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Positive solutions for a one-dimensional Sturm-Liouville semipositone superlinear p-Laplacian problem

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Abstract. We prove the existence of a positive classical solution for the p-Laplacian equation

$$-(r(t)\phi(u'))' = -\lambda h(u) + f(t,u),\ t \in (0,1)$$

with Sturm-Liouville boundary conditions, where $\phi(s) = |s|^{p-2}s, p > 1, r : [0,1] \to (0,\infty), f : (0,1) \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function satisfying a superlinear condition at 0 and ∞ involving the principal eigenvalue of $-(r(t)\phi(u'))'$, $h : (0,\infty) \to (0,\infty)$ is allowed to have infinite semipositone structure at 0, and $\lambda \geq 0$ is a small parameter.

1 Introduction

Consider the one-dimensional p-Laplacian problem

$$\begin{cases} -(r(t)\phi(u'))' = -\lambda g(u) + f(t,u),\ t \in (0,1), \\ au(0) - b\phi^{-1}(r(0))u'(0) = 0,\ cu(1) + d\phi^{-1}(r(1))u'(1) = 0, \end{cases} \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s, p > 1, a, b, c, d$ are nonnegative constants with $\delta < 1, ac + ad + bc > 0,\ r : [0,1] \to (0,\infty),\ f : (0,1) \times [0,\infty) \to \mathbb{R}, g : (0,\infty) \to (0,\infty)$, and λ is a nonnegative parameter.

We are interested in positive classical solution of (1.1) in the superlinear case involving with the principal eigenvalue of the corresponding operator.

When $\lambda = 0$ and $r \equiv 1$, Manásevich, Njoku, and Zanolin [14] used time-mapping estimates to prove the existence of a classical positive solution to (1.1) under Dirichlet boundary conditions when $f(t,z)$ satisfies

$$\limsup_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} < \lambda_1 < \liminf_{z\to\infty} \frac{f(t,z)}{z^{p-1}} \leq \infty \tag{1.2}$$

and $\liminf_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} > -\infty$ uniformly for a.e. $t \in (0,1)$, where $\lambda_1 = 2^p(p-1)\left(\int_0^1 \frac{ds}{(1-s^p)^{1/p}}\right)^p$ is the principal eigenvalue of $-(\phi(u'))'$ with zero boundary conditions (see [6,7]). Their result improved a previous work by Kaper, Knapp, and Kwong [12] where the stronger condition

$$\lim_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} = l \leq 0 \text{ and } \lim_{z\to\infty} \frac{f(t,z)}{z^{p-1}} = \infty$$

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uniformly for $t \in (0, 1)$ had been used. Related results were obtained by Webb and Lan [15] when $p = 2$, in which they gave a general method that covered many boundary conditions including nonlocal ones for both sublinear and superlinear cases involving with the principal eigenvalue of an operator. In the PDE case, the existence of a positive solution to the problem

$$-\Delta u = f(u) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega$$

with $f \geq 0$ satisfying (1.2) was obtained in [10], while similar results for the p-Laplacian

$$-\Delta_p u = f(u) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega$$

with $p > 1$ were obtained in [5] using Granas fixed point index [8]. The existence result in [14] was generalized to the Sturm-Liouville boundary value problem (1.1) with $\lambda = 0$ in [3] and to the singular problem (1.1) with $\lambda \geq 0$ small and $g(u) = u^{-\delta}$ for some $\delta \in [0, 1)$ under Dirichlet boundary condition in [4]. In this note, we shall extend the result in [4] to the Sturm-Liouville boundary condition with a more general $g(u)$ allowing the challenging infinite semipositone case i.e. $\lim_{u \to 0^+} g(u) = -\infty$ (see [13]). We refer to [9,11,16] and the references therein for the literature on existence results in the superlinear/sublinear cases not involving with the principal eigenvalue of the corresponding operator.

