A note on inverse problem for strongly damped wave equation with Gaussian white noise

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Abstract. In this paper, we study for the first time the inverse initial problem for the one-dimensional strongly damped wave with Gaussian white noise data. Under some a priori assumptions on the true solution, we propose the Fourier truncation method for stabilizing the ill-posed problem. Error estimates are given in both the $L^2$- and $H^p$-norms.

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1 Introduction

Let $T$ be a positive number and $D = (0, \pi)$. We are interested in the problem of recovering the initial state $u(x, 0), x \in D$, for the following strongly damped wave equation

$$u_{tt} - \alpha u_{xt} - u_{xx} = 0, \quad (x, t) \in D \times (0, T),$$

subject to the final conditions

$$\begin{cases} u(x, T) = g(x), \quad (x, t) \in D \times (0, T), \\ u_t(x, T) = 0, \quad (x, t) \in D \times (0, T), \end{cases}$$

and the boundary condition

$$u(0, t) = u(\pi, t) = 0, \quad t \in (0, T),$$

where $\alpha > 2$ is a fixed positive constant. In physics, the homogeneous Dirichlet boundary condition (1.3) expresses that two ends of the string are fixed. We can consider other types of
boundary condition instead. In practice, we cannot measure $g$ exactly, but we observe with the presence of a Gaussian white noise process $\xi$

$$g^{\text{obs}}_\epsilon(x) = g(x) + \epsilon \xi(x),$$

(1.4)

where $\epsilon > 0$ is the amplitude of the noise. Moreover, it can only be observed in discretized form:

$$\langle g^{\text{obs}}_\epsilon, \phi \rangle = \langle g, \phi \rangle + \epsilon \langle \xi, \phi \rangle, \quad j = \overline{1,N},$$

(1.5)

where the natural number $N$ is the number of steps of discrete observation and $\phi_j$, $j = \overline{1,N}$, are defined in the next section. For more details on this random model, we refer to [1, 4].

This problem is well-known to be severely ill-posed and regularization methods for it are required. Strongly damped wave equation (SDWE) occurs in a wide range of applications such as modeling motion of viscoelastic materials [2, 7, 8]. From both the theoretical and numerical points of view, the initial value problem has been extensively studied (see e.g., [3, 5, 9]). However, to the best of our knowledge, the final value problem (1.1)-(1.3) with Gaussian white noise data has not been studied. In [6], Lions and Lattes introduced Problem (1.1)-(1.3), but regularization methods haven’t been mentioned. The major object of this paper is to propose a stable regularized solution for the problem (1.1)-(1.3) using the Fourier truncation method.

2 Preliminaries

Let us recall that the problem

$$\begin{aligned}
-\phi_{xx} &= \lambda \phi, \quad x \in D, \\
\phi(0) &= \phi(\pi) = 0,
\end{aligned}$$

(2.6)

admits the eigenvalues $\lambda_j = j^2$, $j = 1, \infty$, and $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ are the corresponding eigenfunctions, which form an orthonormal basis of $L^2(D)$.

**Definition 2.1** Let $H$ be a Hilbert space. The stochastic error is a Hilbert space process, i.e. a bounded linear operator $\xi : H \to L^2(\Omega, \mathcal{A}, P)$, where $(\Omega, \mathcal{A}, P)$ is the underlying probability space and $L^2(\cdot)$ is the space of all square integrable measurable functions.

**Remark 2.1** Let $g \in H$. It follows from the representation (1.4) that

$$\langle g_\epsilon, \chi \rangle = \langle g, \chi \rangle + \epsilon \langle \xi, \chi \rangle, \quad \forall \chi \in H.$$ 

Since $\xi(x)$ is a Gaussian white noise process, we have that $\langle \xi, \chi \rangle \sim N(0, ||\chi||^2_H)$, and for any $\chi_1, \chi_2 \in H$, we have

$$\text{E} (\langle \xi, \chi_1 \rangle \langle \xi, \chi_2 \rangle) = \langle \chi_1, \chi_2 \rangle.$$ 

In (1.5), we consider the test functions $\chi = \phi_j$, $j = \overline{1,N}$, - the eigenfunctions of the problem (2.6), and $H = L^2(D)$. Then, $\xi_i := \langle \xi, \phi_j \rangle \sim N(0,1)$ are i.i.d standard Gaussian random variables.
For $p > 0$, we denote by $H^p(D)$ the closed subspace of $L^2(D)$, given by

$$H^p(D) = \left\{ u \in L^2(D) : \sum_{j=1}^{\infty} \lambda_j^p \langle u, \phi_j \rangle^2 < +\infty \right\},$$

and equipped with the norm

$$\| u \|_{H^p(D)} = \sqrt{\sum_{j=1}^{\infty} \lambda_j^p \langle u, \phi_j \rangle^2},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(D)$. Throughout this paper, denote by $\| \cdot \|$ the norm in $L^2(D)$.

