

A numerical scheme for problems in fractional calculus

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Abstract. In this paper, a new numerical method for solving the fractional differential equations with boundary value problems is presented. The method is based upon hybrid functions approximation. The properties of hybrid functions consisting of block-pulse functions and Bernoulli polynomials are presented. The Riemann-Liouville fractional integral operator for hybrid functions is given. This operator is then utilized to reduce the solution of the boundary value problems for fractional differential equations to a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Hybrid functions; fractional-order differential equations; block-pulse; Bernoulli polynomials; Caputo derivative; numerical solution

1 Introduction

Fractional differential equations (FDEs) are generalized from integer order ones, which are obtained by replacing integer order derivatives with fractional ones. A history of the development of fractional differential operators can be found in [1, 2].

FDEs have attracted considerable interest because of their ability to model complex phenomena such as continuum and statistical mechanics [3] visco-elastic materials [4], solid mechanics [5] and dynamics of interfaces between soft-nanoparticles and rough substrates [6]. In general, most of the FDEs do not have exact solutions. In particular, there is no known method for solving fractional boundary value problems exactly [7].

Due to the extensive applications of FDEs in engineering and science, research in this area has grown significantly, and there has been considerable interest in developing numerical schemes for their solution. These methods include Fourier transforms [8], eigenvector expansion [9], Laplace transforms [10], Adomian decomposition method [11], variational iteration method [12], finite difference method (FDM) [13], the power series method [14], fractional differential transform method (FDTM) [15], and homotopy analysis method [16].

Most of the work published to date concerning exact and numerical solutions for FDEs is devoted to the initial value problems. The theory of boundary value problems for FDEs has received attention quite recently [7].

The available sets of orthogonal functions can be divided into three classes. The first class includes sets of piecewise constant basis functions (e.g., block-pulse, Haar, Walsh, etc.). The second class consists of sets of orthogonal polynomials (e.g., Chebyshev, Laguerre, Legendre, etc.). The third class is the set of sine-cosine functions in the Fourier series. Orthogonal

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functions have been used when dealing with various problems of the dynamical systems. The main advantage, of using orthogonal functions is that they reduce the dynamical system problems to those of solving a system of algebraic equations by using the operational matrix of integration.

In recent years, the hybrid functions consisting of the combination of block-pulse functions with Legendre polynomials, Chebyshev polynomials, Taylor series, Lagrange polynomials or Bernoulli polynomials [17–22] have been shown to be a mathematical power tool for discretization of selected problems. Among these hybrid functions, the hybrid functions of block-pulse and Bernoulli polynomials have been shown to be computationally more effective [21–23]. In 2015, to the best of our knowledge, the Riemann-Liouville fractional integral operator for hybrid of block-pulse functions and Bernoulli polynomials was derived directly for the first time [24,25].

In the present paper, a new numerical method for solving the initial and boundary value problems for fractional order differential equations is presented. The method is based upon Hybrid functions approximation. These hybrid functions, which consist of the hybrid of block-pulse and Bernoulli polynomials are given. We then obtain the Riemann-Liouville fractional integral operator for hybrid of block-pulse functions and Bernoulli polynomials. This operator is then utilized to reduce the solution of the fractional order differential equations to the solution of algebraic equations.

2 Preliminaries and notations

2.1 The fractional derivative and integral

There are various definitions of fractional derivative and integration. The widely used definition of a fractional derivative is the Caputo definition and a fractional integration is the Riemann-Liouville definition.

Definition 1. Caputo’s fractional derivative of order q is defined as [10]

$$(D^q f)(t) = \frac{1}{\Gamma(n - q)} \int_0^t \frac{f^{(n)}(s)}{(t - s)^{q+1-n}} ds, \quad n - 1 < q \leq n, n \in \mathbb{N},$$

where $q > 0$ is the order of the derivative and n is the smallest integer greater than q .

The Caputo fractional derivative is a linear operation, namely

$$D^q(\lambda f(t) + \mu g(t)) = \lambda D^q f(t) + \mu D^q g(t),$$

where λ and μ are constants.

Definition 2. The Riemann-Liouville fractional integral operator of order q is defined as [10]

$$I^q f(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds = \frac{1}{\Gamma(q)} t^{q-1} * f(t), & q > 0, \\ f(t), & q = 0. \end{cases} \quad (1)$$

where $t^{q-1} * f(t)$ is the convolution product of t^{q-1} and $f(t)$. For the Riemann-Liouville fractional integral we have

$$I^q t^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1 + q)} t^{\nu+q}, \quad \nu > -1. \quad (2)$$

The Riemann-Liouville fractional integral is also a linear operation.

