Irreducible polynomials in Int($\mathbb{Z}$)

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Abstract. In order to fully understand the factorization behavior of the ring $\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ of integer-valued polynomials on $\mathbb{Z}$, it is crucial to identify the irreducible elements. Peruginelli [8] gives an algorithmic criterion to recognize whether an integer-valued polynomial $\frac{g}{d}$ is irreducible in the case where d is a square-free integer and $g \in \mathbb{Z}[x]$ has fixed divisor d. For integer-valued polynomials with arbitrary composite denominators, so far there is no algorithmic criterion known to recognize whether they are irreducible. We describe a computational method which allows us to recognize all irreducible polynomials in Int($\mathbb{Z}$). We present some known facts, preliminary new results and open questions.

1 Introduction

The ring $\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ of integer-valued polynomials on $\mathbb{Z}$ is well-known to be a non-unique factorization domain. The investigation of the factorization in this ring has been subject to research over the last three decades, see for example [1], [2], [3], [4], [5] and [8].

Peruginelli [8] described a computational method the determine whether an image-primitive polynomial $\frac{g}{d} \in \text{Int}(\mathbb{Z})$ is irreducible in the case where the d is square-free. His method uses so-called p-coverings and is based on the primary decomposition of the ideal $\{g \in \mathbb{Z}[x] \mid g(\mathbb{Z}) \subseteq p\mathbb{Z}\}$ which he described in [7].

So far, there is no similar computational method to verify the irreducibility of an image-primitive polynomial with a general composite denominator.

In this conference paper we develop such a method, which we refer to as "table method". This is work in progress and we want to emphasize that this method is not very efficient in terms of run-time or memory. It is merely a first step towards recognizing irreducible polynomials in a computational way. A second goal that we pursue in our research is to use the table method to construct irreducible polynomials with prescribed fixed divisors. An overall summary of our current open research questions can be found in Section 4 at the end of this short paper.

As for the structure of this paper, we give a few basic facts on the fixed divisor of a polynomial in Section 1 and then proceed with factorizations in Int($\mathbb{Z}$) in Section 2. Next,

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in Section 3 we explain the table criterion and state the main result together with a few immediate corollaries. Finally, we conclude this short note in Section 4 with open questions that we want to answer in a full version.

Preliminaries

Let $g \in \mathbb{Z}[x]$ be a polynomial. It is easily seen that for $d \in \mathbb{Z}$, $\frac{g}{d}$ is integer-valued on $\mathbb{Z}$ if and only if d divides all values $g(a)$ with $a \in \mathbb{Z}$. This motivates the definition of the fixed divisor.

Definition 1.1. For $f \in \mathrm{Int}(\mathbb{Z})$, we define the *fixed divisor* $\mathrm{d}(f)$ to be

$$\mathrm{d}(f) = (f(a) \mid a \in \mathbb{Z}).$$

We say f is *image-primitive* if $\mathrm{d}(f) = \mathbb{Z}$.

Remark 1.2. It is well-known that there exist primitive polynomials in $\mathbb{Z}[x]$ which are irreducible in both $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ but factor non-trivially in $\mathrm{Int}(\mathbb{Z})$, for example,

$$x^3 - x + 3 = 3 \cdot \frac{x^3 - x + 3}{3}.$$

Note that $x^3 - x + 3$ is primitive, but not image-primitive since $\mathrm{d}(x^3 - x + 3) = 3\mathbb{Z}$. More generally, Chapman and McClain [3, Thm. 2.4] have shown that a primitive polynomial $g \in \mathbb{Z}[x]$ is irreducible in $\mathrm{Int}(\mathbb{Z})$ if and only if g is image-primitive and irreducible in $\mathbb{Z}[x]$.

Remark 1.3. It is easily seen that $\mathrm{d}(fg) \subseteq \mathrm{d}(f)\,\mathrm{d}(g)$ for polynomials $f, g \in \mathrm{Int}(\mathbb{Z})$. In general, equality does not hold. For example, $\mathrm{d}(x) = \mathrm{d}(x-1) = \mathbb{Z}$ but $\mathrm{d}(x(x-1)) = 2\mathbb{Z}$.

