Relationships between quantized algebras and their semi-classical limits

Sei-Qwon Oh[1,*]

[1]Department of Mathematics, Chungnam National University, 99 Daehak-ro, Yuseong-gu, Daejeon 34134, Korea

Abstract. A Poisson $\mathbb{C}$-algebra R appears in classical mechanical system and its quantized algebra appearing in quantum mechanical system is a $\mathbb{C}[[\hbar]]$-algebra $Q = R[[\hbar]]$ with star product $*$ such that for any $a, b \in R \subseteq Q$,

$$a * b = ab + B_1(a, b)\hbar + B_2(a, b)\hbar^2 + \dots$$

subject to

$$\{a, b\} = \hbar^{-1}(a * b - b * a)|_{\hbar=0}, \qquad \cdots \quad (**)$$

where $B_i : R \times R \longrightarrow R$ are bilinear products. The given Poisson algebra R is recovered from its quantized algebra Q by $R = Q/\hbar Q$ with Poisson bracket $(**)$, which is called its semiclassical limit. But it seems that the star product in Q is complicate and that Q is difficult to understand at an algebraic point of view since it is too big. For instance, if λ is a nonzero element of $\mathbb{C}$ then $\hbar - \lambda$ is a unit in Q and thus a so-called deformation of R, $Q/(\hbar - \lambda)Q$, is trivial. Hence it seems that we need an appropriate $\mathbb{F}$-subalgebra A of Q such that A contains all generators of Q, $\hbar \in A$ and A is understandable at an algebraic point of view, where $\mathbb{F}$ is a subring of $\mathbb{C}[[\hbar]]$.
Here we discuss how to find nontrivial deformations from quantized algebras and the natural map in [6] from a class of infinite deformations onto its semi-classical limit. The results are illustrated by examples.

1 Motivation and Quantization

1.1 Star product

A commutative $\mathbb{C}$-algebra R is said to be a *Poisson algebra* if there exists a bilinear product $\{-, -\} : R \times R \to R$, called a *Poisson bracket*, such that $(R, \{-, -\})$ is a Lie algebra satisfying Leibniz's rule $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in R$. A quantization of R is an associative $\mathbb{C}[[\hbar]]$-algebra $R[[\hbar]]$ equipped with a star product $* : R[[\hbar]] \times R[[\hbar]] \to R[[\hbar]]$ such that for all $a, b \in R$,

$$a * b = ab + B_1(a, b)\hbar + B_2(a, b)\hbar^2 + \dots$$

subject to

$$\{a, b\} = \hbar^{-1}(a * b - b * a)|_{\hbar=0}, \tag{1}$$

*e-mail: sqoh@cnu.ac.kr

where $B_i : R \times R \longrightarrow R$ are bilinear products. Denote by $Q = (R[[\hbar]], *)$. We can recover the Poisson algebra R from its quantization Q. That is, $\hbar Q$ is a nontrivial ideal such that $Q/\hbar Q \cong R$ as a commutative $\mathbb{C}$-algebra and the Poisson bracket $\{-, -\}$ in R is obtained by (1). But if λ is a nonzero element of $\mathbb{C}$ then $\hbar - \lambda$ is a unit in Q and thus $Q/(\hbar - \lambda)Q$ is trivial. Moreover, we do not know what $Q_\lambda = \{f|_{\hbar=\lambda} \mid f \in Q\}$ means mathematically. Let us see the following example.

1.2 Poisson Weyl algebra

The Poisson Weyl algebra is the $\mathbb{C}$-algebra $R = \mathbb{C}[x, y]$ with Poisson bracket

$$\{f, g\} = -\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \tag{2}$$

for all $f, g \in R$, namely $\{y, x\} = 1$.

Define a multiplication on the set $R[[\hbar]]$ of formal power series over $R = \mathbb{C}[x, y]$ by

$$yx = xy + \hbar$$

and $\mathbb{C}[[\hbar]]$ is central in $R[[\hbar]]$. Then $Q = R[[\hbar]]$ is an associative $\mathbb{C}[[\hbar]]$-algebra and $\hbar$ is a nonzero, nonunit, non-zero-divisor and central element such that $Q/\hbar Q$ is commutative. Hence $Q/\hbar Q \cong R$ is a Poisson $\mathbb{C}$-algebra with Poisson bracket

$$\{\overline{f}, \overline{g}\} := \overline{\hbar^{-1}(fg - gf)}$$

for all $f, g \in Q$. In particular, $\{\overline{y}, \overline{x}\} = 1$.