The Caputo derivative and Riemann-Liouville integral satisfy the following properties [10]

$$I^\alpha(D^\alpha h(t)) = h(t) - \sum_{k=0}^{n-1} h^k(0) \frac{t^k}{k!}, \tag{3}$$

if $\alpha \in R, n - 1 < \alpha \leq n, n \in N$ then

$$D^\alpha(h(t)) = I^{n-\alpha} D^n h(t). \tag{4}$$

3 Hybrid of block-pulse functions and Bernoulli polynomials

Hybrid functions $b_{nm}(t), n = 1, 2, \dots, N, m = 0, 1, \dots, M$ are defined on the interval $[0, t_f]$ as [24]

$$b_{nm}(t) = \begin{cases} \beta_m(\frac{N}{t_f}t - n + 1), & t \in [\frac{n-1}{N}t_f, \frac{n}{N}t_f), \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where n and m are the order of block-pulse functions and Bernoulli polynomials, respectively. In Eq. (5), $\beta_m(t), m = 0, 1, 2, \dots$ are the Bernoulli polynomials of order m , which can be defined by [26]

$$\beta_m(t) = \sum_{k=0}^m \binom{m}{k} \alpha_{m-k} t^k, \tag{6}$$

where $\alpha_k, k = 0, 1, \dots, m$ are Bernoulli numbers [27]. These polynomials satisfy the following formula [27]

$$\beta_m(1 - x) = (-1)^m \beta_m(x). \tag{7}$$

3.1 Function Approximation

Let $f \in L^2[0, 1]$, the best approximation of f by using hybrid functions is given by

$$f \simeq P_M^N f = \sum_{m=0}^M \sum_{n=1}^N c_{nm} b_{nm}(t) = C^T B(t), \tag{8}$$

where

$$C^T = [c_{10}, c_{20}, \dots, c_{N0}, c_{11}, c_{21}, \dots, c_{N1}, \dots, c_{1M}, c_{2M}, \dots, c_{NM}], \tag{9}$$

and

$$B^T(t) = [b_{10}(t), b_{20}(t), \dots, b_{N0}(t), b_{11}(t), b_{21}(t), \dots, b_{N1}(t), \dots, b_{1M}(t), b_{2M}(t), \dots, b_{NM}(t)]. \tag{10}$$

4 Riemann-Liouville fractional integral operator for hybrid of block-pulse functions and Bernoulli polynomials

We now derive the operator I^α for $B(t)$ in Eq. (10) given by

$$I^\alpha B(t) = \bar{B}(t, \alpha), \tag{11}$$

where

$$\bar{B}(t, \alpha) = [I^\alpha b_{10}(t), \dots, I^\alpha b_{N0}(t), I^\alpha b_{11}(t), \dots, I^\alpha b_{N1}(t), \dots, I^\alpha b_{1M}(t), I^\alpha b_{2M}(t), \dots, I^\alpha b_{NM}(t)]^T. \tag{12}$$

To obtain $I^\alpha b_{nm}(t)$, we use the Laplace transform. By using Eq. (5), we have

$$b_{nm}(t) = \mu_{\frac{n-1}{N}}(t)\beta_m(Nt - n + 1) - \mu_{\frac{n}{N}}(t)\beta_m(Nt - n + 1), \tag{13}$$

where $\mu_c(t)$ is unit step function defined as

$$\mu_c(t) = \begin{cases} 1 & t \geq c, \\ 0 & t < c. \end{cases}$$

By taking the Laplace transform from Eq. (13) and using Eq. (7), we get

$$\begin{aligned} L[b_{nm}(t)] &= e^{-\frac{n-1}{N}s}L[\beta_m(N(t + \frac{n-1}{N}) - n + 1)] - e^{-\frac{n}{N}s}L[\beta_m(N(t + \frac{n}{N}) - n + 1)], \\ &= e^{-\frac{n-1}{N}s}L[\beta_m(Nt)] - (-1)^m e^{-\frac{n}{N}s}L[\beta_m(-Nt)]. \end{aligned} \tag{14}$$