Let λ_1 be the principal eigenvalue of $-(r(t)\phi(u'))'$ on $(0, 1)$ with Sturm-Liouville boundary condition in (1.1) and let $\phi_1 > 0$ be the corresponding eigenfunction with $\|\phi_1\|_\infty = 1$ (see [2]).

We shall assume the following conditions:

(A1) $r : [0, 1] \to (0, \infty)$ is continuous and $g : (0, \infty) \to (0, \infty)$ is continuous, integrable on $(0, 1)$ and decreasing.

(A2) $f : (0, 1) \times [0, \infty)$ is a Carathéodory function, that is $f(., z)$ is measurable for each $z \geq 0$ and $f(t, .)$ is continuous for a.e. $t \in (0, 1)$.

(A3) For each $k > 0$, there exists $\gamma_k \in L^1(0, 1)$ such that

$$|f(t, z)| \leq \gamma_k(t)$$

for a.e. $t \in (0, 1)$ and all $z \in [0, k]$.

(A4) There exists $\gamma \in L^1(0, 1)$ with $\gamma \geq 0$ such that

$$f(t, z) + \gamma(t)z^{p-1} \geq 0$$

for a.e. $t \in (0, 1)$ and all $z \geq 0$.

(A5) There exists a subset $B \subset [0, 1]$ of measure 0 such that

$$\limsup_{z \to 0^+} \frac{f(t, z)}{z^{p-1}} < \lambda_1 < \liminf_{z \to \infty} \frac{f(t, z)}{z^{p-1}}$$

uniformly for $t \in [0, 1] \backslash B$.

By a positive classical solution of (1.1), we mean a function $u \in C^1[0, 1]$ with $u > 0$ on $(0, 1)$, $r(t)\phi(u')$ absolutely continuous on $[0, 1]$ and satisfying (1.1).

Our main result is

Theorem 1.1. *Let (A1)-(A5) hold. Then there exists a constant* $\lambda_0 > 0$ *such that for* $\lambda < \lambda_0$*,(1.1) has a positive classical solution u with* $\inf_{(0,1)} \frac{u}{\omega} > 0$*, where* $\omega(t) = \min(at+b, d+c(1-t))$.

In particular, when f is independent of t, we obtain

Corollary 1.1. *Let r satisfy (A1) and let $f : [0,\infty) \to R$ be continuous with*

$$-\infty < \lim_{z\to 0^+} \frac{f(z)}{z^{p-1}} < \lambda_1 < \lim_{z\to\infty} \frac{f(z)}{z^{p-1}} \le \infty.$$

Then there exists a constant $\lambda_0 > 0$ such that for $\lambda < \lambda_0$, (1.1) has a positive classical solution u with $\inf_{(0,1)} \frac{u}{p} > 0$

2. Preliminary results

We shall denote the norm in $L^q(0,1)$ and $C^1[0,1]$ by $||.||_q$ and $|.|_{C^1}$ respectively. We recall the following fixed point theorem of Krasnoselskii type in a Banach space (see Amann [1, Theorem 12.3]).

Theorem A. [1, Theorem 12.3] *Let E be a Banach space and $A : E \to E$ be a completely continuous operator. Suppose there exist $h \in E, h \neq 0$ and positive constants r, R with $r \neq R$ such that*
(a) If $y \in E$ satisfies $y = \theta Ay$ for some $\theta \in (0,1]$ then $||y|| \neq r$,
(b) If $y \in E$ satisfies $y = Ay + \xi h$ for some $\xi \ge 0$ then $||y|| \neq R$.
Then A has a fixed point $y \in E$ with $\min(r,R) < ||y|| < \max(r,R)$.

Lemma 2.1. *Let $\gamma, h \in L^1(0,1)$ with $\gamma, h \ge 0$ and let $\lambda > 0, u \in C^1[0,1]$ satisfy*

$$\begin{cases} -(r(t)(\phi(u'))' + \gamma(t)\phi(u) \ge -\lambda h \ \text{ on } (0,1), \\ au(0) - b\phi^{-1}(r(0))u'(0) \ge 0, \ cu(1) + d\phi^{-1}(r(1))u'(1) \ge 0. \end{cases}$$

Then there exist constants $\kappa, \sigma > 0$ independent of u, λ, h such that

$$u(t) \ge \left(\kappa ||u||_\infty - \sigma (\lambda ||h||_1)^{\frac{1}{p-1}}\right) p(t)$$

for $t \in [0,1]$.

