# An algebraic method for quasi-linear first-order ODEs

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**Abstract.** In this paper, we consider the class of quasi-linear first-order ODEs of the form $y' = P(x, y)$, where $P$ is a polynomial in $y$ with coefficients in $\mathbb{C}(x)$, and study their algebraic solutions. Our method is intrinsically based on algebraic function field theory. We give an upper bound for the degrees of algebraic solutions which are in a quadratic extension of $\mathbb{C}(x)$.

## 1 Introduction

A quasi-linear first-order ODE is a differential equation of the form $y' = f(x, y)$ for some rational function $f \in \mathbb{C}(x, y)$. In this paper, we are interested in its algebraic solutions lying in a quadratic extension of $\mathbb{C}(x)$.

The study of quasi-linear first-order ODEs has a long history which can be traced back to the works of Fuchs [5] and Poincaré [7]. There are many solution methods for special classes of first-order ODEs. In case $f(x, y)$ is a polynomial of degree 2 in $y$, the differential equation is called a Riccati equation. Riccati equations form the simplest class of nonlinear first-order ODEs. In [6], Kovacic presents a complete algorithm for determining all algebraic solutions of a Riccati equation. The algorithm by Kovacic is based on a systematic study of differential Galois groups for second-order linear ordinary differential equations with rational function coefficients.

For general quasi-linear first-order ODEs, the problem of determining an algebraic solution still remains open. In case an upper bound for the degrees of the algebraic solutions is given, there are efficient algorithms to determine the solutions, such as [8] or [1]. The problem of determining an upper bound for the degrees of the algebraic solutions of a quasi-linear first-order ODE is one of the equivalent form of the Poincaré problem (see [1]). In [2], Carnicer gives an upper bound for the degrees of irreducible invariant algebraic curves of a planar polynomial vector field provided that the singularities of the curves are not too bad. This leads to a generic, but not complete, answer for the Poincaré problem.

In this paper, we modify a method developed in [10] for studying algebraic solutions of quasi-linear first-order ODEs. The method is based on the theory of algebraic function fields. In particular, we view the unknown function $y(x)$ and its derivative $y'(x)$ as elements in a suitable algebraic function field over $\mathbb{C}$. Standing from the algebraic function field theory point of view, since $y(x)$ and $y'(x)$ are algebraically dependent, certain relations between the poles of the principal divisors defined by them can be encoded (see [4]). In [10], the authors use this method to establish an algorithm for determining all solutions in the field $\mathbb{C}(x)$ of

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rational functions, if there is any, of the differential equation. In this paper, we apply this method for a wider class of algebraic solutions, namely the class of algebraic solutions which are in a quadratic extension of $\mathbb{C}(x)$. We obtain a degree bound for algebraic solutions which are in a quadratic extension of $\mathbb{C}(x)$ of a quasi-linear first-order ODE $y' = P(x, y)$, where $P$ is a polynomial of degree at least 5 in $y$ with coefficients in $\mathbb{C}(x)$ (see Theorem 4).

The rest of the paper is organized as follows. In Section 2, we recall several definitions and basic facts from the theory of algebraic functions. Further study for a quadratic extension of $\mathbb{C}(x)$ is presented in Section 3. In Section 4, we apply facts presented in the previous sections to determine a degree bound for algebraic solutions which are in a quadratic extension of $\mathbb{C}(x)$ of a class of quasi-linear first-order ODEs.

## 2 Algebraic function fields

In this section, we recall basic notions and facts from algebraic function field theory. For more details, we refer the readers to [9].

