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A note on inverse problem for strongly damped wave equation with Gaussian white noise

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Abstract. In this paper, we study for the first time the inverse initial problem for the one-dimensional strongly damped wave with Gaussian white noise data. Under some a priori assumptions on the true solution, we propose the Fourier truncation method for stabilizing the ill-posed problem. Error estimates are given in both the L^2- and H^p-norms.
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1 Introduction

Let T be a positive number and $D = (0, \pi)$. We are interested in the problem of recovering the initial state $u(x, 0), x \in D$, for the following strongly damped wave equation

$$u_{tt} - \alpha u_{xxt} - u_{xx} = 0, \quad (x, t) \in D \times (0, T), \tag{1.1}$$

subject to the final conditions

$$\begin{cases} u(x, T) = g(x), & (x, t) \in D \times (0, T), \\ u_t(x, T) = 0, & (x, t) \in D \times (0, T), \end{cases} \tag{1.2}$$

and the boundary condition

$$u(0, t) = u(\pi, t) = 0, \quad t \in (0, T), \tag{1.3}$$

where $\alpha > 2$ is a fixed positive constant. In physics, the homogeneous Dirichlet boundary condition (1.3) expresses that two ends of the string are fixed. We can consider other types of

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boundary condition instead. In practice, we cannot measure g exactly, but we observe with the presence of a Gaussian white noise process ξ

$$g_\epsilon^{obs}(x) = g(x) + \epsilon\xi(x), \tag{1.4}$$

where $\epsilon > 0$ is the amplitude of the noise. Moreover, it can only be observed in discretized form:

$$\langle g_\epsilon^{obs}, \phi_j\rangle = \langle g, \phi_j\rangle + \epsilon\langle \xi, \phi_j\rangle, \quad j = \overline{1, N}, \tag{1.5}$$

where the natural number N is the number of steps of discrete observation and ϕ_j, $j = \overline{1, N}$, are defined in the next section. For more details on this random model, we refer to [1, 4].

This problem is well-known to be severely ill-posed and regularization methods for it are required. Strongly damped wave equation (SDWE) occurs in a wide range of applications such as modeling motion of viscoelastic materials [2, 7, 8]. From both the theoretical and numerical points of view, the initial value problem has been extensively studied (see e.g., [3, 5, 9]). However, to the best of our knowledge, the final value problem (1.1)-(1.3) with Gaussian white noise data has not been studied. In [6], Lions and Lattes introduced Problem (1.1)-(1.3), but regularization methods haven't been mentioned. The major object of this paper is to propose a stable regularized solution for the problem (1.1)-(1.3) using the Fourier truncation method.

2 Preliminaries

Let us recall that the problem

$$\begin{cases} -\phi_{xx} = \lambda\phi, & x \in D, \\ \phi(0) = \phi(\pi) = 0, \end{cases} \tag{2.6}$$

admits the eigenvalues $\lambda_j = j^2, j = \overline{1, \infty}$, and $\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$ are the corresponding eigenfunctions, which form an orthonormal basis of $L^2(D)$.

Definition 2.1 *Let H be a Hilbert space. The stochastic error is a Hilbert space process, i.e. a bounded linear operator $\xi : H \to L^2(\Omega, \mathcal{A}, \mathcal{P})$, where $(\Omega, \mathcal{A}, \mathcal{P})$ is the underlying probability space and $L^2(\cdot)$ is the space of all square integrable measurable functions.*

Remark 2.1 *Let $g \in H$. It follows from the representation* (1.4) *that*

$$\langle g_\epsilon, \chi\rangle = \langle g, \chi\rangle + \epsilon\langle \xi, \chi\rangle, \quad \forall\chi \in H.$$

Since $\xi(x)$ is a Gaussian white noise process, we have that $\langle \xi, \chi\rangle \sim \mathcal{N}\left(0, \|\chi\|_H^2\right)$, and for any $\chi_1, \chi_2 \in H$, we have

$$\mathbf{E}\left(\langle \xi, \chi_1\rangle\langle \xi, \chi_2\rangle\right) = \langle \chi_1, \chi_2\rangle.$$