For $\lambda \in \mathbb{C}$, let Q_λ be the set of formal elements $f|_{\hbar=\lambda}$ for all $f \in Q$. For the case $\lambda = 0$, observe that $Q_0 = R$. For $0 \neq \lambda \in \mathbb{C}$ and $f = 1 + \hbar x + \hbar^2 + \hbar^3 + \ldots \in Q$, we do not know mathematical meaning of the formal form

$$f|_{\hbar=\lambda} = 1 + \lambda x + \lambda^2 + \lambda^3 + \ldots \in Q_\lambda.$$

In particular, we should observe that $Q/(\hbar - \lambda)Q = 0$, since $\hbar - \lambda$ is a unit in Q, and thus $Q_\lambda \neq Q/(\hbar - \lambda)Q$.

The *Weyl algebra* W is the $\mathbb{C}$-algebra generated by x, y subject to the relation

$$yx = xy + 1.$$

Let A be a $\mathbb{C}[\hbar]$-algebra of Q generated by x and y. Then A is a $\mathbb{C}[\hbar]$-algebra generated by x, y subject to the relation $yx - xy = \hbar$. Note that the element $\hbar \in A$ satisfies the following:

- $\hbar$ is a nonzero, nonunit, non-zero-divisor and central element.
- $A/\hbar A$ is commutative.
- $(\hbar - \lambda)A \neq A$ for all $0 \neq \lambda \in \mathbb{C}$.

Hence the bilinear product $\{-, -\}$ on $A/\hbar A \cong R$ defined by

$$\{\overline{f}, \overline{g}\} := \overline{\hbar^{-1}(fg - gf)} \quad (f, g \in A)$$

is well defined and thus $A/\hbar A$ is a Poisson algebra isomorphic to the Poisson Weyl algebra and for each $0 \neq \lambda \in \mathbb{C}$, $A/(\hbar - \lambda)A$ is the $\mathbb{C}$-algebra generated by x, y subject to the relation $yx = xy + \lambda$, which is isomorphic to the Weyl algebra W. In particular,

$$A_\lambda := \{f|_{\hbar=\lambda} \mid f \in A\}$$

makes sense mathematically and is isomorphic to $A/(\hbar - \lambda)A$ as a $\mathbb{C}$-algebra.

2 Deformation and semiclassical limit

2.1 Semiclassical limit

Let A be a $\mathbb{C}$-algebra. An element $\hbar \in A$ is said to be a *regular* element if it is a nonzero, nonunit, non-zero-divisor and central element such that $A/\hbar A$ is commutative.

Let $\hbar$ be a regular element. Then $0 \neq \hbar A$ is a proper ideal such that the factor $\overline{A} := A/\hbar A$ is a Poisson algebra with Poisson bracket

$$\{\overline{a}, \overline{b}\} = \overline{\hbar^{-1}(ab - ba)} \tag{3}$$

for $\overline{a}, \overline{b} \in A/\hbar A$. The Poisson algebra $\overline{A} = A/\hbar A$ is called a *semiclassical limit* of A.

Let R and Q be as in §1.1. Then $\hbar$ is a regular element of Q and the semiclassical limit $Q/\hbar Q$ is isomorphic to R as a Poisson algebra. In §1.2, $\hbar$ is a regular element of A and the semiclassical limit $A/\hbar A$ is isomorphic to the Poisson Weyl algebra $\mathbb{C}[x, y]$ as a Poisson algebra.

2.2 Deformations

Retain the notations in §2.1. Suppose that there is an element $0 \neq \lambda \in \mathbb{C}$ such that $\hbar - \lambda$ is a nonunit in A. Then $(\hbar - \lambda)A$ is a proper ideal of A and thus the factor $A_\lambda := A/(\hbar - \lambda)A$ is a nontrivial $\mathbb{C}$-algebra such that its multiplication is induced by that of A. The factor A_λ is called a *deformation* of $\overline{A}$. For instance, the algebra A_λ in §1.2 is a deformation of the Poisson Weyl algebra.

