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Some strange behaviors of the power series ring $R[[X]]$

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Abstract. Let R be a commutative ring with identity. Let $R[X]$ and $R[[X]]$ be the polynomial ring and the power series ring respectively over R. Being the completion of $R[X]$ (under the X-adic topology), $R[[X]]$ does not always share the same property with $R[X]$. In this paper, we present some known strange behaviors of $R[[X]]$ compared to those of $R[X]$.

1 Introduction

In this paper, a ring means a commutative ring with identity. Let R be a ring. Let $R[X]$ and $R[[X]]$ be the polynomial ring and the power series ring respectively over R. Being the completion of $R[X]$ (under the X-adic topology), $R[[X]]$ does not always share the same property with $R[X]$. In this paper, we present some strange behaviors of $R[[X]]$ compared to those of $R[X]$. Aspects of $R[[X]]$ that will be considered include Krull dimension, transcendence degree, and Noetherian property.

Let R be a ring. If there exists a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of $n+1$ prime ideals of R, but no such chain of $n+2$ prime ideals, then we say that the Krull dimension of R is n (or R is n-dimensional) and write Krull-dim $R = n$. Otherwise, we say that the Krull dimension of R is infinite (or R is infinite-dimensional) and write Krull-dim $R = \infty$. For a cardinal number α, we say that $\dim R = \alpha$ if R has a chain of prime ideals with length α but no longer chains. (The length of a chain $\mathcal{P}$ of prime ideals of R is defined by $|\mathcal{P}| - 1$, where $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$. For two chains $\mathcal{P}$ and $\mathcal{Q}$ of prime ideals, we say that $\mathcal{P}$ is longer than $\mathcal{Q}$ if $|\mathcal{P}| > |\mathcal{Q}|$). We also say that $\dim R \geq \alpha$ if there is a chain of prime ideals of R with length $\geq \alpha$ and that $\dim R \leq \alpha$ if every chain of prime ideals of R has length $\leq \alpha$. Hence, $\dim R = \alpha$ if and only if $\dim R \geq \alpha$ and $\dim R \leq \alpha$. We note that if Krull-dim$R = n < \infty$, then $\dim R =$ Krull-dim R. Furthermore, $\dim R \geq \aleph_0$ implies Krull-dim$R = \infty$ but not vice versa.

The Krull dimension of the polynomial ring $R[X]$ is fairly well-known for a finite-dimensional ring R. For example, $\dim R[X_1, \ldots, X_n] = \dim R + n$ if R is a Noetherian ring or a Prüfer domain [16, 17]. In general, $\dim R + 1 \leq \dim R[X] \leq 2\dim R + 1$ [16]. For the power series ring $R[[X]]$, it is shown by J.T. Arnold that $\dim R[[X]] \geq \aleph_0$ if R is a non-SFT ring. He defines an ideal I of a ring R to be an SFT ideal if there exist a finitely generated ideal $J \subseteq I$ and $k \in \mathbb{N}$ such that $a^k \in J$ for each $a \in I$ and he calls a ring R an SFT ring if every ideal of R is an SFT ideal. Typical examples of non-SFT domains are finite-dimensional nondiscrete valuation domains and non-Noetherian almost Dedekind domains. Kang and Park [9] showed that $\dim V[[X]] \geq |\mathbb{R}|$ and hence $\dim V[[X]] \geq \aleph_1$ for a 1-dimensional nondiscrete

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valuation domain V. Kang and Toan [12] proved that $\dim V[[X]] \geq 2^{\aleph_1}$ for a 1-dimensional nondiscrete valuation domain V. Loper and Lucas [14, 15] also achieved the result that $\dim D[[X]] \geq 2^{\aleph_1}$ for a non-Noetherian almost Dedekind domain or a 1-dimensional nondiscrete valuation domain D. Recently, Toan and Kang [11] generalize the above results by showing that $\dim R[[X]] \geq 2^{\aleph_1}$ if R is a non-SFT domains. In Section 2, we are going to introduce their technique in proving this result.