In (1.5)*, we consider the test functions $\chi = \phi_j$, $j = \overline{1, N}$, - the eigenfunctions of the problem* (2.6)*, and $H = L^2(D)$. Then, $\xi_i := \langle \xi, \phi_j\rangle \sim \mathcal{N}(0, 1)$ are i.i.d standard Gaussian random variables.*

For $p > 0$, we denote by $H^p(D)$ the closed subspace of $L^2(D)$, given by

$$H^p(D) = \left\{ u \in L^2(D) : \sum_{j=1}^{\infty} \lambda_j^p \langle u, \phi_j \rangle^2 < +\infty \right\},$$

and equipped with the norm

$$\|u\|_{H^p(D)} = \sqrt{\sum_{j=1}^{\infty} \lambda_j^p \langle u, \phi_j \rangle^2},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(D)$. Throughout this paper, denote by $\| \cdot \|$ the norm in $L^2(D)$.

Lemma 2.1 *Let $\overline{G}_{\epsilon,N(\epsilon)} \in L^2(D)$ be such that*

$$\overline{G}_{\epsilon,N(\epsilon)} = \sum_{j=1}^{N(\epsilon)} \left\langle g_\epsilon^{obs}, \phi_j \right\rangle \phi_j.$$

Assume that $g \in H^\gamma(D)$ for $\gamma > 0$. Then, we have the following estimate

$$\mathbf{E} \left\| \overline{G}_{\epsilon,N(\epsilon)} - g \right\|^2 \leq \epsilon^2 N(\epsilon) + \frac{1}{N(\epsilon)^{2\gamma}} \|g\|^2_{H^\gamma(D)}.$$

Here, N depends on ϵ and satisfies that $\lim_{\epsilon \to 0} N(\epsilon) = +\infty$.

Proof. By the usual MISE decomposition which involves a variance term and a bias term, we get

$$\begin{aligned}
\mathbf{E} \left\| \overline{G}_{\epsilon,N(\epsilon)} - g \right\|^2 &= \mathbf{E} \left(\sum_{j=1}^{N(\epsilon)} \left\langle g_\epsilon^{obs} - g, \phi_j \right\rangle^2 \right) + \sum_{j \geq N(\epsilon)+1} \left\langle g, \phi_j \right\rangle^2 \\
&= \epsilon^2 \mathbf{E} \left(\sum_{j=1}^{N(\epsilon)} \xi_j^2 \right) + \sum_{j \geq N(\epsilon)+1} \lambda_j^{-\gamma} \lambda_j^{\gamma} \langle g, \phi_j \rangle^2.
\end{aligned}$$

Since $\xi_j = \langle \xi, \phi_j \rangle \sim \mathcal{N}(0, 1)$, it follows that $\mathbf{E}\xi_j^2 = 1$. Therefore,

$$\mathbf{E} \left\| \overline{G}_{\epsilon,N(\epsilon)} - g \right\|^2 \leq \epsilon^2 N(\epsilon) + \frac{1}{N^{2\gamma}(\epsilon)} \|g\|^2_{H^\gamma(D)}.$$

3 Main results

Assume that Problem (1.1)-(1.3) has a unique true solution, we first find its Fourier series representation. From (1.1)-(1.3), we can derive the following ODE with given data at $t = T$

$$\begin{cases}
\dfrac{d^2}{dt^2} u_j(t) + \alpha \lambda_j \dfrac{d}{dt} u_j(t) + \lambda_j u_j(t) = 0, \quad t \in (0, T), \\
u_j(T) = g_j, \quad \dfrac{d}{dt} u_j(T) = 0,
\end{cases}$$

where $u_j(t) = \int_0^\pi u(x,t)\phi_j(x)dx$, $g_j = \int_0^\pi g(x)\phi_j(x)dx$.
For fixed damping $\alpha > 2$, the solution of the latter equation is given by

$$u_j(t) = \underbrace{\frac{\frac{\alpha j^2+\sqrt{\alpha^2 j^4-4j^2}}{2}\exp\left((T-t)\frac{\alpha j^2-\sqrt{\alpha^2 j^4-4j^2}}{2}\right) - \frac{\alpha j^2-\sqrt{\alpha^2 j^4-4j^2}}{2}\exp\left((T-t)\frac{\alpha j^2+\sqrt{\alpha^2 j^4-4j^2}}{2}\right)}{\sqrt{\alpha^2 j^4-4j^2}}}_{\mathcal{A}_j(T-t)} g_j. \tag{3.7}$$