We use the symbol $\mathbb{K}$ to denote an algebraically closed field of characteristic zero. In this paper, $\mathbb{K}$ can be the field $\mathbb{C}$ of complex numbers or the field $\overline{\mathbb{C}(x)}$ of algebraic functions in the variable $x$ over $\mathbb{C}$. A field $F$ containing $\mathbb{K}$ is called an *algebraic function field* over $\mathbb{K}$ if it has transcendence degree one over $\mathbb{K}$. For example, the field $\mathbb{K}(x)$ of rational functions in the variable $x$ with coefficients in $\mathbb{K}$ is an algebraic function field over $\mathbb{K}$. Finite extensions of $\mathbb{K}(x)$ are also algebraic function fields over $\mathbb{K}$.

A *valuation ring* in $F$ (over $\mathbb{K}$) is a ring $O$ such that $\mathbb{K} \subsetneq O \subsetneq F$ and for every $y \in F$ either $y \in O$ or $y^{-1} \in O$. Since a valuation is a local ring, it admits a unique maximal ideal, which is the set of its non-invertible elements. Let $P$ be the unique maximal ideal of $O$. Then $P$ is principal, and every ideal of $O$ is a power of $P$. Assume that $P$ is generated by $t$, i.e. $P = tO$, then every element $y \in F \setminus \{0\}$ can be written uniquely as $y = u \cdot t^n$ for some unit $u$ of $O$ and $n \in \mathbb{Z}$. Note that the power $n$ does not depend on how $t$ is chosen. We call $n$ the *valuation of $y$ at $P$*, and denote it by $\nu_P(y)$. As a convention, we set $\nu_P(0) = \infty$.

A *prime divisor* of $F$ (over $\mathbb{K}$) is defined to be the maximal ideal of a valuation ring of $F$ (over $\mathbb{K}$). By $\mathbb{P}_{F/\mathbb{K}}$ we denote the set of all prime divisors of $F$. Due to [9, Theorem 1.1.13, p. 5], there is a one-to-one correspondence between valuation rings and prime divisors of $F$. Let $P \in \mathbb{P}_F$ be a prime divisor of $F$, and $O_P$ the corresponding valuation ring. Since $\mathbb{K}$ is algebraically closed, the residue field $O_P/P$ is isomorphic to $\mathbb{K}$. Therefore, there is a natural ring projection $O \to O/P \equiv \mathbb{K}$. We usually denote the image of $y \in O$ in $\mathbb{K}$ by $y(P)$. In case $F \subset \overline{\mathbb{K}(x)}$ is a finite extension of $\mathbb{K}(x)$, the projection map is exactly the usual evaluation map.

Let $y \in F$, and $P \in \mathbb{P}_{F/\mathbb{K}}$ a prime divisor of $F$. If $\nu_P(y) > 0$, we say that $y$ has a zero of order $\nu_P(y)$ at $P$, and that $P$ is a zero of $y$. In case $\nu_P(y) < 0$, we say that $y$ has a pole of order $-\nu_P(y)$ at $P$, and that $P$ is a pole of $y$. Every elements of $F$ has only finitely many zeros and poles. Furthermore, the number of zeros of $y$ is always equal to the number of poles of $y$, counting multiplicities.

Let $\mathbf{Div}_{\mathbb{K}}(F) = \bigoplus_{P \in \mathbb{P}_{F/\mathbb{K}}} \mathbb{Z}P$ be the set of all finitely many linear combinations of prime divisors of $F$. $\mathbf{Div}_{\mathbb{K}}(F)$ forms a free abelian group with the basis $\mathbb{P}_{F/\mathbb{K}}$. Elements of $\mathbf{Div}_{\mathbb{K}}(F)$ are called *divisors* of $F$ (over $\mathbb{K}$). For a divisor $\delta = n_1 P_1 + n_2 P_2 + \ldots + n_r P_r$, we define $\deg \delta = n_1 + \ldots + n_r$ as the degree of $\delta$. We define a partial order in $\mathbf{Div}_{\mathbb{K}}(F)$ as follows: for divisors $\delta, \delta'$, we say $\delta \geq \delta'$ if all coefficients of $\delta - \delta'$ are non-negative. A divisor whose coefficients are all non-negative is called *effective*.