From the definition of Bernoulli polynomials in Eq. (6), we have

$$\begin{aligned} L[b_{nm}(t)] &= e^{-\frac{n-1}{N}s}L[\sum_{k=0}^m \binom{m}{k} \alpha_{m-k} N^k t^k] - (-1)^m e^{-\frac{n}{N}s}L[\sum_{k=0}^m \binom{m}{k} \alpha_{m-k} (-N)^k t^k] \\ &= e^{-\frac{n-1}{N}s} \sum_{k=0}^m \binom{m}{k} \alpha_{m-k} N^k \frac{\Gamma(k+1)}{s^{k+1}} - (-1)^m e^{-\frac{n}{N}s} \sum_{k=0}^m \alpha_{m-k} \binom{m}{k} (-N)^k \frac{\Gamma(k+1)}{s^{k+1}}. \end{aligned} \tag{15}$$

From Eq. (1),

$$\begin{aligned} L[I^\alpha b_{nm}(t)] &= L[\frac{1}{\Gamma(\alpha)t^{1-\alpha}} * b_{nm}(t)] = \\ &= e^{-\frac{n-1}{N}s} \sum_{k=0}^m \binom{m}{k} \alpha_{m-k} N^k \frac{\Gamma(k+1)}{s^{k+1+\alpha}} - (-1)^m e^{-\frac{n}{N}s} \sum_{k=0}^m \binom{m}{k} \alpha_{m-k} (-N)^k \frac{\Gamma(k+1)}{s^{k+1+\alpha}}. \end{aligned} \tag{16}$$

Taking the inverse Laplace transform of Eq. (16) yields [24,25]

$$I^\alpha b_{nm}(t) = \begin{cases} 0, & t \in (-\infty, \frac{n-1}{N}), \\ (t - \frac{n-1}{N})^\alpha d_{nm}(t) & t \in [\frac{n-1}{N}, \frac{n}{N}), \\ (t - \frac{n-1}{N})^\alpha d_{nm}(t) - (-1)^m (t - \frac{n}{N})^\alpha \bar{d}_{n+1m}(t) & t \in [\frac{n}{N}, \infty), \end{cases} \tag{17}$$

where

$$d_{nm}(t) = \sum_{k=0}^m \binom{m}{k} N^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} (t - \frac{n-1}{N})^k,$$

$$\bar{d}_{nm}(t) = \sum_{k=0}^m (-1)^k \binom{m}{k} N^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} (t - \frac{n}{N})^k.$$

5 Problem statement

We focus on the following problem:

Caputo fractional differential equation

$$D^\alpha f(t) = h(t, f(t), D^\beta f(t)), \quad 1 < \alpha \leq 2, \quad 0 \leq \beta \leq 1, \quad (18)$$

with the boundary conditions

$$f(0) = f_0, \quad f(1) = f_1. \quad (19)$$

For this problem we have:

Lemma 1. Assume that $h : [0, 1] \times R \times R \rightarrow R$ is continuous. Then, $f \in C[0, 1]$ is a solution of the boundary value problem in Eq. (18) and Eq. (19) if, and only if, $f(t)$ is the solution of [28]

$$f(t) = I^\alpha h(t, f(t), D^\beta f(t)) - tI^\alpha h(1, f(1), D^\beta f(1)) + (f_1 - f_0)t + f_0 \quad (20)$$

$$= \int_0^1 G(t, s)h(s, f(s), D^\beta f(s))ds + (f_1 - f_0)t + f_0,$$

where $G(t, s)$ is the Green function given by

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1} - t(1-s)^{\alpha-1}, & 0 \leq s \leq t, \\ -t(1-s)^{\alpha-1}, & t < s \leq 1. \end{cases}$$

The existence and uniqueness results for this problem are given in [28].

6 Numerical method

In this section, we use the hybrid of block-pulse functions and Bernoulli polynomials for solving problem given in Eqs. (18) and (19). Here, we expand $f(t)$ by the hybrid functions as

$$f(t) = A^T B(t), \quad (21)$$

by using Eq. (11), we get

$$I^{\alpha-\beta} f(t) = A^T \bar{B}(t, \alpha - \beta) \quad (22)$$

Substituting Eqs. (21) and (22) in Eq. (20), collocating the resulting equation at the Newton-cotes nodes t_i , given by

$$t_i = \frac{i+1}{2N(M+1)}, \quad i = 0, 1, \dots, 2N(M+1) - 2. \quad (23)$$

These equations give $N(M+1)$ algebraic equations, which can be solved for the unknown vector A^T using Newton's iterative method.

7 Illustrative Example

In this section, two examples are given to demonstrate the applicability and accuracy of our method.