Hence, the solution of problem (1.1)-(1.3) can be represented as

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t)\phi_j(x).$$

For fixed t and $\beta_{N(\epsilon)}$, let us define the operator $Q_{\beta_{N(\epsilon)}}(t) : L_2(D) \to L_2(D)$

$$Q_{\beta_{N(\epsilon)}}(t)\varphi = \sum_{\lambda_j \le \beta_{N(\epsilon)}} \mathcal{A}_j(t)\left\langle \varphi, \phi_j \right\rangle \phi_j.$$

Now, we sate and prove the following lemma, which is needed for our analysis.

Lemma 3.1 *For fixed $t \in [0,T]$, we have*

$$\left\|Q_{\beta_{N(\epsilon)}}(t)\varphi\right\| \le \sqrt{2 + \frac{8T^2}{\alpha^2}} e^{\alpha t \beta_{N(\epsilon)}} \|\varphi\|. \tag{3.8}$$

Moreover,

$$\left\|Q_{\beta_{N(\epsilon)}}(t)\varphi\right\|_{H^p(D)} \le \sqrt{\beta^p_{N(\epsilon)}\left(2 + \frac{8T^2}{\alpha^2}\right)} e^{\alpha t \beta_{N(\epsilon)}} \|\varphi\|. \tag{3.9}$$

Proof. Using Parseval's equality, we obtain

$$\left\|Q_{\beta_{N(\epsilon)}}(t)\varphi\right\|^2 = \sum_{\lambda_j \le \beta_{N(\epsilon)}} \mathcal{A}_j^2(t)\left\langle \varphi, \phi_j \right\rangle^2. \tag{3.10}$$

From the equality

$$\frac{\alpha j^2 + \sqrt{\alpha^2 j^4 - 4j^2}}{2} = \sqrt{\alpha^2 j^4 - 4j^2} + \frac{\alpha j^2 - \sqrt{\alpha^2 j^4 - 4j^2}}{2},$$

$\mathcal{A}_j(t)$ can be rewritten as

$$\begin{aligned}
\mathcal{A}_j(t) &= \frac{\frac{\alpha j^2+\sqrt{\alpha^2 j^4-4j^2}}{2}\exp\left(t\frac{\alpha j^2-\sqrt{\alpha^2 j^4-4j^2}}{2}\right) - \frac{\alpha j^2-\sqrt{\alpha^2 j^4-4j^2}}{2}\exp\left(t\frac{\alpha j^2+\sqrt{\alpha^2 j^4-4j^2}}{2}\right)}{\sqrt{\alpha^2 j^4-4j^2}} \\
&= \exp\left(t\frac{\alpha j^2 - \sqrt{\alpha^2 j^4 - 4j^2}}{2}\right) \\
&\quad + \left(\alpha j^2 - \sqrt{\alpha^2 j^4 - 4j^2}\right)\frac{\exp\left(t\frac{\alpha j^2-\sqrt{\alpha^2 j^4-4j^2}}{2}\right) - \exp\left(t\frac{\alpha j^2+\sqrt{\alpha^2 j^4-4j^2}}{2}\right)}{2\sqrt{\alpha^2 j^4-4j^2}}.
\end{aligned}$$

Applying the inequalities $(a+b)^2 \leq 2a^2 + 2b^2$ and $|e^a - e^b| \leq \max\{e^a, e^b\}|a-b|$, we obtain

$$
\begin{aligned}
\mathcal{A}_j^2(t) &\leq 2\exp\left[t\left(\alpha j^2 - \sqrt{\alpha^2 j^4 - 4j^2}\right)\right] + \frac{\left(\alpha j^2 - \sqrt{\alpha^2 j^4 - 4j^2}\right)^2 t^2 \exp\left[t\left(\alpha j^2 + \sqrt{\alpha^2 j^4 - 4j^2}\right)\right]}{2} \\
&\leq \exp\left(2\alpha j^2 t\right)\left(2 + \frac{8j^4 t^2}{\left(\alpha j^2 + \sqrt{\alpha^2 j^4 - 4j^2}\right)^2}\right) \\
&\leq \exp\left(2\alpha\beta_{N(\epsilon)} t\right)\left(2 + \frac{8T^2}{\alpha^2}\right). \qquad (3.11)
\end{aligned}
$$