**Lemma 2.1** Let $\overline{G}_{\epsilon,N(\epsilon)} \in L^2(D)$ be such that

$$\overline{G}_{\epsilon,N(\epsilon)} = \sum_{j=1}^{N(\epsilon)} \left( g^\text{obs}_\epsilon, \phi_j \right) \phi_j.$$

Assume that $g \in H^p(D)$ for $\gamma > 0$. Then, we have the following estimate

$$E \| \overline{G}_{\epsilon,N(\epsilon)} - g \|^2 \leq \epsilon^2 N(\epsilon) + \frac{1}{N(\epsilon)^{2\gamma}} \| g \|_{H^p(D)}^2.$$

Here, $N$ depends on $\epsilon$ and satisfies that $\lim_{\epsilon \to 0} N(\epsilon) = +\infty$.

**Proof.** By the usual MISE decomposition which involves a variance term and a bias term, we get

$$E \| \overline{G}_{\epsilon,N(\epsilon)} - g \|^2 = E \left( \sum_{j=1}^{N(\epsilon)} \left( g^\text{obs}_\epsilon - g, \phi_j \right)^2 \right) + \sum_{j \geq N(\epsilon)+1} \left( g, \phi_j \right)^2$$

$$= \epsilon^2 E \left( \sum_{j=1}^{N(\epsilon)} \xi_j^2 \right) + \sum_{j \geq N(\epsilon)+1} \lambda_j^\gamma \lambda_j^\gamma \langle g, \phi_j \rangle^2.$$

Since $\xi_j = \langle \xi, \phi_j \rangle \sim N(0,1)$, it follows that $E \xi_j^2 = 1$. Therefore,

$$E \| \overline{G}_{\epsilon,N(\epsilon)} - g \|^2 \leq \epsilon^2 N(\epsilon) + \frac{1}{N^{2\gamma}(\epsilon)} \| g \|_{H^p(D)}^2.$$

**3 Main results**

Assume that Problem (1.1)-(1.3) has a unique true solution, we first find its Fourier series representation. From (1.1)-(1.3), we can derive the following ODE with given data at $t = T$

$$\begin{cases}
\frac{d^2}{dt^2} u_j(t) + \alpha \lambda_j \frac{d}{dt} u_j(t) + \lambda_j u_j(t) = 0, & t \in (0, T), \\
u_j(T) = g_j, & \frac{d}{dt} u_j(T) = 0,
\end{cases}$$
where \( u_j(t) = \int_0^\pi u(x,t)\phi_j(x)\,dx, \) \( g_j = \int_0^\pi g(x)\phi_j(x)\,dx. \)

For fixed damping \( \alpha > 2, \) the solution of the latter equation is given by

\[
\frac{\alpha_j^2 + \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \exp\left( (T - t) \frac{\alpha_j^2 - \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \right) - \frac{\alpha_j^2 - \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \exp\left( (T - t) \frac{\alpha_j^2 + \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \right)
\]

\[
\frac{\sqrt{\alpha^2 j^4 - 4 j^2}}{\mathcal{A}_j(T-t)} \quad g_j.
\]

Hence, the solution of problem (1.1)-(1.3) can be represented as

\[
u(x,t) = \sum_{j=1}^{\infty} u_j(t)\phi_j(x).
\]

For fixed \( t \) and \( \beta_{N(t)} \), let us define the operator \( Q_{\beta_{N(t)}}(t) : L_2(D) \to L_2(D) \)

\[
Q_{\beta_{N(t)}}(t) = \sum_{\lambda_j \leq \beta_{N(t)}} \mathcal{A}_j(t) \langle \varphi, \phi_j \rangle \varphi_j.
\]

Now, we state and prove the following lemma, which is needed for our analysis.