Proof. Let $v \in C^1[0,1]$ satisfy

$$\begin{cases} -(r(t)(\phi(v'))' + \gamma(t)\phi(v) = -\lambda h \ \text{ on } (0,1), \\ av(0) - b\phi^{-1}(r(0))v'(0) = 0, \ cv(1) + d\phi^{-1}(r(1))v'(1) = 0. \end{cases}$$

By [3, Lemma 2.4], $v \le 0$ and $|v|_{C^1} \le M (\lambda ||h||_1)^{\frac{1}{p-1}}$, where $M > 0$ is independent of u and λ. By [3, Lemma 2.5], $u \ge v$ on $[0,1]$. Since there exists a constant $k_0 > 0$ such that $|v(t)| \le k_0 |v|_{C^1} \omega(t)$ for all $t \in [0,1]$ and $v \in C^1[0,1]$ satisfying the Sturm-Liouville boundary conditions in (1.1), it follows that $v(t) \ge -Mk_0 (\lambda ||h||_1)^{\frac{1}{p-1}} \omega(t)$ for $t \in [0,1]$. If $||u||_\infty \le C (\lambda ||h||_1)^{\frac{1}{p-1}}$ for some $C > 0$ then

$$u(t) \ge -Mk_0 (\lambda ||h||_1)^{\frac{1}{p-1}} \omega(t) \ge \left(||u||_\infty - (C + Mk_0) (\lambda ||h||_1)^{\frac{1}{p-1}}\right) \omega(t) \tag{2.1}$$

for $t \in [0,1$. Suppose $||u||_\infty > C (\lambda ||h||_1)^{\frac{1}{p-1}}$, where $C > M$. Let $||u||_\infty = |u(\tau)|$ for some $\tau \in [0,1]$. Then $u(\tau) > 0$ for otherwise $||u||_\infty = -u(\tau) \le -v(\tau) \le M (\lambda ||h||_1)^{\frac{1}{p-1}}$, a contradiction.

Let $z \in C^1[0,\tau]$ be the solution of

$$\begin{cases} -(r(t)\phi(z'))' + \gamma(t)\phi(z) = -\lambda h \ \text{ on } (0,\tau), \\ az(0) - b\phi^{-1}(r(0))z'(0) = 0, \ \ z(\tau) = ||u||_\infty \end{cases}$$

Note that $u \geq z \geq v \geq -M(\lambda\|h\|_1)^{\frac{1}{p-1}}$ on $[0,\tau]$. Let $z_1(t) = z(t) + M(\lambda\|h\|_1)^{\frac{1}{p-1}}$. Then $z_1 \geq 0$ on $[0,1]$ and

$$z_1(t) = z_1(0) + \int_0^t \phi^{-1}\left(\frac{r(0)\phi(z_1'(0) + \int_0^s(\gamma(\xi)\phi(z) + \lambda h)d\xi}{r(s)}\right). \tag{2.2}$$

If $b = 0$ then $z(0) = v(0) = 0$ and so $z_1'(0) = z'(0) \geq v'(0) \geq -M(\lambda\|h\|_1)^{\frac{1}{p-1}}$. On the other hand, if $b \neq 0$ then $z_1'(0) = \frac{a}{b\phi^{-1}(r(0))}z(0) \geq -\frac{Ma}{b\phi^{-1}(r(0))}(\lambda\|h\|_1)^{\frac{1}{p-1}}$. Hence

$$z_1'(0) \geq -M_1(\lambda\|h\|_1)^{\frac{1}{p-1}}, \tag{2.3}$$

where $M_1 = M$ if $b = 0$ and $M_1 = \frac{Ma}{b\phi^{-1}(r(0))}$ if $b \neq 0$. Since $\phi(z_1'(0)) \leq \phi\left(z_1'(0) + M_1(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)$ and