Let D be an integral domain with quotient field K. It is easy to see that the polynomial rings $D[X]$ and $K[X]$ have the same quotient field. However, as mentioned in [3], it is rare that $D[[X]]$ and $K[[X]]$ have the same quotient field. Except for the trivial case when D is a field, the only example showing that $D[[X]]$ and $K[[X]]$ have the same quotient field is given by Gilmer in [7]. For two integral domains $D_1 \subseteq D_2$, denote by $tr.d.(D_2/D_1)$ the transcendence degree of the quotient field of D_2 over that of D_1. Hence, for a cardinal number α, $tr.d.(D_2/D_1) \geq \alpha$ if there exists a subset of the quotient field of D_2 with cardinality at least α that is algebraically independent over the quotient field of D_1. Sheldon showed in [18] that if D contains a nonzero element a such that $\cap_{i=1}^{\infty} a^i D = (0)$, then $tr.d.(D[[x/a]]/D[[X]]) \geq \aleph_0$ and hence $tr.d.(K[[X]]/D[[X]]) \geq \aleph_0$ since $D[[x/a]] \subseteq D[1/a][[X]] \subseteq K[[X]]$. In [2], Arnold and Boyd made a great improvement of this result by showing that if $K[[X]]$ and $D[[X]]$ have different quotient fields, then $tr.d.(K[[X]]/D[[X]]) \geq \aleph_0$. We present here the technique in [8], where the authors showed that whenever $K[[X]]$ and $D[[X]]$ have different quotient fields, $tr.d.(K[[X]]/D[[X]]) \geq \aleph_1$. Note that in general the bound $\aleph_1$ is the greatest lower bound that one can obtain since under the continuum hypothesis the cardinality of the quotient field of $K[[X]]$ is exactly $\aleph_1$ provided that K is countable.

Let α be an infinite cardinal number (e.g., $\alpha = \aleph_0, \aleph_1, \ldots$). An ideal I of a ring R is called an α-generated ideal if I can be generated by a set with cardinality $\leq \alpha$. R is called an α-generated ring if every ideal of R is an α-generated ideal. By definition, an $\aleph_0$-generated ring is a ring whose ideals are countably generated. Trivial examples of $\aleph_0$-generated rings are those that have only countably many elements (so that each ideal has itself as a countable generating set). Every Noetherian ring is obviously an $\aleph_0$-generated ring. However, the converse does not hold. Polynomial rings $R[X_1, X_2, \ldots, X_n, \ldots]$ in countably infinite indeterminates over countable rings R, the ring $\mathcal{O}$ of algebraic integers, the ring $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials on $\mathbb{Z}$, and 1-dimensional nondiscrete valuation domains are good examples of $\aleph_0$-generated rings that are not Noetherian rings.

Even though the class of $\aleph_0$-generated rings is strictly larger than the class of Noetherian rings, it is shown in [10] that when restricted to power series rings, they are actually the same. In other words, the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same for the power series ring $R[\![X]\!]$. This shows a strange behavior of the power series ring $R[\![X]\!]$ compared to that of the polynomial ring $R[X]$. Indeed, for any infinite cardinal number α, R is an α-generated ring if and only if $R[X]$ is an α-generated ring, which is an analogue of Hilbert Basis Theorem stating that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring.

2 Krull dimension of $R[[X]]$

2.1 Krull dimension of $R[X]$

The aim of this section is to show that $\dim R[[X]] \geq 2^{\aleph_1}$ if R is a non-SFT domain, which contrasts with the fact that the Krull dimension of the polynomial ring $R[X]$ is always finite provided that R is. The following result is from [16, Theorems 2 and 9].

Theorem 2.1 *If R is a finite-dimensional ring, then*

$$\dim R + 1 \leq \dim R[X] \leq 2 \dim R + 1.$$

Furthermore, if R is a Noetherian ring, then $\dim R[X] = \dim R + 1$.

When R is a Prüfer domain, one has the following.

Theorem 2.2 ([17, Theorem 4]) *If R is a finite-dimensional Prüfer domain, then*

$$\dim R[X_1, \ldots, X_n] = \dim R + n.$$

2.2 Construction of an η_1-set $\mathcal{A}$

We first construct an η_1-set $\mathcal{A}$. This η_1-set will be the index set of an infinite chain of prime ideals in $D[\![X]\!]$ when D is a non-SFT domain.

- Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers and let $\mathcal{U}$ be the set of all subsets U of $\mathbb{N}$ such that $U = \{n, n+1, \ldots\}$ for some $n \in \mathbb{N}$.
- For two sequences of positive integers $s = \{s_n\}$, $t = \{t_n\}$, define $s \gg t$ (or $t \ll s$) if for each positive integer k, there is a set $U \in \mathcal{U}$ (depending on k) such that $s_n > kt_n$ for each $n \in U$, i.e., $s_n > kt_n$ for all large n.
- Let $\mathcal{S}$ be the collection of all $\mathcal{A}$ such that $\mathcal{A}$ has the following properties.
  - $\mathcal{A}$ is a nonempty collection of strictly increasing sequences $s = \{s_n\}$ of positive integers.
  - If $s \in \mathcal{A}$, then $s \gg b$, where $b = \{b_n\}$ is the sequence defined by $b_n := n$ for all n.
  - If $s, t \in \mathcal{A}$ and $s \neq t$, then $s \gg t$ or $t \gg s$.

If u is the sequence defined by $u_n := b_n^2$ for each n, then it is easy to see that $u \gg b$. It follows that the set $\mathcal{S}$ is nonempty. Order $\mathcal{S}$ by set-theoretic inclusion. By Zorn's Lemma, there exists a maximal element in $\mathcal{S}$.

Definition 2.3 Let $(\mathcal{Y}, \underline{\ll})$ be a totally ordered set and B, C be subsets of $\mathcal{Y}$. We say that $B \ll C$ if $b \ll c$ for each $b \in B$ and each $c \in C$. A totally ordered set $(\mathcal{Y}, \underline{\ll})$ is an η_1-set if for any two countable subsets B, C of $\mathcal{Y}$ such that $B \ll C$, there exists an element $a \in \mathcal{Y}$ such that $B \ll a \ll C$, i.e., $b \ll a \ll c$ for each $b \in B$ and each $c \in C$.

Let $\mathcal{A}$ be a maximal element in $\mathcal{S}$. This choice of $\mathcal{A}$ will be fixed through the rest of this section. For $s, t \in \mathcal{A}$, we define $s \underline{\ll} t$ if and only if $s = t$ or $s \ll t$. Then $(\mathcal{A}, \underline{\ll})$ becomes a totally order set.

Theorem 2.4 ([11, Theorem 2.3]) *The set $(\mathcal{A}, \underline{\ll})$ is an η_1-set.*

2.3 Chains of prime ideals in $D[[X]]$

Suppose that D is a non-SFT domain. Then there is an ideal I of D which is not SFT. As in [1, p. 300], there exists a sequence $a_0, a_1, \ldots, a_n, \ldots$ of $I \subseteq D$ such that

$$a_m^m \notin (a_0, a_1, \ldots, a_{m-1})$$

for each $m \geq 1$. Fix this sequence $a_0, a_1, \ldots, a_n, \ldots$. For an element $s = \{s_n\} \in \mathcal{A}$, let

$$f_s := a_0 + a_1 X^{s_1} + \cdots + a_n X^{s_n} + \cdots$$

in $D[[X]]$. For elements $v_1, v_2, \ldots, v_n$ (not necessarily distinct) in $\mathcal{A}$, we let $I_{v_1, v_2, \ldots, v_n}$ be the ideal of $D[[X]]$ generated by all power series f_s such that $s \gg v_i$ for each $i = 1, 2, \ldots, n$,

$$I_{v_1, v_2, \ldots, v_n} := \langle \{ f_s \mid s \in \mathcal{A} \text{ and } s \gg \{v_1, v_2, \ldots, v_n\} \} \rangle.$$

Define

$$\frac{I_{v_1,v_2,\ldots,v_n}}{f_{v_1}f_{v_2}\cdots f_{v_n}} := \left\{ \frac{f}{f_{v_1}f_{v_2}\cdots f_{v_n}} \,\middle|\, f \in I_{v_1,v_2,\ldots,v_n} \right\}$$

and

$$S := \bigcup_{\substack{n\in\mathbb{N} \\ v_1,v_2,\ldots,v_n\in\mathcal{A}}} \frac{I_{v_1,v_2,\ldots,v_n}}{f_{v_1}f_{v_2}\cdots f_{v_n}}.$$