An algorithm to obtain a deformation is given as follows. Let $\mathbb{F}$ be a subring of $\mathbb{C}[[\hbar]]$ containing $\mathbb{C}[\hbar]$ and let A be an $\mathbb{F}$-algebra generated by $x_1, \ldots, x_n$ subject to relations $f_1, \ldots, f_r$. For $\lambda \in \mathbb{C}$, assume that $f_i|_{\hbar=\lambda}$ makes sense mathematically for each $i = 1, \ldots, r$. The $\mathbb{C}$-algebra generated by $x_1, \ldots, x_n$ subject to the relations $f_1|_{\hbar=\lambda}, \ldots, f_r|_{\hbar=\lambda}$, still denoted by A_λ, is deeply related to A. If $g|_{\hbar=\lambda}$ makes sense mathematically for all $g \in A$ then the evaluation map

$$\varphi_\lambda : A \longrightarrow A_\lambda, \quad g \mapsto g|_{\hbar=\lambda}$$

is a $\mathbb{C}$-algebra epimorphism and thus $A_\lambda \cong A/\ker\varphi_\lambda$ and the multiplication of A_λ is induced by that of A. If $\hbar$ is a regular element then A_λ is a deformation of $\overline{A}$. An example of this algorithm is given in §3.

2.3 Weyl algebra

As shown in §1.2, let A be the $\mathbb{C}[\hbar]$-subalgebra of $Q = R[[\hbar]]$ generated by x and y. Then $\hbar \in A$ is a regular element and its semiclassical limit $\overline{A} = A/\hbar A$ is the Poisson Weyl algebra $R = \mathbb{C}[x, y]$ with the Poisson bracket $\{y, x\} = 1$. For each $0 \neq \lambda \in \mathbb{C}$, deformation $A_\lambda = A/(\hbar - \lambda)A$ is the $\mathbb{C}$-algebra generated by x and y subject to the relation

$$yx - xy = \lambda.$$

Note that every deformation A_λ is isomorphic to the Weyl algebra W by [4, Proposition 3.4].

2.4 Another deformation of Poisson Weyl algebra

Here we recall [5, Example 3.8]. Let Q be a $\mathbb{C}[[\hbar]]$-algebra generated by x, y subject to the relation

$$yx = (\cos\hbar)xy + \sin\hbar.$$

Denote by A the $\mathbb{C}[\hbar]$-subalgebra of Q generated by $\cos\hbar, \sin\hbar$ and x, y. Consider the composition of two natural maps

$$\phi : A \longrightarrow Q \longrightarrow Q/\hbar Q \cong R = \mathbb{C}[x, y].$$

Then $\hbar$ is a regular element in A, since $\hbar$ is regular in Q, and $A/\ker\phi \cong \mathbb{C}[x, y]$ as a $\mathbb{C}$-algebra. Give a Poisson bracket in $A/\ker\phi$ by (3). Then the Poisson bracket in $A/\ker\phi$ is

$$\{\overline{y}, \overline{x}\} = -\overline{(\sin\hbar|_{\hbar=0})xy + \cos\hbar|_{\hbar=0}} = 1$$

and thus $A/\ker\phi \cong \mathbb{C}[x, y]$, the Poisson Weyl algebra as a Poisson $\mathbb{C}$-algebra.

For each $0 \neq \lambda \in \mathbb{C}$, let A_λ be the $\mathbb{C}$-algebra generated by $\cos\lambda, \sin\lambda, x, y$ subject to the relation

$$yx = (\cos\lambda)xy + \sin\lambda.$$

Then the evaluation map

$$\varphi_\lambda : A \longrightarrow A_\lambda, \quad g \mapsto g|_{\hbar=\lambda}$$

is an algebra ephimorphism and thus $A/\ker\varphi_\lambda \cong A_\lambda$. In particular, the $\mathbb{C}$-algebra A_π for the case $\lambda = \pi$ is generated by x, y subject to the relation

$$yx + xy = 0,$$

which can be considered as a deformation of the Poisson Weyl algebra.

