except p but not of type $(0, 0, 0 \ldots)$.
2. If char$(K) = p > 0$, then $K[G]$ is a weakly factorial domain.
3. If char$(K) \neq p$ (e.g., char$(K) = 0$), then $K[G]$ is not a WFD.

Hence, $\mathbb{R}[G]$ is not a WFD, while $\mathbb{Z}_p[G]$ is a WFD, where $\mathbb{Z}_p$ is a field of p elements.

Proof. For (1), see [25, Proposition 12]. For (2) and (3), see (1) and Theorems 3.5 and 3.15. □

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