Remark 1.4. Assume that $g \in \mathbb{Z}[x]$ is a primitive polynomial with fixed divisor $\mathrm{d}(g) = d\mathbb{Z}$ for an integer d. If p is a prime number with $p \mid d$, then $g(a) \equiv 0 \mod p$ for all $a \in \mathbb{Z}$ or equivalently, the residue class of g modulo p has p different roots in the field $\mathbb{Z}/p\mathbb{Z}$.

In particular, this implies that the prime decomposition of the fixed divisor of a primitive polynomial $g \in \mathbb{Z}[x]$ can only contain prime numbers p with $p \leq \deg(g)$.

This observation follows from a result of Pólya [9], cf. [10, Thm. 3.1].

2 Factorizations in $\mathrm{Int}(\mathbb{Z})$ and the σ notation

Notation 2.1. Let $\mathbb{P} \subseteq \mathbb{Z}$ denote the set of (positive) prime elements and for each prime number $p \in \mathbb{P}$, we write ν_p for the p-adic valuation.

Every polynomial $f \in \mathrm{Int}(\mathbb{Z})$ can be written as

$$f = a \frac{\prod_{i \in I} g_i}{\prod_{p \in \mathbb{P}} p^{k_p}} \tag{1}$$

where $a \in \mathbb{Z}$, $G = (g_i)_{i \in I}$ is a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ and $k_p \in \mathbb{N}_0$ such that $k_p = 0$ for all but finitely many $p \in \mathbb{P}$.

Remark 2.2. It follows from Remark 1.2 that irreducible polynomials in $\mathrm{Int}(\mathbb{Z})$ are necessarily image-primitive, that is, $a = \pm 1$ in Equation (1). In the context of factorizations, the sign of a is not important. From now on, we assume $a = 1$ whenever we consider image-primitive polynomials in $\mathrm{Int}(\mathbb{Z})$.

Notation 2.3. Let $G = (g_i)_{i\in I}$ be a finite family of polynomials $g_i \in \mathbb{Q}[x]$ and $J \subseteq I$ a subset. For a prime $p \in \mathbb{P}$ and $a \in \mathbb{Z}$, we set

$$\sigma_G^p(J) = \min_{a\in\mathbb{Z}} \sum_{j\in J} \nu_p(g_j(a)).$$

Lemma 2.4. *Let $G = (g_i)_{i\in I}$ be a finite family of polynomials $g_i \in \mathbb{Z}[x]$, $p \in \mathbb{P}$ a prime number and $J \subseteq I$. Then the following assertions hold*

(i) $\sigma_G^p(J) = \nu_p\left(\mathrm{d}\left(\prod_{j\in J} g_j\right)\right)$ *and*

(ii) $\sigma_G^p(I) \geq \sigma_G^p(J) + \sigma_G^p(I \setminus J)$.

Proof. (*i*): For $k \in \mathbb{N}_0$, $k = \nu_p\left(\mathrm{d}\left(\prod_{j\in J} g_j\right)\right)$ if and only if for all $a \in \mathbb{Z}$, $\nu_p\left(\prod_{j\in J} g_j(a)\right) \geq k$ and there exists $b \in \mathbb{Z}$ such that $\nu_p\left(\prod_{j\in J} g_j(b)\right) < k + 1$. It follows immediately that

$$\nu_p\left(\mathrm{d}\left(\prod_{j\in J} g_j\right)\right) = \min_{a\in\mathbb{Z}}\left(\nu_p\left(\prod_{j\in J} g_j(a)\right)\right) = \min_{a\in\mathbb{Z}} \sum_{j\in J} \nu_p(g_j(a)) = \sigma_G^p(J).$$

(*ii*): Since $\sum_{i\in I} \nu_p(g_i(a)) = \sum_{j\in J} \nu_p(g_j(a)) + \sum_{j\in I\setminus J} \nu_p(g_j(a))$ for all $a \in \mathbb{Z}$, the claim follows. □

Lemma 2.4 states that $\sigma_G^p(I)$ is a short-cut notation for the p-adic valuation of the fixed divisor of the polynomial $\prod_{i\in I} g_i$. This notation turns out to be useful. The next lemma states an easy but important result.

Lemma 2.5. *Let $G = (g_i)_{i\in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ and let $f = \frac{\prod_{i\in I} g_i}{\prod_{p\in\mathbb{P}} p^{k_p}} \in \mathbb{Q}[x]$.*

Then

(i) $f \in \mathrm{Int}(\mathbb{Z})$ *if and only if* $\sigma_G^p(I) \geq k_p$ *for all* $p \in \mathbb{P}$ *and*

(ii) $f \in \mathrm{Int}(\mathbb{Z})$ *is image-primitive if and only if* $\sigma_G^p(I) = k_p$ *for all* $p \in \mathbb{P}$.