Then S is a subset of the quotient field of $D[[X]]$. For elements $v_1, \ldots, v_n$ (not necessarily distinct) in $\mathcal{A}$, let $B = \{v_1, \ldots, v_n\}$ be a multiset. We also let $\|B\| := n$. Define

$$I_B := I_{v_1,\ldots,v_n} \quad \text{and} \quad f_B := \prod_{v\in B} f_v = f_{v_1}\cdots f_{v_n}.$$

Then the set S can be rewritten as

$$S = \bigcup \frac{I_B}{f_B},$$

where the union is taken over all multisets B consisting of elements in $\mathcal{A}$. Let $D[[X]][S]$ be the ring generated by S over $D[[X]]$. Denote by $SD[[X]][S]$ the ideal of $D[[X]][S]$ generated by S. Then this ideal is not the unit ideal ([11, Lemma 3.4]).

Theorem 2.5 ([6, Theorem 19.6]) *Let P be a prime ideal of an integral domain R. There exists a valuation overring W of R with maximal ideal Q such that $Q \cap R = P$.*

Let $R := D[[X]][S]$. Since SR is not the unit ideal of R, one can choose a prime ideal P of R such that P contains SR. By Theorem 2.5, there exists a valuation overring W of R with maximal ideal Q such that $Q \cap R = P$. For each $\alpha \in \mathcal{A}$, let $I_\alpha W$ be the ideal of W generated by I_α and let

$$P_\alpha := \sqrt{I_\alpha W} \cap D[[X]].$$

Since W is a valuation domain, $\sqrt{I_\alpha W}$ is a prime ideal of W. Hence, P_α is a prime ideal of $D[[X]]$. For $\alpha, \beta \in \mathcal{A}$, it can be shown that $\alpha \ll \beta$ if and only if $P_\alpha \supset P_\beta$. Hence, the set $\mathcal{P} := \{P_\alpha\}_{\alpha\in\mathcal{A}}$ is a chain of prime ideals in $D[[X]]$. Furthermore, it is an η_1-set since $\mathcal{A}$ is.

A totally ordered set A is *Dedekind-complete* provided that every nonempty subset of A that has an upper bound has a supremum. Finally we can now present the main result of [11].

Theorem 2.6 *If D is a non-SFT domain, then* $\dim D[[X]] \geq 2^{\aleph_1}$.

Proof. Let

$$\mathcal{P}^* := \{\cup_{\alpha\in\mathcal{D}} P_\alpha \mid \emptyset \neq \mathcal{D} \subseteq \mathcal{A}\}.$$

Then $\mathcal{P}^*$ is a chain of prime ideals in $D[[X]]$ and $\mathcal{P}^*$ is Dedekind-complete. $\mathcal{P}^*$ contains an η_1-set, namely, $\mathcal{P}$. Therefore, by [5, Corollary 13.24], $\mathcal{P}^*$ has cardinality at least $2^{\aleph_1}$.

3 Transcendence degree in power series rings

3.1 Transcendence degree in polynomial rings

Theorem 3.1 *Let D be an integral domain with quotient field K. Then $D[X]$ and $K[X]$ have the same quotient field. Hence,*

$$tr.d.(K[X]/D[X]) = 1.$$

Proof. This is obvious.

3.2 Construction of an upper fathomless set $\mathcal{B}$

In this subsection, we construct an uncountable set $\mathcal{B}$. In the next subsection, we will construct a set $\{\boldsymbol{a}_f\}_{f\in\mathcal{B}}$ of power series in $K[[x]]$ that is algebraically independent over the quotient field of $D[[x]]$ when $K[[x]]$ and $D[[x]]$ have different quotient fields.

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers. Set $\mathcal{S}$ be the collection of all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the following three conditions (see [18]).

- $f(i) > i$ for all $i \in \mathbb{N}$.
- $f(i + 1) > f(i) + 1$ for all $i \in \mathbb{N}$.
- For each $k \geq 1$, there exists an $I \geq 1$ (depending on k) such that $f(i + 1) > k \cdot f(i)$ for all $i \geq I$.

The function $f(i) = 2^{2^i}$ satisfies these three conditions. Hence, the set $\mathcal{S}$ is nonempty. For a function f in $\mathcal{S}$, denote

$$\mathrm{im}(f|_{\geq I}) = \{f(i) \mid i \geq I\}.$$

Hence, $\mathrm{im}(f|_{\geq 1}) = \mathrm{im}(f)$ is the image of f.