$$(x+y)^q \leq 2^{(q-1)^+}(x^q + y^q) \text{ for } x, y \geq 0, q > 0, \tag{2.4}$$

it follows from (2.2) that

$$z_1(t) \leq m_1\left(z_1(0) + z_1'(0) + M_1(\lambda\|h\|_1)^{\frac{1}{p-1}} + \phi^{-1}\left(\int_0^t(\gamma(\xi)\phi(z_1) + \lambda h)d\xi\right)\right),$$

where $m_1 > 0$ is a constant depending only on p and r. Hence using (2.4) again and the inequality $\phi(x) + \phi(y) \leq 2\phi(x+y)$ for $x, y \geq 0$, we deduce that

$$\phi(z_1(t)) \leq m_2\left(\phi\left(z_1(0) + z_1'(0) + M_1(\lambda\|h\|_1)^{\frac{1}{p-1}}\right) + \lambda\|h\|_1 + \int_0^t \gamma(s)\phi(z_1)ds\right)$$

$$\leq 2m_2\phi\left(z_1(0) + z_1'(0) + M_2(\lambda\|h\|_1)^{\frac{1}{p-1}}\right) + m_2\int_0^t \gamma(s)\phi(z_1)ds$$

for $t \in [0,\tau]$, where $M_2 = M_1 + 1$ and $m_2 = 2^{(p-2)^+}\phi(m_1)$.

By Gronwall's inequality,

$$\phi(z_1(t)) \leq 2m_2\phi\left(z_1(0) + z_1'(0) + M_2(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)e^{m_2\int_0^t \gamma(s)ds}$$

for $t \in [0,\tau]$. In particular when $t = \tau$, we obtain

$$z(0) + z'(0) + (M_2 + M)(\lambda\|h\|_1)^{\frac{1}{p-1}} \geq \kappa_0\|u\|_\infty, \tag{2.5}$$

where $\kappa_0 = \left(e^{-m_2\|\gamma\|_1}/2m_2\right)^{1/(p-1)}$. Since

$$(r(t)\phi(z'))' = \gamma(t)\phi(z) + \lambda h \geq -\gamma(t)\lambda M^{p-1}\|h\|_1 \text{ on } [0,\tau],$$

it follows upon integrating on $[0,t]$ that

$$r(t)\phi(z'(t)) \geq r(0)\phi(z'(0)) - \lambda M_3\|h\|_1,$$

where $M_3 = M^{p-1}\|\gamma\|_1$, which implies

$$z'(t) \geq \phi^{-1}\left(\frac{r(0)\phi(z'(0)) - \lambda M_3\|h\|_1}{r(t)}\right) \tag{2.6}$$

for $t \in [0, \tau]$. Using the inequality $\psi(x - y) \geq c_1\psi(x) - c_2\psi(y)$ for $x, y \geq 0$ and $\psi \in \{\phi, \phi^{-1}\}$, where $c_1 = \psi(1/2), c_2 = 2\psi(2)$, we deduce from (2.3) and (2.6) that

$$z'(t) \geq M_4 z'(0) - M_5(\lambda\|h\|_1)^{\frac{1}{p-1}} \geq -M_6(\lambda\|h\|_1)^{\frac{1}{p-1}} \tag{2.7}$$

where M_4, M_5 are positive constants depending only on p and r, and $M_6 = M_1M_4 + M_5$. If $b = 0$ then $z(0) = 0$ and (2.5) and (2.7) give

$$\begin{aligned} z(t) = \int_0^t z' &\geq \left(M_4\left((\kappa_0\|u\|_\infty - (M + M_2)(\lambda\|h\|_1)^{\frac{1}{p-1}}\right) - M_5(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)t \\ &= \left(\kappa_1\|u\|_\infty - \sigma_1(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)(at + b) \end{aligned} \tag{2.8}$$