2.5 Poisson affine 2-space

Let $\mathbb{F} = \mathbb{C}[\hbar, \hbar^{-1}]$ and let A be the $\mathbb{F}$-algebra generated by x, y subject to the relation

$$yx = \hbar xy.$$

Note that $\hbar - 1$ is a regular element in A and the semiclassical limit $\overline{A} = A/(\hbar - 1)$ is the Poisson algebra $\mathbb{C}[x, y]$ with Poisson bracket

$$\{y, x\} = xy.$$

Then for each $0, \pm 1 \neq q \in \mathbb{C}$, the deformation $A_q = A/(\hbar - q)A$ is the $\mathbb{C}$-algebra generated by x, y subject to the relation

$$yx = qxy$$

and thus A_q is the so-called coordinate ring $O_q(\mathbb{C}^2)$ of affine 2-space.

3 A natural map from a class of infinite deformations onto its semiclassical limit

We assume the following conditions (i)-(vii) in [6, Notation 1.1].

(i) Assume that $\mathbf{K}$ is an infinite subset of the set $\mathbb{C} \setminus \{0\}$.

(ii) Assume that $\mathbb{F}$ is a subring of the ring of regular functions on $\mathbf{K} \cup \{0\}$ containing $\mathbb{C}[\hbar]$. That is,

$$\mathbb{C}[\hbar] \subseteq \mathbb{F} \subseteq \{f/g \in \mathbb{C}(\hbar) | f, g \in \mathbb{C}[\hbar] \text{ such that } g|_{\hbar=\lambda} \neq 0 \ \forall \lambda \in \mathbf{K} \cup \{0\}\}. \tag{4}$$

(iii) Let $\mathbb{F}\langle x_1, \ldots, x_n\rangle$ be the free $\mathbb{F}$-algebra on the set $\{x_1, \ldots, x_n\}$. A finite product $\mathbf{x}$ of x_i's (repetitions allowed) is called a monomial. For each $i = 1, \ldots, r$, let f_i be an $\mathbb{F}$-linear combination of monomials

$$f_i = \sum_{\mathbf{x}} a^i_{\mathbf{x}}(\hbar)\mathbf{x}, \quad a^i_{\mathbf{x}}(\hbar) \in \mathbb{F}.$$

Set $A = \mathbb{F}\langle x_1, \ldots, x_n\rangle/I$, where I is the ideal of $\mathbb{F}\langle x_1, \ldots, x_n\rangle$ generated by $f_1, \ldots, f_r$. That is, A is the $\mathbb{F}$-algebra generated by $x_1, \ldots, x_n$ subject to the relations

$$f_1, \ldots, f_r.$$

(iv) Assume that $\hbar$ is a regular element. Hence there exists the semiclassical limit $A/\hbar A$. Denote by φ_0 the canonical projection

$$\varphi_0 : A \longrightarrow A/\hbar A, \quad f \mapsto f|_{\hbar=0}.$$

(v) For each $\lambda \in \mathbf{K}$, let A_λ be the $\mathbb{C}$-algebra generated by $x_1, \ldots, x_n$ subject to the relations

$$f_1|_{\hbar=\lambda}, \ldots, f_r|_{\hbar=\lambda}.$$

Note that $a^i_{\mathbf{x}}(\lambda)$ is a well-defined element of $\mathbb{C}$ by (4) and that the evaluation map

$$\varphi_\lambda : A \longrightarrow A_\lambda, \quad f \mapsto f|_{\hbar=\lambda}$$

is an epimorphism of $\mathbb{C}$-algebras. In other words, A_λ is a deformation of the Poisson algebra $A/\hbar A$.

(vi) Let $\prod_{\lambda\in\mathbf{K}} A_\lambda$ be the product of infinite deformations A_λ and let φ be the homomorphism of $\mathbb{C}$-algebras defined by

$$\varphi : A \longrightarrow \prod_{\lambda\in\mathbf{K}} A_\lambda, \quad \varphi(a) = (\varphi_\lambda(a))_{\lambda\in\mathbf{K}}. \tag{5}$$

Note that $\varphi(\hbar)$ is an invertible element of $\prod_{\lambda\in\mathbf{K}} A_\lambda$ since $0 \notin \mathbf{K}$.