Principal divisors are essential examples of divisors. For each $x \in F^*$, the principal divisor of $x$ in $F$ is denoted by $(x) = \sum_{P \in \mathbb{P}_F} \nu_P(x) P$. Note that, since every element in $F^*$ has only finitely many zeros and poles, the sum is always finite. It is important for us to gather information about poles of $x$. In order to do that, we set

$$(x)_\infty = - \sum_{\substack{P \in \mathbb{P}_F \\ \nu_P(x) < 0}} \nu_P(x) P.$$

The simplest algebraic function field over $\mathbb{K}$ is the field $\mathbb{K}(x)$ of rational functions for some $x$ transcendental over $\mathbb{K}$. In this case, there is a one-to-one correspondence between prime divisors of $\mathbb{K}(x)$ and the set $\mathbb{K} \cup \{\infty\}$. Therefore, we might identify $\mathbb{P}_{\mathbb{K}(x)}$ with $\mathbb{K} \cup \{\infty\}$. For a rational function $y(x) \in \mathbb{K}(x)$, an element $x_0 \in \mathbb{K}$ is a pole of $y(x)$ if it is a root of the denominator of $y(x)$. The multiplicity of the root is the order of the pole. The infinity $\infty$ is a pole of $y(x)$ if and only if the degree of the denominator of $y(x)$ is larger than the degree of the numerator. In this case, the difference of the degrees is the order of the pole.

## 3 Quadratic extensions of $\mathbb{C}(x)$

In this section, we study the number of poles of an algebraic function in a quadratic extension of $\mathbb{C}(x)$. The result of this section is essential in the determination of an upper bound for algebraic solutions of nonlinear first-order differential equations in the next section. We also revisit a result in [10] which is a key step in the algebraic function field approach for differential equations.

**Lemma 1.** *Let $F \subset \overline{\mathbb{C}(x)}$ be a quadratic extension of $\mathbb{C}(x)$, and $y = y(x) \in F \setminus \mathbb{C}(x)$. Let $P(X, Y) \in \mathbb{C}[X, Y]$ be an irreducible polynomial such that $P(x, y) = 0$. Then*

1. $\deg_Y P = 2$,
2. $\deg(y)_\infty = \deg_X P$, *and*
3. $\deg(\frac{dy}{dx})_\infty \leq 4 \deg_X P$.

*Proof.* 1. This item follows from the fact that $F$ is an extension of degree 2 over $\mathbb{C}(x)$.

2. By [9, Theorem 1.4.11, p. 19], we have $\deg(y)_\infty = [F : \mathbb{C}(y)]$. Since $y \notin \mathbb{C}(x)$, $F = \mathbb{C}(x, y)$. Therefore $\deg(y)_\infty$ is equal to the degree of a minimal polynomial of $x$ over $\mathbb{C}(y)$, which is $\deg_X P$.

3. Taking the derivative both sides of the equation $P(x, y) = 0$, we obtain

$$\frac{\partial P}{\partial X}(x, y) + \frac{dy}{dx} \cdot \frac{\partial P}{\partial Y}(x, y) = 0.$$

Let $R(X, Z)$ be the resultant of $P(X, Y)$ and $\frac{\partial P}{\partial X}(X, Y) + Z \cdot \frac{\partial P}{\partial Y}(X, Y)$ with respect to $Y$. From the definition of the resultant, since $\deg_Y P = 2$, the polynomial $R(X, Z)$ is the determinant of a Sylvester matrix of size $4 \times 4$ (see [3, Definition 2, p. 162]). The entries of the matrix are coefficients of the polynomials $P(X, Y)$ and $\frac{\partial P}{\partial X}(X, Y) + Z \cdot \frac{\partial P}{\partial Y}(X, Y)$ which are treated as polynomials in $Y$ with coefficients in $\mathbb{C}[X, Z]$. Therefore, $\deg_X R(X, Z) \leq 4 \deg_X P(X, Y)$. By [3, Proposition 5, p. 164], we have $R(x, \frac{dy}{dx}) = 0$. Therefore, the minimal polynomial of $x$ over $\mathbb{C}\left(\frac{dy}{dx}\right)$ is a factor of $R$. Hence, the degree of the divisor $(\frac{dy}{dx})_\infty$, which is equal to the degree of the minimal polynomial of $x$ over $\mathbb{C}\left(\frac{dy}{dx}\right)$, is at most $4 \deg_X P$.