for $t \in [0, \tau]$, where $\kappa_1 = a^{-1}M_4\kappa_0, \sigma_1 = a^{-1}(M_4(M + M_2) + M_5)$. On the other hand, if $b > 0$ then $z'(0) = \frac{a}{b\phi^{-1}(r(0))}z(0)$ from which (2.5) becomes

$$z(0) \geq \tilde{\kappa}_1\|u\|_\infty - M_7(\lambda\|h\|_1)^{\frac{1}{p-1}},$$

where $\tilde{\kappa}_1 = \kappa_0\left(1 + \frac{a}{b\phi^{-1}(r(0))}\right)^{-1}$, $M_7 = (M + M_2)\left(1 + \frac{a}{b\phi^{-1}(r(0))}\right)^{-1}$. Hence it follows upon integrating (2.7) that

$$\begin{aligned} z(t) \geq z(0) - M_6(\lambda\|h\|_1)^{\frac{1}{p-1}}t &\geq \tilde{\kappa}_1\|u\|_\infty - (M_6 + M_7)(\lambda\|h\|_1)^{\frac{1}{p-1}} \\ &\geq \left(\kappa_1\|u\|_\infty - \sigma_1(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)(at + b) \end{aligned} \tag{2.9}$$

for $t \in [0, \tau]$, where $\kappa_1 = \frac{\tilde{\kappa}_1}{a+b}$, $\sigma_1 = \frac{M_6+M_7}{b}$. Combining (2.8) and (2.9), we see that in any case there exist constants κ_1 and $\sigma_1 > 0$ independent of u, z, λ, h such that

$$u(t) \geq z(t) \geq \left(\kappa_1\|u\|_\infty - \sigma_1(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)(at + b) \tag{2.10}$$

for $t \in [0, \tau]$. Next, let $w \in C^1[\tau, 1]$ be the unique solution of

$$\begin{cases} -(r(t)\phi(w'))' + \gamma(t)\phi(w) = -\lambda h \ \text{ on } (\tau, 1), \\ w(\tau) = \|u\|_\infty, \ \ cw(1) + d\phi^{-1}(r(1))w'(1) = 0. \end{cases}$$

Then $u \geq w \geq v$ on $[\tau, 1]$. Let $w_1 = w + M(\lambda\|h\|_1)^{\frac{1}{p-1}}$. Then $w_1 \geq 0$ and by using the integral formula

$$w_1(t) = w_1(1) + \int_t^1 \phi^{-1}\left(\frac{-r(1)\phi(w_1'(1) + \int_s^1(\gamma(\xi)\phi(w) + \lambda h)d\xi}{r(s)}\right)ds$$

for $t \in [\tau, 1]$ with the same arguments as above, we obtain

$$u(t) \geq w(t) \geq \left(\kappa_2\|u\|_\infty - \sigma_2(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)(d + c(1 - t)) \tag{2.11}$$

for $t \in [\tau, 1]$, where $\kappa_2 > 0$ is a constant independent of u, w, λ, h. Hence if $C > \max(\sigma_1/\kappa_1, \sigma_2/\kappa_2, M)$ then it follows from (2.1),(2.10), and (2.11) that $u(t) \geq \left(\kappa\|u\|_\infty - \sigma(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)\omega(t)$ for $t \in [0, 1]$, where $\kappa = \min(\kappa_1, \kappa_2, 1), \sigma = \max(\sigma_1, \sigma_2, C + Mk_0)$, which completes the proof of Lemma 2.1.