Proof. This follows immediately from Lemma 2.4 and Definition 1.1. □

Remark 2.6. Let $G = (g_i)_{i\in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ such that $f = \frac{\prod_{i\in I} g_i}{\prod_{p\in\mathbb{P}} p^{k_p}} \in \mathrm{Int}(\mathbb{Z})$ is image-primitive.

Assume that f is reducible in $\mathrm{Int}(\mathbb{Z})$, that is, $f = h\tilde{h}$ factors into non-units $h, \tilde{h} \in \mathrm{Int}(\mathbb{Z})$. Since both h and $\tilde{h}$ divide f in $\mathbb{Q}[x]$ which is a unique factorization domain, it follows that h and $\tilde{h}$ are of the form

$$h = c\prod_{j\in J} g_j \quad \text{and} \quad \tilde{h} = \tilde{c}\prod_{j\in I\setminus J} g_j$$

for a subset $J \subseteq I$ and some $c, \tilde{c} \in \mathbb{Q}$ with $c\tilde{c} = \prod_{p\in\mathbb{P}} p^{-k_p}$.

Furthermore, f is image-primitive, which implies that neither h nor $\tilde{h}$ is a constant and hence $\emptyset \neq J \subsetneq I$.

Moreover, according to Lemmas 2.4 and 2.5 it follows that

$$k_p = \sigma_G^p(I) \geq \sigma_G^p(J) \geq -\nu_p(c)$$

and similarly

$$k_p \geq -\nu_p(\tilde{c})$$

for all prime numbers $p \in \mathbb{P}$.

On the other hand, $\nu_p(c) + \nu_p(\tilde{c}) = -k_p$ holds for all primes $p \in \mathbb{P}$ which implies that

$$-k_p \leq \nu_p(c), \nu_p(\tilde{c}) \leq 0.$$

We can conclude that every factor of f in $\mathrm{Int}(\mathbb{Z})$ which is a non-unit is of the form

$$\frac{\prod_{j \in J} g_j}{\prod_{p \in \mathbb{P}} p^{m_p}}$$

where $\emptyset \neq J \subseteq I$ and $0 \leq m_p \leq k_p$ for all prime numbers $p \in \mathbb{P}$.

3 Table method

In this section, we develop a computational method to recognize whether an image-primivite polynomial $f \in \mathrm{Int}(\mathbb{Z})$ is irreducible. Moreover, if f is reducible, this technique can be used to find all factorizations of f in $\mathrm{Int}(\mathbb{Z})$.

Following Remark 2.2, we can assume that f is of the form

$$f = \frac{\prod_{i \in I} g_i}{\prod_{p \in \mathbb{P}} p^{k_p}}.$$

Remark 3.1. We can determine $\sigma_G^p(J)$ by computing the values $\sum_{j \in J} \nu_p(g_j(a))$ for only finitely many $a \in \mathbb{Z}$ because every polynomial $g \in \mathbb{Z}[x]$ respects congruences, that is, if $a \equiv b \mod p^k$, then $g(a) \equiv g(b) \mod p^k$. In particular, this implies that if $\sigma_G^p(J) = n$, then there exists a $0 \leq a \leq p^{n+1} - 1$ with $\sum_{j \in J} \nu_p(g_j(a)) = n$.

A strategy that we pursue in this work is to create a table as in Table 1. The last entry of

Table 1. Template for table method

	g_1	g_2	$\dots$	$\sum$
0	$\nu_p(g_1(0))$	$\nu_p(g_2(0))$	$\dots$	$\sum_{j \in J} \nu_p(g_j(0))$
1	$\nu_p(g_1(1))$	$\nu_p(g_2(1))$	$\dots$	$\sum_{j \in J} \nu_p(g_j(1))$
$\vdots$		$\vdots$		$\vdots$
a	$\nu_p(g_1(a))$	$\nu_p(g_2(a))$	$\dots$	$\sum_{j \in J} \nu_p(g_j(a))$
$\vdots$		$\vdots$		$\vdots$

every row is greater than or equal to $\sigma_G^p(J)$. Say, we encounter the value n at the right end of one row. Then we only need to compute the table rows for $0 \leq a \leq p^n - 1$. Note that it is possible that this value is only reached in a row with $a \geq p^n$ in which case we can stop the computation, see Example 3.2.