7.1 Example 1

Consider the boundary value problem for inhomogeneous linear fractional differential equation [7]

$$D^\alpha f(t) + af(t) = g(t), \quad t \in [0, 1], \tag{24}$$

$$f(0) = 0, \quad f(1) = \frac{1}{\Gamma(\alpha + 2)}, \tag{25}$$

where $1 < \alpha \leq 2$, $a \in R$. For $g(t) = t + \frac{at^\alpha}{\Gamma(\alpha+2)}$, the exact solution of this problem is $f(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$.

Here, we solve this problem by using hybrid functions with $N = 1$ and $M = 1$. Let

$$D^\alpha f(t) = A^T B(t) = [a_{10}, a_{11}] \begin{bmatrix} b_{10}(t) \\ b_{11}(t) \end{bmatrix}. \tag{26}$$

Then, by using Eqs. (11) and (26), we have

$$f(t) = A^T \bar{B}(t, \alpha) + f'(0)t, \tag{27}$$

by substituting Eqs. (26) and (27) in Eq. (24), we get

$$A^T B(t) + a(A^T \bar{B}(t, \alpha) + f'(0)t) = g(t) \tag{28}$$

By collocating Eq. (28) at the Newton-cotes nodes given in Eq. (23), we get $N(M + 1) + 1$ unknown with $N(M + 1)$ equations. Also, by using Eqs. (25) and (27), we have

$$f(1) = A^T \bar{B}(1, \alpha) + f'(0), \tag{29}$$

From Eqs. (28) and (29), we get $a_{10} = 0.5$, $a_{11} = 1$, and $f'(0) = 0$. Then, by using Eq. (27), we have the exact solution.

7.2 Example 2

Consider boundary value problems for a class of fractional differential equations with three-point boundary conditions [7],

$$D^\alpha f(t) + af(t) = g(t), \quad t \in [0, 1], \tag{30}$$

$$f(0) = 0, \quad f(1) = \xi f(\eta),$$

where $\eta \in (0, 1)$, $a, \xi \in R$ and $\Delta := 1 - \xi\eta \neq 0$.

Integrating Eq. (30) and using the boundary conditions, we have the following integral representation

$$f(t) = -aI^\alpha f(t) - \frac{at^{\alpha-1}}{\Delta} (\xi I^\alpha f(\eta) - I^\alpha f(1)) + F(t), \tag{31}$$

where

$$F(t) = I^\alpha g(t) + \frac{at^{\alpha-1}}{\Delta} (\xi I^\alpha g(\eta) - I^\alpha g(1)).$$

We solve this problem by using hybrid functions with $N = 1$ and $M = 5$. Let

$$f(t) = A_1^T B(t), \quad I^\alpha f(\eta) = A_2^T B(t), \quad I^\alpha f(1) = A_3^T B(t), \quad F(t) = K^T B(t), \tag{32}$$

then by using Eq. (11), we have

$$I^\alpha f(t) = A_1^T \bar{B}(t, \alpha). \tag{33}$$

Substituting Eqs. (32) and (33) in Eq. (31), we get

$$A_1^T B(t) = -aA_1^T \bar{B}(t, \alpha) - \frac{at^{\alpha-1}}{\Delta} (\xi A_2^T B(t) - A_3^T B(t)) + K^T B(t). \tag{34}$$

By collocating Eq. (34) at the Newton-cotes nodes given in Eq. (23), we get A_1 . Taking

$$\alpha = \frac{3}{2}, a = \frac{e^{-3\pi}}{\sqrt{\pi}}, \xi = -\frac{125}{196}, \eta = \frac{2}{5}, f(1) = -\frac{1}{40}$$

and

$$g(t) = \frac{e^{-3\pi}}{\sqrt{\pi}} (t^5 - \frac{37}{20}t^3 + \frac{33}{40}t^2) + \frac{1}{\sqrt{\pi}} (\frac{128}{7}t^{\frac{7}{2}} - \frac{74}{5}t^{\frac{3}{2}} + \frac{33}{10}t^{\frac{1}{2}}),$$

the exact solution is

$$f(t) = (t^3 - \frac{37t}{20} + \frac{33}{40})t^2.$$

For these values, the exact solution is obtained by the present method.

8 Conclusion

A general formulation for the Riemann-Liouville fractional integral operator for hybrid of block-pulse functions and Bernoulli polynomials has been given. This operator is used to approximate numerical solution of FDEs. Our numerical findings are compared with exact solutions and with the solutions obtained by some other numerical methods. The solution obtained using the present method shows that this approach can solve the problem effectively.