**Lemma 3.1** For fixed \( t \in [0,T] \), we have

\[
\|Q_{\beta_{N(t)}}(t)\varphi\| \leq \sqrt{2 + \frac{8T^2}{\alpha^2} e^{\alpha \beta_{N(t)}}} \|\varphi\|. \tag{3.8}
\]

Moreover,

\[
\|Q_{\beta_{N(t)}}(t)\varphi\|_{H_p(D)} \leq \beta_{N(t)}^p \left( 2 + \frac{8T^2}{\alpha^2} \right) e^{\alpha \beta_{N(t)}} \|\varphi\|. \tag{3.9}
\]

**Proof.** Using Parseval’s equality, we obtain

\[
\|Q_{\beta_{N(t)}}(t)\varphi\|^2 = \sum_{\lambda_j \leq \beta_{N(t)}} \mathcal{A}_j(t) \langle \varphi, \phi_j \rangle^2. \tag{3.10}
\]

From the equality

\[
\frac{\alpha_j^2 + \sqrt{\alpha^2 j^4 - 4 j^2}}{2} = \sqrt{\alpha^2 j^4 - 4 j^2} + \frac{\alpha_j^2 - \sqrt{\alpha^2 j^4 - 4 j^2}}{2},
\]

\( \mathcal{A}_j(t) \) can be rewritten as

\[
\mathcal{A}_j(t) = \frac{\alpha_j^2 + \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \exp\left( (T - t) \frac{\alpha_j^2 - \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \right) - \frac{\alpha_j^2 - \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \exp\left( (T - t) \frac{\alpha_j^2 + \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \right)
\]

\[
= \exp\left( \frac{\alpha_j^2 - \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \right)
\]

\[
+ \left( \frac{\alpha_j^2 - \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \right) \exp\left( \frac{\alpha_j^2 + \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \right) - \exp\left( \frac{\alpha_j^2 + \sqrt{\alpha^2 j^4 - 4 j^2}}{2} \right).
\]
Applying the inequalities \((a + b)^2 \leq 2a^2 + 2b^2\) and \(|e^a - e^b| \leq \max\{e^a, e^b\}|a - b|\), we obtain

\[
\begin{align*}
\mathcal{A}^2_j(t) &\leq 2 \exp \left[ t \left( \alpha \frac{j}{2} - \sqrt{\alpha^2 \frac{j^2}{4} - 4} \right) \right] + \frac{\left( \alpha \frac{j}{2} + \sqrt{\alpha^2 \frac{j^2}{4} - 4} \right)^2}{2} t^2 \exp \left[ t \left( \alpha \frac{j}{2} + \sqrt{\alpha^2 \frac{j^2}{4} - 4} \right) \right] \\
&\leq \exp \left( 2 \alpha \frac{j}{2} t \right) \left( 2 + \frac{8 j^2}{\alpha^2} \right) \\
&\leq \exp \left( 2 \alpha \beta_{N(e)} t \right) \left( 2 + \frac{8 T^2}{\alpha^2} \right), \quad \text{(3.11)}
\end{align*}
\]

From (3.10) and (3.11), we can deduce (3.8).

For the second statement of Lemma 3.1, from (3.8), we have

\[
\|Q_{\beta_{N(e)}}(t)\varphi\|^2_{H^p(D)} = \sum_{\lambda_j < \beta_{N(e)}} \lambda_j^2 \mathcal{A}^2_j(t) \langle \varphi, \phi_j \rangle^2 \leq \beta_{N(e)}^p \exp \left( 2 \alpha \beta_{N(e)} t \right) \left( 2 + \frac{8 T^2}{\alpha^2} \right).
\]

The proof of Lemma 3.1 is complete.

Define the truncation operator \(I_{\beta_{N(e)}} : L^2(D) \to L^2(D)\) such that

\[
I_{\beta_{N(e)}} v = \sum_{\lambda_j < \beta_{N(e)}} \langle v, \phi_j \rangle \phi_j, \quad \text{for all } v \in L^2(D).
\]

Next, we formulate a regularized problem for Problem (1.1)-(1.3) as follows

\[
\begin{align*}
W^e_{\alpha}(x, t) - \alpha W^e_{\alpha}(x, t) - W^e_{\alpha}(x, t) = 0, \quad (x, t) \in D \times (0, T), \\
W^e_{\alpha}(0, t) = W^e_{\alpha}(\pi, t) = 0, \quad t \in (0, T), \\
W^e_{\alpha}(x, T) = I_{\beta_{N(e)}} (T - t) \Theta_{\epsilon, N(e)}(x), \quad (x, t) \in D \times (0, T), \\
W^e_{\alpha}(x, t) = 0, \quad (x, t) \in D \times (0, T),
\end{align*}
\]

where \(\beta_{N(e)}\) plays the role of regularization parameter. Using the same techniques as the beginning of this section for the problem (1.1)-(1.3), we can easily show that Problem (3.12) has a unique solution \(W^e_{\alpha}(x, t) \in C([0, T], L^2(D))\) given by

\[
W^e_{\alpha}(x, t) = Q_{\beta_{N(e)}} (T - t) \Theta_{\epsilon, N(e)}(x).
\]

We are now in position to state the main results of this paper.