Note that the starting value $a = 0$ is chosen randomly.

Example 3.2. We use Lemma 2.5 to determine whether $\frac{x(x-1)(x-2)}{9}$ is in $\mathrm{Int}(\mathbb{Z})$. Let $G = (g_1, g_2, g_3)$ with $g_1 = x$, $g_2 = x - 1$ and $g_3 = x - 2$. We compute $\sigma_G^3(\{1, 2, 3\})$ using the table method explained in Remark 3.1, see Table 2. Once we reach the fourth row we can stop further computations and conclude that $\sigma_G^3(\{1, 2, 3\}) = 1$. Hence $\frac{x(x-1)(x-2)}{9}$ is not integer-valued on $\mathbb{Z}$.

Example 3.3. From Table 2 and Lemma 2.5 we can conclude that $\frac{x(x-1)(x-2)}{3}$ is in $\mathrm{Int}(\mathbb{Z})$. It is also easily checked (for example using an additional table), that $\mathrm{d}(x(x-1)(x-2)) = 6\mathbb{Z}$ and hence $\frac{x(x-1)(x-2)}{6}$ is an image-primitive polynomial in $\mathrm{Int}(\mathbb{Z})$.

Table 2. Table method for Example 3.2

	g_1	g_2	g_3	$\sum$
0	∞	0	0	∞
1	0	∞	0	∞
2	0	0	∞	∞
3	1	0	0	1

Notation 3.4. For a finite family $G = (g_i)_{i\in I}$ of polynomials $g_i \in \mathbb{Z}[x]$ and $J \subseteq I$, we set

$$\sigma_G(J) = \sum_{p\in\mathbb{P}} \sigma_G^p(J).$$

Keep in mind that the sum above is finite, since $\sigma_G^p(J) = 0$ for all prime numbers $p \in \mathbb{P}$ with $p > \sum_{j\in J} \deg(g_j)$, see Remark 1.4.

Theorem 1. *Let $G = (g_i)_{i\in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ such that $f = \frac{\prod_{i\in I} g_i}{\prod_{p\in\mathbb{P}} p^{k_p}} \in \mathrm{Int}(\mathbb{Z})$ is image-primitive.*

Then the following assertions are equivalent:

(i) f is irreducible in $\mathrm{Int}(\mathbb{Z})$.

(ii) For all subsets $\emptyset \neq J \subsetneq I$ there exists a prime number $p \in \mathbb{P}$ such that

$$\sigma_G^p(J) + \sigma_G^p(I \setminus J) < k_p.$$

(iii) For all subsets $\emptyset \neq J \subsetneq I$,

$$\sigma_G(J) + \sigma_G(I \setminus J) < \sum_{p\in\mathbb{P}} k_p.$$

Proof. $(ii) \Leftrightarrow (iii)$: It follows from Lemmas 2.4 and 2.5 that for all $p \in \mathbb{P}$ and all $J \subseteq I$,

$$\sigma_G^p(J) + \sigma_G^p(I \setminus J) \leq \sigma_G^p(I) = k_p$$

holds. The equivalence between assertions (ii) and (iii) follows.

$(iii) \Rightarrow (i)$: Assume that f is reducible in $\mathrm{Int}(\mathbb{Z})$, that is,

$$f = \frac{\prod_{j\in J} g_j}{\prod_{p\in\mathbb{P}} p^{m_p}} \cdot \frac{\prod_{j\in I\setminus J} g_j}{\prod_{p\in\mathbb{P}} p^{k_p - m_p}}$$

for some subset $\emptyset \neq J \subsetneq I$ and natural numbers $m_p \leq k_p$ (cf. Remark 2.6). By Lemma 2.5 it follows that $\sigma_G^p(J) \geq m_p$ and $\sigma_G^p(I \setminus J) \geq k_p - m_p$ for all primes $p \in \mathbb{P}$. Hence

$$\sigma_G(J) + \sigma_G(I \setminus J) = \sum_{p\in\mathbb{P}} \sigma_G^p(J) + \sigma_G^p(I \setminus J) \geq \sum_{p\in\mathbb{P}} k_p.$$