□

The following theorem is the key step in the computation of rational solutions of nonlinear first-order ODEs in [10]. A different version of it can be found in [4].

**Theorem 2** (See [11]). *Let $K$ be an algebraic function field over $\overline{\mathbb{C}(x)}$, and $u, v \in K$ such that $(u)_\infty \leq (v)_\infty$. Let $G(Y,Z) \in \overline{\mathbb{C}(x)}[Y,Z]$ be an irreducible polynomial such that $G(u,v) = 0$. Consider a finite extension $L \subset \overline{\mathbb{C}(x)}$ of $\mathbb{C}(x)$ containing coefficients of $G$. There exists an effective divisor $\delta \in \mathbf{Div}_{\mathbb{C}}(L)$, depending only on $u$ and $v$, such that for every prime divisor $P \in \mathbb{P}_{K/\overline{\mathbb{C}(x)}}$, if $u(P)$ and $v(P)$ lie in $L$, then*

$$(u(P))_\infty \leq (v(P))_\infty + \delta,$$

*as divisors in $\mathbf{Div}_{\mathbb{C}}(L)$.*

In [10], the authors apply the rational function version of the above theorem to the problem of determining rational solutions of differential equations. The following is another consequence applying to the class of algebraic functions in a quadratic extension of $\mathbb{C}(x)$.

**Corollary 3.** *Let $u = u(x,y)$, $v = v(x,y) \in \mathbb{C}(x,y)$ such that $(u)_\infty \leq (v)_\infty$ as divisors in $\mathbf{Div}_{\overline{\mathbb{C}(x)}}(\overline{\mathbb{C}(x)}(y))$. Let $F \subset \overline{\mathbb{C}(x)}$ be a quadratic extension of $\mathbb{C}(x)$. Then there exists a constant $C$, depending only on $u$ and $v$, such that for every $y(x) \in F$, if $u(x,y(x))$ and $v(x,y(x))$ are defined then*

$$\deg(u(x,y(x)))_\infty \leq \deg(v(x,y(x)))_\infty + C.$$

## 4 A class of nonlinear first-order ODEs and their algebraic solutions in a quadratic extension of $\mathbb{C}(x)$

In this section, we consider a class of nonlinear first-order ODEs and study their solutions lying in a quadratic extension of $\mathbb{C}(x)$. We associate for each algebraic function $y(x) \in \overline{\mathbb{C}(x)}$ an irreducible polynomial $P(X,Y) \in \mathbb{C}[X,Y]$ such that $P(x,y(x)) = 0$. The polynomial $P$ will be called the *annihilating polynomial* of $y(x)$. Note that the annihilating polynomial of an algebraic function is unique up to multiplication by a nonzero complex number. In this case, we shall call the degree of $P$ in $X$ to be the degree of $y(x)$, i.e. $\deg y(x) := \deg_X P$.