3. Proof of the main result

Proof of Theorem 1.1. Since $\limsup_{z\to 0^+} \frac{f(t,z)}{z^{p-1}} < \lambda_1$, there exist constants $r, \bar{\lambda} > 0$ with $\bar{\lambda} < \lambda_1$ such that

$$f(t,z) \le \bar{\lambda} z^{p-1} \tag{3.1}$$

for $z \le r$ and a.e. $t \in (0,1)$. Let κ, σ be given by Lemma 2.1 with $h(t) = g(r_0\omega(t))$, where $r_0 = \kappa r/2$, and let $\lambda > 0$ be small enough so that $\sigma(\lambda \|h\|_1)^{\frac{1}{p-1}} < r_0$. For $v \in E = C[0,1]$, we have $0 \le f(t,|v|) + \gamma(t)|v|^{p-1} \in L^1(0,1)$ in view of (A2) and (A4). Hence the problem

$$\begin{cases} -(r(t)\phi(u'))' + \gamma(t)\phi(u) = -\lambda g(\max(v, r_0\omega)) + f(t,|v|) + \gamma(t)|v|^{p-1} \text{ on } (0,1), \\ au(0) - b\phi^{-1}(r(0))u'(0) = 0, \ cu(1) + d\phi^{-1}(r(1))u'(1) = 0, \end{cases}$$

has a unique solution $u = Av \in C^1[0,1]$. Since $A = T_0 \circ S_0$, where $S_0 : C[0,1] \to L^1(0,1)$ is defined by $(S_0 v)(t) = -\lambda g(\max(v, r_0\omega)) + f(t,|v|) + \gamma(t)|v|^{p-1}$ and T_0 is defined in Lemma A with $\alpha = \beta = 0$, we see that $A : E \to E$ is completely continuous. We shall verify that

(i) $u = \theta A u, \ \theta \in (0,1] \Longrightarrow \|u\|_\infty \ne r.$

Indeed, let $u \in E$ satisfy $u = \theta A u$ for some $\theta \in (0,1]$ and suppose $\|u\|_\infty = r$. Then $u \in C^1[0,1]$ and

$$\begin{aligned} -(r(t)\phi(u'))' + \gamma(t)\phi(u) &= \theta^{p-1}\left(-\lambda g(\max(u, r_0\omega)) + f(t,|u|) + \gamma(t)|u|^{p-1}\right) \\ &\ge -\lambda h(t) \end{aligned}$$

on $(0,1)$, from which Lemma 2.1 gives

$$u(t) \ge \left(\kappa \|u\|_\infty - \sigma(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)\omega(t) \ge r_0\omega(t) > 0 \tag{3.2}$$

for $t \in (0,1)$. Hence

$$-(r(t)\phi(u'))' \le \theta^{p-1} f(t,u) \tag{3.3}$$

on $(0,1)$. Let $\delta_0 = \sup\limits_{(0,1)} \frac{u}{\phi_1} \in (0,\infty)$. Then it follows from (3.1) and (3.3) that

$$-(r(t)\phi(u'))' \le \bar{\lambda} u^{p-1} \le \bar{\lambda}\delta_0^{p-1}\phi_1^{p-1} \text{ on } (0,1),$$

from which the weak comparison principle gives

$$u \le (\bar{\lambda}\delta_0^{p-1}/\lambda_1)^{\frac{1}{p-1}}\phi_1$$

on $[0,1]$, a contradiction with the definition of δ_0. Thus $\|u\|_\infty \ne r$ i.e. (i) holds.

Next, we claim that

(ii) There exists a constant $R > r$ such that $u = Au + \xi, \ \xi \ge 0 \Longrightarrow \|u\|_\infty \ne R$.

Let $u \in E$ satisfy $u = Au + \xi$ for some $\xi \in [0,\infty)$. Then $u - \xi = Au$ and therefore

$$-(r(t)\phi(u'))' + \gamma(t)\phi(u-\xi) = -\lambda g(\max(u, r_0\omega)) + f(t,|u|) + \gamma(t)|u|^{p-1} \tag{3.4}$$

on $(0,1)$, which implies

$$-(r(t)\phi(u'))' + \gamma(t)\phi(u) \ge -\lambda h(t) \tag{3.5}$$

on $(0,1)$. Since $\liminf\limits_{z\to\infty} \frac{f(t,z)-\lambda g(z)}{z^{p-1}} > \lambda_1$ uniformly for a.e. $t \in (0,1)$, there exist positive constants $L, \tilde{\lambda}$ with $\tilde{\lambda} > \lambda_1$ such that $f(t,z) - \lambda g(z) \ge \tilde{\lambda} z^{p-1}$ for a.e. $t \in (0,1)$ and $z > L$.