Definition 3.2 For two functions f and g in $\mathcal{S}$, we define $g \gg f$ (we also write $f \ll g$) if there exists an integer $I \geq 1$ such that the following hold.

- $\mathrm{im}(g|_{\geq I}) \subseteq \mathrm{im}(f)$ and $g(i) > f(i)$ for all $i \geq I$.
- If $s_1 < s_2$ are both in $\mathrm{im}(g|_{\geq I})$, then they are not adjacent in $\mathrm{im}(f)$, i.e., there is no i such that $s_1 = f(i)$ and $s_2 = f(i + 1)$.

Remark 3.3 For $f, g, h \in \mathcal{S}$, if $f \ll g$ and $g \ll h$, then $f \ll h$.

Let $C(\mathcal{S})$ denote the collection of all nonempty subsets $\mathcal{B}$ of $\mathcal{S}$ satisfying: for any two different functions f and g in $\mathcal{B}$, either $f \gg g$ or $g \gg f$. We order $C(\mathcal{S})$ by set-theoretic inclusion. By Zorn's Lemma, there exists a maximal element in $C(\mathcal{S})$. Let $\mathcal{B}$ be a maximal element in $C(\mathcal{S})$. This choice of $\mathcal{B}$ will be fixed through the rest of the section. For $f, g \in \mathcal{B}$, we define $f \underline{\ll} g$ if and only if $f = g$ or $f \ll g$. Then $(\mathcal{B}, \underline{\ll})$ becomes a totally ordered set.

Definition 3.4 A totally ordered set $(\mathcal{Y}, \underline{\ll})$ is called an *upper fathomless set* if for every nonempty countable subset C of $\mathcal{Y}$, there exists an element $y \in \mathcal{Y}$ such that $y \gg c$ for all $c \in C$.

By definition, every upper fathomless set is an uncountable set. The following is [8, Theorem 7].

Theorem 3.5 *The set $(\mathcal{B}, \underline{\ll})$ is an upper fathomless set and hence is uncountable.*

3.3 Transcendence degree in power series rings

Suppose that D is an integral domain with quotient field K such that the quotient field of $K[[X]]$ properly contains the quotient field of $D[[X]]$. Then by [7, Theorem 1], there exists a sequence $\{a_i\}_{i=1}^{\infty}$ of nonzero elements of D such that $\cap_{i=1}^{\infty}(a_i) = (0)$. Replacing $\{(a_i)\}_{i=1}^{\infty}$ by a (suitable) strictly descending subsequence if necessary, we may assume that the sequence $\{(a_i)\}_{i=1}^{\infty}$ is strictly descending (see [7, Remark 1]), i.e.,

$$(a_1) \supset (a_2) \supset \cdots \supset (a_i) \supset \cdots .$$

Therefore, if $i_1 \geq i_2$ then $(a_{i_1}) \subseteq (a_{i_2})$ and hence a_{i_1} is a multiple of a_{i_2}. Let $\mathcal{B}$ be the upper fathomless set constructed in Subsection 3.2. For each $f \in \mathcal{B}$, define a power series $\boldsymbol{a}_f$ in $K[[X]]$ by

$$\boldsymbol{a}_f = \sum_{i=1}^{\infty} (X/a_i)^{f(i)}.$$

We will give sketch of proof of the following main result of the section. For a power series $\boldsymbol{p}$ in $K[[X]]$, the support of $\boldsymbol{p}$, denoted by supp$(\boldsymbol{p})$, is the set of all nonnegative integer s for which the coefficient of X^s in $\boldsymbol{p}$ is nonzero.