**Theorem 4.** *Consider the differential equation*

$$y' = P(x,y), \tag{1}$$

*where $P(x,y) \in \mathbb{C}[x,y]$ and $\deg_y P \geq 5$. Then there exists a constant $C$, depending only on $P$, such that for every solution $y(x)$ lying in a quadratic extension of $\mathbb{C}(x)$, we have $\deg y(x) \leq C$.*

*Proof.* Let $n = \deg_x P$ and consider two rational functions $u(x,y) = y^n$ and $v(x,y) = P(x,y)$ in $\mathbb{C}(x,y)$. As elements in the algebraic function field $\overline{\mathbb{C}(x)}(y)$ over $\overline{\mathbb{C}(x)}$, $u$ and $v$ have pole only at infinity with the same degree. Therefore they meet the requirement of Corollary 3. Let us fix a quadratic extension $F \subset \overline{\mathbb{C}(x)}$ of $\mathbb{C}(x)$. Then $F$ is an algebraic function field over $\mathbb{C}$. Due to Corollary 3, there exists a constant $\tilde{C}$, depending only on $P$, such that for every algebraic function $y(x) \in F$, we have

$$\deg(u(x,y(x)))_\infty \leq \deg(v(x,y(x)))_\infty + \tilde{C}.$$

Or equivalently,

$$n \cdot \deg(y(x))_\infty \leq \deg(P(x,y(x)))_\infty + \tilde{C}. \tag{2}$$

If, furthermore, $y(x)$ is a solution of the differential equation (1), then the right hand side of the above inequality is equal to $\deg(\frac{dy(x)}{dx})_\infty + \tilde{C}$. By Lemma 1, it can be upper bounded by $4 \cdot \deg(y(x))_\infty + \tilde{C}$. Hence, from (2), we obtain

$$\deg(y(x))_\infty \leq \frac{\tilde{C}}{n-4}. \tag{3}$$

In case $y(x) \in \mathbb{C}(x)$, we have

$$\deg(y(x))_\infty = [F : \mathbb{C}(y(x))] = [F : \mathbb{C}(x)] \cdot [\mathbb{C}(x) : \mathbb{C}(y(x))] = 2 \deg y(x).$$

Otherwise, Lemma 1 yields $\deg(y(x))_\infty = \deg y(x)$. In any case, we can choose the constant $C = \frac{\tilde{C}}{n-4}$. □

**Remark 4.1.** In the proof of Theorem 4, the constant $\tilde{C}$ can be determined effectively (see [10]). Then so is the constant $C$. Theorem 4 provides an upper bound for the degree of the annihilating polynomial of an algebraic solution which belongs to a quadratic extension of $\mathbb{C}(x)$. As a consequence, there is an algorithm for computing all algebraic solutions lying in a quadratic extension of $\mathbb{C}(x)$ of the differential equation (1). Note that, an algorithm for determining all rational solutions of the given differential equation is provided in [10].

**Example 4.1.** Consider the differential equation

$$y' = -xy^5 + y^3. \tag{4}$$

We view $u = y^5$ and $v = -xy^5 + y^3$ as elements in the algebraic function field $\overline{\mathbb{C}(x)}(y)$ over $\overline{\mathbb{C}(x)}$. Then we see that they only have pole at infinity with the same order. By following the lines of the proof of [10, Theorem 4.3], we can choose $\tilde{C} = 2$. This means that for every quadratic extension $F \subset \overline{\mathbb{C}(x)}$ of $\mathbb{C}(x)$ and for every $y(x) \in F$, we always have

$$\deg(u(x, y(x)))_\infty \leq \deg(v(x, y(x)))_\infty + 2. \tag{5}$$

If furthermore $y(x)$ is a solution of the given differential equation, then we have

$$\deg(v(x, y(x)))_\infty = \deg(y'(x))_\infty \leq 4 \deg(y(x))_\infty.$$

By combining with (5), we obtain $\deg(y(x))_\infty \leq 2$. Hence, the minimal polynomial of $y(x)$ over $\mathbb{C}(x)$ must be a polynomial of degree at most 2 in $x$ and at most 2 in $y$. Using algorithms in [1], we find all algebraic solutions of the given differential equation which are lying in a quadratic extension of $\mathbb{C}(x)$. They are $y(x) = 0$ and $\pm\frac{1}{\sqrt{2x}}$.

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