For $z \le L$, (A2) gives $f(t,z) - \lambda g(\max(z, r_0\omega(t))) \ge -\tilde{\gamma}_L(t)$ for a.e. $t \in (0,1)$ and all $z \ge 0$, where $\tilde{\gamma}_L = \gamma_L + \lambda h$. Suppose $\|u\|_\infty = R > r$. Then

$$u(t) \ge \left(\kappa R - \sigma(\lambda\|h\|_1)^{\frac{1}{p-1}}\right)p(t) \ge \frac{\kappa R}{2}\omega(t) \ge \frac{\kappa R c}{2}\phi_1(t)$$

for $t \in [0,1]$ in view of Lemma 2.1, where $c = \inf_{(0,1)} \frac{\omega}{\phi_1}$. In particular, $u(t) \geq r_0\omega(t)$ for $t \in [0,1]$. Let $\delta_1 = \inf_{(0,1)} \frac{u}{\phi_1}$. Then $\delta_1 \geq 1$ if $R > 2/\kappa c$, which we assume. Hence

$$-(r(t)\phi(u'))' \geq f(t,u) - \lambda g(u) \geq \begin{cases} \tilde{\lambda} u^{p-1} & \text{if } u > L \\ -\tilde{\gamma}_L & \text{if } u \leq L \end{cases}.$$

Since $u(t) > L$ if $\omega(t) > 2L/\kappa R$, it follows that

$$-\left(r(t)\phi\left(\frac{u'}{\delta_1}\right)\right)' \geq \begin{cases} \tilde{\lambda}\phi_1^{p-1} & \text{in } D_R, \\ -\tilde{\gamma}_L & \text{in } (0,1)\backslash D_R \end{cases} \equiv h_{1R},$$

where $D_R = \{t : \omega(t) > 2L/\kappa R\}$. Let $u_{1R}, u_2 \in C^1[0,1]$ satisfy

$$-\left(r(t)\phi\left(u_{1R}'\right)\right)' = h_{1R} \text{ on } (0,1),$$

and

$$-(r(t)\phi(u_2'))' = \tilde{\lambda}{\phi_1}^{p-1} \equiv h_2 \text{ on } (0,1).$$

with Sturm-Liouville boundary condition in (1.1). Note that $u_2 = (\tilde{\lambda}/\lambda_1)^{\frac{1}{p-1}}\phi_1$. Since

$$\|h_{1R} - h_2\|_1 = \int_{(0,1)\backslash D_R} (\tilde{\lambda}\phi_1^{p-1} + \tilde{\gamma}_L) \to 0$$

as $R \to \infty$, it follows that $|u_{1R} - u_2|_{C^1} \to 0$ as $R \to \infty$ (see e.g. [11, Lemma 3.1]). Let $\varepsilon > 0$ be such that $\left(\tilde{\lambda}/\lambda_1\right)^{\frac{1}{p-1}} - \varepsilon c_0 \equiv c_1 > 1$, where $c_0 = \sup_{(0,1)} \frac{\omega}{\phi_1} > 0$. Then, if R is large enough (independent of u), we get

$$u_{1R} \geq u_2 - \varepsilon\omega \geq u_2 - \varepsilon c_0\phi_1 = c_1\phi_1 \text{ on } (0,1),$$

and consequently, $u \geq \delta_1 u_{1R} \geq \delta_1 c_1 \phi_1$ on $(0,1)$, a contradiction with the definition of δ_1. Thus $\|u\|_\infty \neq R$ if R is large enough i.e. (ii) holds.

By Lemma A, A has a fixed point $u \in E$ with $\|u\|_\infty > r$, which is a classical positive solution of (1.1) in view of (3.2) and Lemmas 2.2. This completes the proof of Theorem 1.1.

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