$(i) \Rightarrow (iii)$: Assume $\emptyset \neq J \subsetneq I$ is a subset such that $\sigma_G(J) + \sigma_G(I \setminus J) \geq \sum_{p\in\mathbb{P}} k_p$. For $p \in \mathbb{P}$, we set $m_p = \sigma_G^p(J)$ and $\tilde{m}_p = \sigma_G^p(I \setminus J)$. By Lemma 2.5 it follows that

$$h = \frac{\prod_{j\in J} g_j}{\prod_p p^{m_p}} \quad \text{and} \quad \tilde{h} = \frac{\prod_{j\in J} g_j}{\prod_p p^{\tilde{m}_p}}$$

are elements of Int($\mathbb{Z}$).

Next, $m_p + \tilde{m}_p \leq \sigma_G^p(I) = k_p$ for all primes $p \in \mathbb{P}$ according to Lemmas 2.4 and 2.5. On the other hand,

$$\sum_{p\in\mathbb{P}} m_p + \tilde{m}_p = \sigma_G(J) + \sigma_G(I \setminus J) \geq \sum_{p\in\mathbb{P}} k_p$$

holds, and therefore it follows that $m_p + \tilde{m}_p = k_p$ for all $p \in \mathbb{P}$.

Hence

$$f = h \cdot \tilde{h}$$

is a proper factorization of f in Int($\mathbb{Z}$), that is, f is reducible in Int($\mathbb{Z}$). □

Theorem 1 yields some immediate corollaries. The next one is merely a restatement of the theorem in case only one prime factor appears in the denominator.

Corollary 3.5. *Let $G = (g_i)_{i\in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$, $p \in \mathbb{P}$ a prime number and $k \in \mathbb{N}$ such that $f = \frac{\prod_{i\in I} g_i}{p^k} \in$ Int($\mathbb{Z}$) is image-primitive.*

Then f is irreducible in Int($\mathbb{Z}$) *if and only if for all subsets $\emptyset \neq J \subsetneq I$*

$$\sigma_G^p(J) + \sigma_G^p(I \setminus J) < k.$$

The next corollary summarizes how to find all factorizations of a polynomial $f \in$ Int($\mathbb{Z}$) into not necessarily irreducible factors.

Corollary 3.6. *Let $G = (g_i)_{i\in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ such that $f = \frac{\prod_{i\in I} g_i}{\prod_{p\in\mathbb{P}} p^{k_p}} \in$ Int($\mathbb{Z}$) is image-primitive.*

Then every factorization of f (into not necessarily irreducibles) is of the form

$$f = \left(\frac{\prod_{j\in J_1} g_j}{\prod_{p\in\mathbb{P}} p^{m_{1,p}}}\right) \cdots \left(\frac{\prod_{j\in J_t} g_j}{\prod_{p\in\mathbb{P}} p^{m_{t,p}}}\right)$$

where $I = J_1 \uplus \cdots \uplus J_t$ is a partition of I such that for each prime number $p \in \mathbb{P}$ the following conditions hold

(i) $\sigma_G^p(J_i) = m_{i,p}$ *for all* $1 \leq i \leq t$,

(ii) $m_{1,p} + \cdots + m_{t,p} = k_p$ *and*

Remark 3.7. We can use Corollary 3.6 also to find the factorizations into irreducibles of $f \in$ Int($\mathbb{Z}$). For this purpose, we need to find all partitions of I which satisfy the conditions but cannot be "refined" any further.

Example 3.8. We find all factorizations of

$$f = \frac{(x^3 + 2x + 1)(x^2 + 2x + 1)(x^3 - 3x^2 + 8x + 20)(x^3 + 5)}{4}$$

and consider the family of polynomials $G = (g_1, g_2, g_3, g_4)$ with $g_1 = x^3 + 2x + 1$, $g_2 = x^2 + 2x + 1$, $g_3 = x^3 - 3x^2 + 8x + 20$ and $g_4 = x^3 + 5$.

The degree of the product is 11, which means that possible prime factors of the fixed divisor are 2, 3, 5, 7 and 11. By plugging in 0, we can exclude 3 and 7 and by plugging in 1, we can exclude 5 and 11.

We create a table for $p = 2$, see Table 3.

It follows that the $\sigma_G(\{1, 2, 3, 4\}) = \sigma_G^2(\{1, 2, 3, 4\}) = 2$. Therefore, f is image-primitive.