(vii) Assume that there exists an $\mathbb{F}$-basis $\{\xi_i \mid i \in I\}$ of A such that $\{\varphi_0(\xi_i)|i \in I\}$ and $\{\varphi_\lambda(\xi_i)|i \in I\}$ are $\mathbb{C}$-bases of $A/\hbar A$ and A_λ, respectively, for each $\lambda \in \mathbf{K}$. Hence every element $a \in A$ is expressed uniquely by

$$a = \sum_i a_i(\hbar)\xi_i, \quad a_i(\hbar) \in \mathbb{F}$$

and, for each $\lambda \in \mathbf{K}$,

$$\varphi_\lambda(a) = \textstyle\sum_i a_i(\lambda)\varphi_\lambda(\xi_i), \quad \varphi_0(a) = \sum_i a_i(0)\varphi_0(\xi_i).$$

Note that $a_i(\lambda)$ and $a_i(0)$ are well-defined elements of $\mathbb{C}$ by (4).

(viii) Let $\widehat{q}$ be the parameter taking values in $\mathbf{K}$. That is, $\widehat{q}$ is a function defined by

$$\widehat{q} : \mathbf{K} \longrightarrow \mathbb{C}, \ \ \widehat{q}(\lambda) = \lambda.$$

(iv) Let $A_{\widehat{q}}$ be the $\mathbb{C}$-algebra obtained by replacing $\hbar$ in A by $\widehat{q}$ and let $\widehat{\ }$ be the map defined by

$$\widehat{\ } : A_{\widehat{q}} \longrightarrow \prod_{\lambda\in\mathbf{K}} A_\lambda, \quad f \mapsto \widehat{f} = (f|_{\widehat{q}=\lambda})_{\lambda\in\mathbf{K}}.$$

3.1 Theorem [Oh, 2017]

Here we write a natural map from a class of infinite deformations onto its semiclassical limit in [6, §1]. Set $\widehat{A} = \widehat{\ }^{-1}(\varphi(A)) \subset A_{\widehat{q}}$, the inverse image of $\varphi(A)$ by the map $\widehat{\ }$. Since φ is a monomorphism of $\mathbb{C}$-algebras by [6, Lemma 1.2], there exists the composition

$$\gamma : \widehat{A} \longrightarrow \varphi(A) \left(\subseteq \prod_{\lambda\in\mathbf{K}} A_\lambda \right) \longrightarrow A \longrightarrow A/\hbar A, \quad f \mapsto (\varphi_0 \circ \varphi^{-1})(\widehat{f}), \tag{6}$$

which is an epimorphism of $\mathbb{C}$-algebras such that $\gamma(\widehat{q}) = 0$ and $\gamma(x_i) = \varphi_0(x_i)$ for $i = 1, \ldots, r$.

Here we write applications of the map γ in (6).

• One should observe [6, Example 1.6] in which the map γ in (6) induces a homeomorphism between the primitive spectrum of the coordinate ring $O_q(\mathbb{C}^2)$ of quantized affine 2-space and the Poisson primitive spectrum of its semiclassical limit $O(\mathbb{C}^2)$.

• One should observe [4, Theorem 4.2] in which the map γ in (6) induces a monomorphism from the group of automorphisms of Weyl algebra into the group of Poisson automorphisms of Poisson Weyl algebra.

Acknowledgment The author is supported by National Research Foundation of Korea, NRF-2017R1A2B4008388.

References

[1] Eun-Hee Cho and Sei-Qwon Oh, *Semiclassical limits of Ore extensions and a Poisson generalized Weyl algebra*, Lett. Math. Phys. **106** (2016), no. 7, 997–1009.

[2] K. R. Goodearl, *Semiclassical limits of quantized coordinate rings*, in Advances in Ring Theory (D. V. Huynh and S. R. Lopez-Permouth, Eds.) Basel Birkhäuser (2009), 165–204.

[3] M. Kontsevich, *Deformation quantization of Poisson manifolds*, arXiv:q-alg/9709040.

[4] No-Ho Myung and Sei-Qwon Oh, *Automorphism groups of Weyl algebras*, arXiv: 1710.00432v2, (2018)

[5] No-Ho Myung and Sei-Qwon Oh, *A construction of an iterated Ore extension*, arXiv: 1707.05160v2, (2018)

[6] Sei-Qwon Oh, *A natural map from a quantized space onto its semiclassical limit and a multi-parameter Poisson Weyl algebra*, Comm. Algebra **45** (2017), 60–75.