From (3.10) and (3.11), we can deduce (3.8).
For the second statement of Lemma 3.1, from (3.8), we have

$$
\left\|Q_{\beta_{N(\epsilon)}}(t)\varphi\right\|^2_{H^p(D)} = \sum_{\lambda_j \leq \beta_{N(\epsilon)}} \lambda_j^p \mathcal{A}_j^2(t)\left\langle\varphi, \phi_j\right\rangle^2 \leq \beta^p_{N(\epsilon)} \exp\left(2\alpha\beta_{N(\epsilon)} t\right)\left(2 + \frac{8T^2}{\alpha^2}\right).
$$

The proof of Lemma 3.1 is complete.

Define the truncation operator $\mathbf{I}_{\beta_{N(\epsilon)}} : L^2(D) \to L^2(D)$ such that

$$
\mathbf{I}_{\beta_{N(\epsilon)}} v = \sum_{\lambda_j \leq \beta_{N(\epsilon)}} \langle v, \phi_j\rangle \phi_j, \quad \text{for all v} \in L^2(D).
$$

Next, we formulate a regularized problem for Problem (1.1)-(1.3) as follows

$$
\begin{cases}
W_{tt}^{\epsilon,N(\epsilon)} - \alpha W_{xxt}^{\epsilon,N(\epsilon)} - W_{xx}^{\epsilon,N(\epsilon)} = 0, & (x,t) \in D \times (0,T), \\
W^{\epsilon,N(\epsilon)}(0,t) = W^{\epsilon,N(\epsilon)}(\pi,t) = 0, & t \in (0,T), \\
W^{\epsilon,N(\epsilon)}(x,T) = \mathbf{I}_{\beta_{N(\epsilon)}} \overline{G}_{\epsilon,N(\epsilon)}(x), & (x,t) \in D \times (0,T), \\
W_t^{\epsilon,N(\epsilon)}(x,T) = 0, & (x,t) \in D \times (0,T),
\end{cases} \qquad (3.12)
$$

where $\beta_{N(\epsilon)}$ plays the role of regularization parameter. Using the same techniques as the beginning of this section for the problem (1.1)-(1.3), we can easily show that Problem (3.12) has a unique solution $W^{\epsilon,N(\epsilon)} \in C([0,T], L^2(D))$ given by

$$
W^{\epsilon,N(\epsilon)}(x,t) = Q_{\beta_{N(\epsilon)}}(T-t)\overline{G}_{\epsilon,N(\epsilon)}(x).
$$

We are now in position to state the main results of this paper.

Theorem 3.1 *Let us choose $N(\epsilon)$ and $\beta_{N(\epsilon)}$ such that*

$$
\lim_{\epsilon\to 0} N(\epsilon) = +\infty, \quad \lim_{\epsilon\to 0} \beta_{N(\epsilon)} = +\infty, \quad \lim_{\epsilon\to 0} \frac{e^{\alpha T \beta_{N(\epsilon)}}}{N^\gamma(\epsilon)} = 0, \quad \lim_{\epsilon\to 0} e^{\alpha T\beta_{N(\epsilon)}} \epsilon \sqrt{N(\epsilon)} = 0. \qquad (3.13)
$$

Assume that Problem (1.1)-(1.3) *has a unique solution $u \in H^{2\theta}(D)$ and $g \in H^\gamma(D)$ for $\theta, \gamma >$ 0. Then, there exists $C_0 = C_0(\alpha, T) > 0$ such that*

$$
\mathbf{E}\left\|u(\cdot,t) - W^{\epsilon,N(\epsilon)}(\cdot,t)\right\|^2 \leq C_0 e^{2\alpha T\beta_{N(\epsilon)}}\left(\epsilon^2 N(\epsilon) + \frac{1}{N^{2\gamma}(\epsilon)}\|g\|^2_{H^\gamma(D)}\right) + \frac{2\|u(\cdot,t)\|^2_{H^{2\theta}(D)}}{\beta^{2\theta}_{N(\epsilon)}}. \qquad (3.14)
$$