References

- [1] Oldham K.B, Spanier J. The Fractional Calculus, Academic Press: New York. 1974.
- [2] Miller K.S, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley: New York. 1993.
- [3] Mainardi F. Fractional calculus: Some basic problems in continuum and statistical mechanics', in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag: New York. 1997; 291–348.
- [4] Bagley R.L, Torvik P.J. Fractional calculus in the transient analysis of viscoelastically damped structures, AIAA J. 1985; 23: 918–925.
- [5] Rossikhin Y.A, Shitikova M.V. Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids, Appl.Mech. Rev. 1997; 50: 15–67.
- [6] Chow T.S. Fractional dynamics of interfaces between soft-nanoparticles and rough substrates, Phys. Lett. A. 2005; 342: 148–155.
- [7] Rehman M.U, Ali khan R. A numerical method for solving boundary value problems for fractional differential equations, Appl. Math. Model. 2012; 36: 894–907.
- [8] Gaul L, Klein P, Kemple S. Damping description involving fractional operators, Mech. Syst. Signal. Pr. 1991; 5: 81–88.

- [9] Suarez L, Shokooh A. An eigenvector expansion method for the solution of motion containing fractional derivatives, *J. Appl. Mech.* 1997; 64: 629–735.
- [10] Podlubny I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Academic Press, New York. 1998.
- [11] Momani S, Al-Khaled K. Numerical solutions for systems of fractional differential equations by the decomposition method, *Appl. Math. Comput.* 2005; 162: 1351–1365.
- [12] Odibat Z, Momani S. Application of variational iteration method to nonlinear differential equations of fractional order, *Int. J. Nonlinear Sci. Numer. Simul.* 2006; 7: 27–34.
- [13] Meerschaert M, Tadjeran C. Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.* 2006; 56: 80–90.
- [14] Odibat Z, Shawagfeh N. Generalized Taylor's formula, *Appl. Math. Comput.* 2007; 186: 286–293.
- [15] Arikoglu A, Ozkol I. Solution of fractional integro-differential equations by using fractional differential transform method, *Chaos Solitons Fract* 2009; 40: 521–529.
- [16] Hashim I, Abdulaziz O, Momani S. Homotopy analysis method for fractional IVPs, *Commun. Nonlinear Sci. Numer. Simul.* 2009; 14: 674–684.
- [17] Razzaghi M, Marzban H. Hybrid analysis direct method in the calculus of variations, *Int. J. Comput. Math.* 2000; 75: 259–269.
- [18] Razzaghi M, Marzban H. Direct method for variational problems via hybrid of block-pulse and Chebyshev functions, *Math. Prob. Eng.* 2000; 6: 85-97.
- [19] Marzban H, Razzaghi M. Solution of multi-delay systems using hybrid of block-pulse functions and Taylor series, *J. Soun. Vibr.* 2006; 292: 954–963.
- [20] Tabrizidooz H.R, Marzban H, Razzaghi M. A composite collocation method for the nonlinear mixed Volterra-Fredholm-Hammerstein integral equations, *Commun. Nonlin. Sci. Numeric. Simul.* 2011; 16: 1186–1194.
- [21] Haddadi N, Ordokhani Y, Razzaghi M. Optimal Control of Delay Systems by Using a Hybrid Functions Approximation, *J. Optimi. Theo. Appli.* 2012; 153: 338–356.
- [22] Mashayekhi S, Ordokhani Y, Razzaghi M. Hybrid functions approach for nonlinear constrained optimal control problems, *Commun. Nonlin. Sci. Numer. Simulat.* 2012; 17: 1831–1843.
- [23] Mashayekhi S, Ordokhani Y, Razzaghi M. Hybrid functions approach for optimal control of systems described by integro-differential equations, *Appl. Math. Model.* 2013; 37: 3355–3368.
- [24] Mashayekhi, S., and Razzaghi, M., 2015, "Numerical solution of nonlinear fractional integro-differential equations by hybrid functions," *Eng. Analysis Bound. Elem.*, 56, pp. 81–89.
- [25] Mashayekhi, S., and Razzaghi, M., 2016, "Numerical solution of distributed order fractional differential equations by hybrid functions," *J. Comput. Physics.*, 315, pp. 169–181.
- [26] Costabile F, Dellaccio F, Gualtieri M.I. A new approach to Bernoulli polynomials, *Rendiconti di Matematica, Serie VII.* 2006; 26: 1–12.
- [27] Arfken G. *Mathematical Methods for Physicists*, Third edition, Academic Press: San Diego. 1985.
- [28] Mujeeb U.R, Ali Rahmat K. The Legendre wavelet method for solving fractional differential equations, *Commun Nonlinear Sci. Numer. Simulat.* 2011; 16: 4163–4173.