Table 3. Tables for Example 3.8

	g_1	g_2	g_3	g_4	$\sum \nu_2$
0	0	0	2	0	2
1	2	2	1	1	6
2	0	0	5	0	5
3	1	4	2	5	12

There are 3 partitions of $\{1, 2, 3, 4\}$ which satisfy the conditions of Corollary 3.6. These are

$$\begin{aligned}\{1, 2, 3, 4\} &= \{1, 3\} \uplus \{2\} \uplus \{4\}\\ &= \{2, 3\} \uplus \{1\} \uplus \{4\}\\ &= \{3, 4\} \uplus \{1\} \uplus \{2\}\end{aligned}$$

It follows that

$$\begin{aligned}f &= \frac{(x^3 + 2x + 1)(x^3 - 3x^2 + 8x + 20)}{4}(x^2 + 2x + 1)(x^3 + 5)\\ &= \frac{(x^2 + 2x + 1)(x^3 - 3x^2 + 8x + 20)}{4}(x^3 + 2x + 1)(x^3 + 5)\\ &= \frac{(x^3 - 3x^2 + 8x + 20)(x^3 + 5)}{4}(x^3 + 2x + 1)(x^2 + 2x + 1)\end{aligned}$$

are all factorizations of f into irreducibles in $\mathrm{Int}(\mathbb{Z})$.

Corollary 3.9. *Let $G = (g_i)_{i\in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ such that $f = \frac{\prod_{i\in I} g_i}{\prod_{p\in\mathbb{P}} p^{k_p}} \in \mathrm{Int}(\mathbb{Z})$ is image-primitive.*

If there exists a prime number $p \in \mathrm{Int}(\mathbb{Z})$ such that $\sigma_G^p(J) + \sigma_G^p(I \setminus J) < k_p$ for all subsets $\emptyset \neq J \subsetneq I$, then f is irreducible.

The following example shows that the implication in Corollary 3.9 cannot be reversed in general. A similar observation has already been made by Peruginelli [8] in the case of square-free denominators.

Example 3.10. We consider the polynomial

$$f = \frac{(x^2 + 4)(x^2 - 2x + 5)(x^3 - 5x^2 + 6x + 5)}{20}.$$

We set $g_1 = x^2 + 4$, $g_2 = x^2 - 2x + 5$ and $g_3 = x^3 - 5x^2 + 6x + 5$, $G = (g_1, g_2, g_3)$ and compute a table with respect to ν_2 and ν_5 using the table method which we explained in Remark 3.1, see Table 4. Possible prime factors of the fixed divisors of the numerator are 2, 3, 5 and 7. By plugging in 0, one can see that 3 and 7 cannot occur.

We can read off these tables that $\sigma_G^2(\{1, 2\}) + \sigma_G^2(\{3\}) = 2 = k_2$ and $\sigma_G^5(\{2\}) + \sigma_G^2(\{1, 3\}) = 1 = k_5$. Hence the lefthand side of the implication in Corollary 3.9 does not hold.

However, f is irreducible according to Theorem 1, since

$$\begin{aligned}\sigma_G(\{1\}) + \sigma_G(\{2, 3\}) &= 0 < 3 = k_2 + k_5\\ \sigma_G(\{2\}) + \sigma_G(\{1, 3\}) &= 1 < 3\\ \sigma_G(\{3\}) + \sigma_G(\{1, 2\}) &= 2 < 3\end{aligned}$$

Table 4. Tables for Example 3.10

	g_1	g_2	g_3	$\sum \nu_2$
0	2	0	0	2
1	0	2	0	2
2	3	0	0	3
3	0	3	0	3

	g_1	g_2	g_3	$\sum \nu_5$
0	0	1	1	2
1	1	0	0	1
2	0	1	1	2
3	0	0	1	1
4	1	0	0	1

Since for all non-empty subsets J of an index set of size 3, either J or its complement is a singleton, there are no more subsets to be checked.

4 Open questions

Absolutely irreducible polynomials in $\mathrm{Int}(\mathbb{Z})$

We are interested in a certain subclass of irreducible polynomials in $\mathrm{Int}(\mathbb{Z})$.