**Theorem 3.1** Let us choose \(N(e)\) and \(\beta_{N(e)}\) such that

\[
\lim_{\epsilon \to 0} N(e) = +\infty, \quad \lim_{\epsilon \to 0} \beta_{N(e)} = +\infty, \quad \lim_{\epsilon \to 0} e^{\alpha T \beta_{N(e)}} = 0, \quad \lim_{\epsilon \to 0} e^{\alpha T \beta_{N(e)}} e^{\sqrt{N(e)}} = 0. \quad (3.13)
\]

Assume that Problem (1.1)-(1.3) has a unique solution \(u \in H^{2\theta}(D)\) and \(g \in H^{\gamma}(D)\) for \(\theta, \gamma > 0\). Then, there exists \(C_0 = C_0(\alpha, T) > 0\) such that

\[
E \left\| u(\cdot, t) - W^e_{\alpha}(\cdot, t) \right\|^2 \leq C_0 e^{2 \alpha T \beta_{N(e)}} \left( e^2 N(e) + \frac{1}{N^2 \gamma(e)} \|g\|_{H^{\gamma}(D)}^2 + \frac{2 \|u(\cdot, t)\|^2_{H^{2\theta}(D)}}{\beta_{N(e)}^2} \right). \quad (3.14)
\]
Remark 3.1 We now give an example for the choice of $N(\epsilon)$ and $\beta_{N(\epsilon)}$, which satisfy the condition (3.13). Since $\lambda_N(\epsilon) = N^2(\epsilon)$, we can choose $\beta_{N(\epsilon)}$ such that $e^{2\alpha T\beta_{N(\epsilon)}} = N(\epsilon)^a$ for $0 < a < \gamma$. Then, we have $\beta_{N(\epsilon)} = \frac{a}{2\alpha T} \ln (N(\epsilon))$. The number $N(\epsilon)$ is chosen as follows

$$N(\epsilon) = \left( \frac{1}{\epsilon} \right)^{\frac{2b}{a+b}}$$

for $0 < b < 1$. With $N(\epsilon)$ chosen as above, $E \| u(\cdot, t) - U^{e,N(\epsilon)}(\cdot, t) \|$ is of order

$$\frac{1}{\ln^b \left( \frac{1}{\epsilon} \right)}.$$

Proof of Theorem 3.1. It follows from (3.7) that

$$V^{e,N(\epsilon)}(x, t) := I_{\beta_{N(\epsilon)}}u(x, t) = \sum_{\lambda_j \leq \beta_{N(\epsilon)}} u_j(t) \phi_j (x) = Q_{\beta_{N(\epsilon)}}(T - t) g(x).$$

Applying the inequality $(a + b)^2 \leq 2a^2 + 2b^2$, Parseval’s equality, from (3.8) and Lemma 2.1, we have

$$E \| u(\cdot, t) - W^{e,N(\epsilon)}(\cdot, t) \|^2 \leq 2 \| u(\cdot, t) - V^{e,N(\epsilon)}(\cdot, t) \|^2 + 2E \| W^{e,N(\epsilon)}(\cdot, t) - V^{e,N(\epsilon)}(\cdot, t) \|^2$$

$$= 2 \sum_{\lambda_j > \beta_{N(\epsilon)}} u_j^2(t) + 2E \left\| Q_{\beta_{N(\epsilon)}}(T - t) \left( G_{e,N(\epsilon)} - g \right) \right\|^2$$

$$\leq 2 \sum_{\lambda_j > \beta_{N(\epsilon)}} \lambda_j^{2\gamma} \lambda_j^2 u_j^2(t) + \left( 4 + \frac{16T^2}{\alpha^2} \right) e^{2\alpha(T-t)\beta_{N(\epsilon)}} E \| G_{e,N(\epsilon)} - g \|^2$$

$$\leq 2\beta_{N(\epsilon)}^2 \| u(\cdot, t) \|^2_{H^{2\gamma}(D)} + \left( 4 + \frac{16T^2}{\alpha^2} \right) e^{2\alpha(T-t)\beta_{N(\epsilon)}} \left( \epsilon^2 N(\epsilon) + \frac{1}{N^{2\gamma}(\epsilon)} \| g \|^2_{H^{\gamma}(D)} \right).$$

From the above inequality, we can imply (3.14). This completes the proof of Theorem 3.1. The next theorem gives the error estimate in the $H^p(\epsilon)$--norm.