Remark 3.1 *We now give an example for the choice of* $N(\epsilon)$ *and* $\beta_{N(\epsilon)}$*, which satisfy the condition* (3.13). *Since* $\lambda_N(\epsilon) = N^2(\epsilon)$*, we can choose* $\beta_{N(\epsilon)}$ *such that* $e^{\alpha T \beta_{N(\epsilon)}} = N(\epsilon)^a$ *for* $0 < a < \gamma$*. Then, we have* $\beta_{N(\epsilon)} = \frac{a}{\alpha T} \ln(N(\epsilon))$*. The number* $N(\epsilon)$ *is chosen as follows*

$$N(\epsilon) = \left(\frac{1}{\epsilon}\right)^{\frac{2b}{2a+1}}$$

for $0 < b < 1$*. With* $N(\epsilon)$ *chosen as above,* $\mathbf{E}\left\|u(\cdot,t) - U^{\epsilon,N(\epsilon)}(\cdot,t)\right\|$ *is of order*

$$\frac{1}{\ln^{\theta}\left(\frac{1}{\epsilon}\right)}.$$

Proof of Theorem 3.1. It follows from (3.7) that

$$V^{\epsilon,N(\epsilon)}(x,t) := \mathbf{I}_{\beta_{N(\epsilon)}} u(x,t) = \sum_{\lambda_j \le \beta_{N(\epsilon)}} u_j(t)\phi_j(x) = Q_{\beta_{N(\epsilon)}}(T-t)g(x).$$

Applying the inequality $(a+b)^2 \le 2a^2 + 2b^2$, Parseval's equality, from (3.8) and Lemma 2.1, we have

$$\begin{aligned}
\mathbf{E}\left\|u(\cdot,t) - W^{\epsilon,N(\epsilon)}(\cdot,t)\right\|^2 &\le 2\left\|u(\cdot,t) - V^{\epsilon,N(\epsilon)}(\cdot,t)\right\|^2 + 2\mathbf{E}\left\|W^{\epsilon,N(\epsilon)}(\cdot,t) - V^{\epsilon,N(\epsilon)}(\cdot,t)\right\|^2 \\
&= 2\sum_{\lambda_j > \beta_{N(\epsilon)}} u_j^2(t) + 2\mathbf{E}\left\|Q_{\beta_{N(\epsilon)}}(T-t)\left(\overline{G}_{\epsilon,N(\epsilon)} - g\right)\right\|^2 \\
&\le 2\sum_{\lambda_j > \beta_{N(\epsilon)}} \lambda_j^{-2\theta}\lambda_j^{2\theta} u_j^2(t) + \left(4 + \frac{16T^2}{\alpha^2}\right) e^{2\alpha(T-t)\beta_{N(\epsilon)}} \mathbf{E}\left\|\overline{G}_{\epsilon,N(\epsilon)} - g\right\|^2 \\
&\le 2\beta_{N(\epsilon)}^{-2\theta}\|u(\cdot,t)\|^2_{H^{2\theta}(D)} + \left(4 + \frac{16T^2}{\alpha^2}\right) e^{2\alpha(T-t)\beta_{N(\epsilon)}} \left(\epsilon^2 N(\epsilon) + \frac{1}{N^{2\gamma}(\epsilon)}\|g\|^2_{H^{\gamma}(D)}\right).
\end{aligned}$$

From the above inequality, we can imply (3.14). This completes the proof of Theorem 3.1.
The next theorem gives the error estimate in the $H^p(D)$–norm.