Definition 4.1. A polynomial $f \in \mathrm{Int}(\mathbb{Z})$ is *absolutely irreducible* if for all $n \in \mathbb{N}$, f^n has exactly one factorization into irreducibles, that is,

$$f^n = \underbrace{f \cdots f}_{n \text{ times}}$$

Remark 4.2. It is easily seen that $\mathrm{d}(g^n) = \mathrm{d}(g)^n$. Hence, if $g \in \mathbb{Z}[x]$ is primitive and irreducible with fixed divisor $d\mathbb{Z}$, then $f = \frac{g}{d}$ is absolutely irreducible.

Polynomials of the form described in Remark 4.2 are not the only types of absolutely irreducible polynomial as the following example shows.

Example 4.3. The polynomial $f = \frac{x(x-1)}{2}$ is absolutely irreducible: According to Remark 2.6, any factorization of f^n into two non-units in $\mathrm{Int}(\mathbb{Z})$ is of the form

$$\left(\frac{x^k(x-1)^\ell}{2^t}\right) \cdot \left(\frac{x^{n-k}(x-1)^{n-\ell}}{2^{n-t}}\right)$$

However, $\mathrm{d}(x) = \mathrm{d}(x^m) = \mathbb{Z}$ and $\mathrm{d}(x-1) = \mathrm{d}((x-1)^m) = \mathbb{Z}$ for all m. Therefore, it follows that $t = \ell = k$ which implies that f is absolutely irreducible.

The following example shows that not every irreducible polynomials is also absolutely irreducible.

Example 4.4. Let $f = \frac{x(x^2+2x+5)}{2}$. Using Remark 3.1, we create tables for the prime numbers 2, see Table 5. The prime number 3 can be excluded by plugging in 1. From this table we

Table 5. Tables for Example 4.4

	x	x^2+2x+5	$\sum \nu_2$
0	∞	0	∞
1	0	3	3
2	1	0	1

can see, that f is irreducible in $\mathrm{Int}(\mathbb{Z})$. However, it is not absolutely irreducible since

$$f^2 = f \cdot f = \frac{x^2(x^2+2x+5)}{4} \cdot (x^2+2x+5)$$

Question: Can we use the table method to recognize absolutely irreducible polynomials? For example, assume f is an irreducible polynomial in $\mathrm{Int}(\mathbb{Z})$ such that there are two polynomials in the numerator $g_i \neq g_j$ whose table columns look alike. Then, it is easily seen that they can be interchanged in every factorization of f^n with $n > 1$ and hence such a polynomial cannot be absolutely irreducible.

If not the table method, is there some other way to recognize absolutely irreducible polynomials?

A Irreducible polynomials in $\mathrm{Int}(\mathbb{Z}_{(p)})$

The *length* of factorization of, here a polynomial $f \in \mathrm{Int}(\mathbb{Z}_{(p)})$, into irreducibles is the number of irreducible factors. The collection of all possible lengths of an element is called its *set of lengths*. When we speak of multisets of lengths, then we simply take multiplicities of lengths into account. Note that we want to count multiplicities only for essentially different factorizations, that is, if the factors of the two factorizations are not pairwise associated. For details on factorizations, we refer to Geroldinger and Halter-Koch's textbook [6].

Question. Which multisets of lengths can occur in $\mathrm{Int}(\mathbb{Z}_{(p)})$?

It is known that every finite (multi-)set of natural numbers occurs as (multi-)set of a polynomial in $\mathrm{Int}(\mathbb{Z})$, see Frisch [4] and Frisch, Nakato and Rissner [5]. The construction that was used in these papers cannot immediately be adapted to the local (or semilocal case) because it strongly relies on the existence of infinitely many prime numbers.

In order to overcome the difficulties, we are trying to answer the following question.

Question. How to construct irreducible polynomials in $\mathrm{Int}(\mathbb{Z}_{(p)})$ with a prescribed fixed divisor?

Another question that comes up in that context is whether $\mathrm{Int}(\mathbb{Z}_{(p)})$ is *transfer Krull*, that is, there exists a transfer homomorphism to a block monoid. For definitions and further details, we refer to [6].

It has been shown in [4] that no such homomorphism can exist for $\mathrm{Int}(\mathbb{Z})$. More generally, it is known that $\mathrm{Int}(D)$ is not transfer Krull if D is a Dedekind domain with infinitely many maximal ideals, all of which have finite index, see [5].

Question. Does there exist a transfer homomorphism from the multiplicative monoid of $\mathrm{Int}(\mathbb{Z}_{(p)})$ to a block monoid?

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