Theorem 3.6 ([8, Theorem 12]) *Suppose that D is an integral domain with quotient field K such that $D[[X]]$ and $K[[X]]$ have different quotient fields. Then the quotient field of $K[[X]]$ has uncountable transcendence degree over the quotient field of $D[[X]]$, i.e, t.r.$(K[[X]]/D[[X]]) \geq \aleph_1$.*

Sketch of proof. We show that the set $\{\boldsymbol{a}_f\}_{f\in\mathcal{B}}$ is algebraically independent over the quotient field of $D[[X]]$. Suppose on the contrary that there are some $f_1, f_2, \ldots, f_J \in \mathcal{B}$ such that $\boldsymbol{a}_{f_1}, \boldsymbol{a}_{f_2}, \ldots, \boldsymbol{a}_{f_J}$ are algebraically dependent over the quotient field of $D[[X]]$. We may assume that

$$f_1 \ll f_2 \ll \cdots \ll f_J.$$

Let $P(t_1, \ldots, t_J)$ be a nonzero polynomial with coefficients in the quotient field of $D[[X]]$ such that

$$P(\boldsymbol{a}_{f_1}, \ldots, \boldsymbol{a}_{f_J}) = 0.$$

Multiplying both sides of this equation by a suitable element in $D[[X]]$, we may assume that the coefficients of $P(t_1, \ldots, t_J)$ are in $D[[X]]$. Let M be the set of all power series of the form $\boldsymbol{a}_{f_1}^{e(1)} \ldots \boldsymbol{a}_{f_J}^{e(J)}$ corresponding to those monomials $t_1^{e(1)} \ldots t_J^{e(J)}$ which occur in $P(t_1, \ldots, t_J)$ with nonzero coefficient. Then the relation $P(\boldsymbol{a}_{f_1}, \ldots, \boldsymbol{a}_{f_J}) = 0$ is actually a linear combination of the elements of M with coefficients in $D[[X]]$. Order the elements of M lexicographically, using the exponents of $\boldsymbol{a}_{f_1}$, then $\boldsymbol{a}_{f_2}$, and so forth. Write $M = \{\boldsymbol{m}_0, \boldsymbol{m}_1, \ldots, \boldsymbol{m}_R\}$, where $\boldsymbol{m}_0$ is the largest term in this ordering. Then $P(\boldsymbol{a}_{f_1}, \ldots, \boldsymbol{a}_{f_J}) = 0$ has the form

$$\boldsymbol{d}_0\boldsymbol{m}_0 + \boldsymbol{d}_1\boldsymbol{m}_1 + \cdots + \boldsymbol{d}_R\boldsymbol{m}_R = 0$$

for some $\boldsymbol{d}_r \in D[[X]]$ $(r = 0, 1, \ldots, R)$, $\boldsymbol{d}_0 \neq 0$. Denote by k the index of the first nonzero coefficient $d_{0,k}$ of $\boldsymbol{d}_0$. The proof is finished by showing a contradiction that $0 \neq d_{0,k} \in \cap_{i=1}^{\infty}(a_i)$, which is technically complicated and is omitted.

4 $\aleph_0$-generated power series rings

The purpose of this section is to show that the power series ring $R[\![X]\!]$ is an $\aleph_0$-generated ring if and only if $R[\![X]\!]$ is a Noetherian ring (and hence) if and only if R is a Noetherian ring. In order to prove this result, we only need to show that if R is a non-Noetherian ring, then the power series ring $R[\![X]\!]$ is not an $\aleph_0$-generated ring since if R is a Noetherian ring, then $R[\![X]\!]$ is also a Noetherian ring (for example, see [13, Theorem 71]) and hence an $\aleph_0$-generated ring. Suppose that R is a non-Noetherian ring. Our task is to construct an ideal J of $R[\![X]\!]$ that cannot be generated by any countable subset of J. Using the η_1-set $(\mathcal{A}, \underline{\ll})$ in Section 2, we construct generators for the J of $R[\![X]\!]$ as follows.

Let R be a non-Noetherian ring. Then there exists a sequence $a_0, a_1, \ldots, a_m, \ldots$ of elements in R such that

$$a_m \notin (a_0, a_1, \ldots, a_{m-1})$$

for each $m \geq 1$. For each integer $m \geq 0$, we let $I_m := (a_0, a_1, \ldots, a_m)$. Then $a_m \notin I_{m-1}$ for each $m \geq 1$. For each sequence $s = \{s_n\} \in \mathcal{A}$, we define

$$f_s := a_0 + a_1 X^{s_1} + a_2 X^{s_2} + \cdots + a_n X^{s_n} + \cdots \in R[\![X]\!].$$

We let J be the ideal of $R[\![X]\!]$ generated by all f_s with $s \in \mathcal{A}$. The following is the main result of [10].