Theorem 3.2 Let us choose $N(\epsilon)$ and $\beta_{N(\epsilon)}$ such that

$$\lim_{\epsilon \to 0} N(\epsilon) = +\infty, \quad \lim_{\epsilon \to 0} \beta_{N(\epsilon)} = +\infty, \quad \lim_{\epsilon \to 0} \frac{\sqrt{\beta_{N(\epsilon)}} e^{2\alpha T \beta_{N(\epsilon)}}}{N^{\gamma}(\epsilon)} = 0, \quad \lim_{\epsilon \to 0} e^{2\alpha T \beta_{N(\epsilon)}} \sqrt{\beta_{N(\epsilon)}^{2\gamma}(\epsilon)} N(\epsilon) = 0.$$

Suppose that the problem (1.1)-(1.3) has a unique solution $u \in H^{p+2\gamma}(D)$ and $g \in H^p(D)$ for $p, \gamma > 0$. Then,

$$E \left\| W^{e,N(\epsilon)}(\cdot, t) - u(\cdot, t) \right\|^2_{H^p(D)}$$

$$\leq 2\beta_{N(\epsilon)}^{2\gamma} \| u(\cdot, t) \|^2_{H^{p+2\gamma}(D)} + C_0 \beta_{N(\epsilon)} e^{2\alpha T \beta_{N(\epsilon)}} \left( \epsilon^2 N(\epsilon) + \frac{1}{N^{2\gamma}(\epsilon)} \| g \|^2_{H^p(D)} \right). \quad (3.15)$$

Remark 3.2 We give one example for the choice of $N(\epsilon)$. Choose $\beta_{N(\epsilon)}$ such that $\beta_{N(\epsilon)} = \frac{a}{2\alpha T} \ln (N(\epsilon))$ for $0 < a < \gamma$. The number $N(\epsilon)$ is chosen as follows

$$N(\epsilon) = \left( \frac{1}{\epsilon} \right)^{\frac{2b}{a+b}}$$

for $0 < b < 1$. Then, $E \left\| W^{e,N(\epsilon)}(\cdot, t) - u(\cdot, t) \right\|^2_{H^p(D)}$ is of order

$$\frac{1}{\ln^b \left( \frac{1}{\epsilon} \right)}.$$
Proof of Theorem 3.2. Indeed, using (3.9) and Lemma 2.1, we have

$$
\mathbb{E} \left\| W^{e,N(e)}(\cdot, t) - u(\cdot, t) \right\|_{H^p(D)}^2 \\
\leq 2 \left\| u(\cdot, t) - V^{e,N(e)}(\cdot, t) \right\|_{H^p(D)}^2 + 2 \mathbb{E} \left\| W^{e,N(e)}(\cdot, t) - V^{e,N(e)}(\cdot, t) \right\|_{H^p(D)}^2 \\
= 2 \sum_{ \lambda_j > \beta_{N(e)} } \lambda_j^2 u_j^2(t) + 2 \mathbb{E} \left\| Q_{\beta_{N(e)}}(T-t) \left( \overline{G}(e) - g \right) \right\|_{H^p(D)}^2 \\
\leq 2 \sum_{ \lambda_j > \beta_{N(e)} } \lambda_j^{-2} \lambda_j^{20} u_j^2(t) + \left( 4 + \frac{16T^2}{\alpha^2} \right) \beta_{N(e)}^p e^{20(T-t)\beta_{N(e)}} \mathbb{E} \left\| \overline{G}(e) - g \right\|^2 \\
\leq 2 \beta_{N(e)}^p \left\| u(\cdot, t) \right\|_{H^p,2\alpha(D)}^2 + \left( 4 + \frac{16T^2}{\alpha^2} \right) \beta_{N(e)}^p e^{20(T-t)\beta_{N(e)}} \left( e^2 N(e) + \frac{1}{N^2(\epsilon)} \left\| g \right\|_{H^p(D)}^2 \right),
$$

from which we can deduce (3.15). This completes the proof.

References