Theorem 3.2 *Let us choose* $N(\epsilon)$ *and* $\beta_{N(\epsilon)}$ *such that*

$$\lim_{\epsilon\to 0} N(\epsilon) = +\infty, \quad \lim_{\epsilon\to 0} \beta_{N(\epsilon)} = +\infty, \quad \lim_{\epsilon\to 0} \frac{\sqrt{\beta^p_{N(\epsilon)}}\, e^{\alpha T \beta_{N(\epsilon)}}}{N^{\gamma}(\epsilon)} = 0, \quad \lim_{\epsilon\to 0} e^{\alpha T \beta_{N(\epsilon)}} \epsilon \sqrt{\beta^p_{N(\epsilon)} N(\epsilon)} = 0.$$

Suppose that the problem (1.1)-(1.3) *has a unique solution* $u \in H^{p+2\theta}(D)$ *and* $g \in H^{\gamma}(D)$ *for* $p, \gamma > 0$*. Then,*

$$\begin{aligned}
&\mathbf{E}\left\|W^{\epsilon,N(\epsilon)}(\cdot,t) - u(\cdot,t)\right\|^2_{H^p(D)} \\
&\le 2\beta_{N(\epsilon)}^{-2\theta}\|u(\cdot,t)\|^2_{H^{p+2\theta}(D)} + C_0 \beta^p_{N(\epsilon)} e^{2\alpha T \beta_{N(\epsilon)}} \left(\epsilon^2 N(\epsilon) + \frac{1}{N^{2\gamma}(\epsilon)}\|g\|^2_{H^{\gamma}(D)}\right).
\end{aligned} \tag{3.15}$$

Remark 3.2 *We give one example for the choice of* $N(\epsilon)$*. Choose* $\beta_{N(\epsilon)}$ *such that* $\beta_{N(\epsilon)} = \frac{a}{\alpha T}\ln(N(\epsilon))$ *for* $0 < a < \gamma$*. The number* $N(\epsilon)$ *is chosen as follows*

$$N(\epsilon) = \left(\frac{1}{\epsilon}\right)^{\frac{2b}{2a+1}}$$

for $0 < b < 1$*. Then,* $\mathbf{E}\left\|W^{\epsilon,N(\epsilon)}(\cdot,t) - u(\cdot,t)\right\|^2_{H^p(D)}$ *is of order*

$$\frac{1}{\ln^{\theta}\left(\frac{1}{\epsilon}\right)}.$$

Proof of Theorem 3.2. Indeed, using (3.9) and Lemma 2.1, we have

$$\begin{aligned}
&\mathbf{E}\left\|W^{\epsilon,N(\epsilon)}(\cdot,t)-u(\cdot,t)\right\|^2_{H^p(D)}\\
&\le 2\left\|u(\cdot,t)-V^{\epsilon,N(\epsilon)}(\cdot,t)\right\|^2_{H^p(D)}+2\mathbf{E}\left\|W^{\epsilon,N(\epsilon)}(\cdot,t)-V^{\epsilon,N(\epsilon)}(\cdot,t)\right\|^2_{H^p(D)}\\
&=2\sum_{\lambda_j>\beta_{N(\epsilon)}}\lambda_j^p u_j^2(t)+2\mathbf{E}\left\|Q_{\beta_{N(\epsilon)}}(T-t)\left(\overline{G}_{\epsilon,N(\epsilon)}-g\right)\right\|^2_{H^p(D)}\\
&\le 2\sum_{\lambda_j>\beta_{N(\epsilon)}}\lambda_j^{-2\theta}\lambda_j^{2\theta+p}u_j^2(t)+\left(4+\frac{16T^2}{\alpha^2}\right)\beta^p_{N(\epsilon)}e^{2\alpha(T-t)\beta_{N(\epsilon)}}\mathbf{E}\left\|\overline{G}_{\epsilon,N(\epsilon)}-g\right\|^2\\
&\le 2\beta^{-2\theta}_{N(\epsilon)}\|u(\cdot,t)\|^2_{H^{p+2\theta}(D)}+\left(4+\frac{16T^2}{\alpha^2}\right)\beta^p_{N(\epsilon)}e^{2\alpha(T-t)\beta_{N(\epsilon)}}\left(\epsilon^2N(\epsilon)+\frac{1}{N^{2\gamma}(\epsilon)}\|g\|^2_{H^\gamma(D)}\right),
\end{aligned}$$

from which we can deduce (3.15). This completes the proof.

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