Theorem 4.1 ([10, Theorem 13]) *For a ring R, the following are equivalent.*

(1) $R[\![X]\!]$ *is an* $\aleph_0$*-generated ring.*

(2) $R[\![X]\!]$ *is a Noetherian ring.*

(3) R is a Noetherian ring.

Proof. We only need to prove that (1) implies (3). Suppose that R is not a non-Noetherian ring. We show that $R[\![X]\!]$ is not an $\aleph_0$-generated ring. It suffices to show that the ideal J constructed above is not a countably generated ideal. Suppose on the contrary that J is countably generated. Then there exists a countable subset $\mathcal{B}$ of $\mathcal{A}$ such that J is generated by $\{f_s \mid s \in \mathcal{B}\}$. Since $\mathcal{A}$ is a fathomless set, there exists a sequence $v \in \mathcal{A}$ such that $v \ll \mathcal{B}$. Since $f_v \in J$, f_v is a finite sum of elements of the form $h(s)f_s$,

$$f_v = \sum_s h(s) f_s, \tag{1}$$

where $h(s) \in R[\![X]\!]$ and $s \in \mathcal{B}$. Since $v \ll \mathcal{B}$, by taking a finite intersection of members of $\mathcal{U}$, we can find a set $U \in \mathcal{U}$ such that $v_m < s_m$ for each $m \in U$ and for each s appearing in the finite sum (1). Choose any number $m \in U$. Since $v_m < s_m$, the coefficient of f_s at X^j belongs to I_{m-1} for all $j \leq v_m$. It follows that the coefficient of $h(s)f_s$ at X^{v_m} belongs to I_{m-1}. This holds for every s appearing in the finite sum (1). Therefore, the coefficient of $\sum_s h(s) f_s$ at X^{v_m} belongs to I_{m-1}. This is a contradiction since the coefficient of $f_v = \sum_s h(s) f_s$ at X^{v_m} is a_m and $a_m \notin I_{m-1}$.

Corollary 4.2 *If R is an* $\aleph_0$*-generated ring, then the power series ring* $R[\![X]\!]$ *is not necessarily an* $\aleph_0$*-generated ring.*

Proof. Let R be any $\aleph_0$-generated ring which is not a Noetherian ring (for example, let $R = \mathbb{Q}[X_1, X_2, \ldots, X_n, \ldots]$, the polynomial ring in countably infinite indeterminates over the field of rational numbers $\mathbb{Q}$). Then $R[\![X]\!]$ is not an $\aleph_0$-generated ring by Theorem 4.1.

The above corollary contrasts with the following famous result for polynomial rings.

Theorem 4.3 *(Hilbert Basis Theorem.) If R is a Noetherian ring, then so is the polynomial ring* $R[X]$.

In comparing with Theorem 4.1, on has the following result, whose its proof follows the standard one of Hilbert Basis Theorem.

Theorem 4.4 ([10, Theorem 22]) *For any infinite cardinal number* α*, a ring R is an* α*-generated ring if and only if the polynomial ring* $R[X]$ *is an* α*-generated ring.*

5 Conclusion

Being the completion of the polynomial ring $R[X]$, the power series ring $R[[X]]$ however has very strange behaviors. The paper presents the following three examples.

- If $\dim R$ is finite, then so is $\dim R[X]$. However, $\dim R[[X]]$ can be infinite, in fact $\geq 2^{\aleph_1}$, even if $\dim R$ is finite.
- If D is an integral domain with quotient field K, then $D[X]$ and $K[X]$ share the same quotient field. However, the quotient field of $K[[X]]$ is much larger than that of $D[[X]]$. In fact, the quotient field of $K[[X]]$ is often has uncountable transcendence degree over that of $D[[X]]$.
- The concepts "Noetherian ring" and "$\aleph_0$-generated ring" are the same for the power series ring $R[[X]]$ while they are different for the polynomial ring $R[X]$. In fact, the polynomial ring $R[X]$ is an α-generated ring if and only if R is an α-generated ring, which is an analogue of Hilbert Basis Theorem stating that the polynomial ring $R[X]$ is a Noetherian ring if and only if R is a Noetherian ring.

Techniques involved in the proofs of the above results are quite complicated. By exploring these, we hope that similar techniques can be further developed to have a better understand of the power series ring $R[[X]]$.

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