Regularity for the second order Riesz transforms associated with Schrödinger operators on weighted BMO type spaces

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Abstract. Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, where V is a nonnegative potential satisfying the suitable reverse Hölder's inequality. In this paper, we study the boundedness of the second order Riesz transforms such as $\mathcal{L}^{-1}\nabla^2$ on the spaces of BMO type for weighted case. We generalized the known results to the weighted case.

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1 Introduction and preliminaries

The study of both function spaces and the boundedness of singular integral operators associated with Schrödinger operators arose from practical applications in some mathematical fields, such as harmonic analysis and partial differential equations, and has become an active area of research in the last few years. These type of questions have been already considered by many authors and they are very important questions which appeal to various techniques from partial differential equations and harmonic analysis.

In recent years, many authors have been interested in the problems of harmonic analysis associated with Schrödinger operators on $\mathbb{R}^n$, see [1–3, 5, 6, 12, 14–17, 19, 21, 25]. More specifically, boundedness of singular integral versions on stratified Lie group was also attracted the attention, see [12, 14–16]. Very recently, D. Yang developed the theory of localized non-weighted BMO spaces associated to admissible functions in spaces of homogeneous type and eastablished the resuts for the boundedness of the singular integrals on these space, see [22–24].

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Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, where the nonnegative potential V belongs to the reverse Hölder class RH_q for $q \geq \frac{n}{2}$, i.e., V satisfies the reverse Hölder's inequality: there exists $C > 0$ such that

$$\Big(\frac{1}{|B|}\int\limits_B V(x)^q dx\Big)^{\frac{1}{q}} \leq \frac{C}{|B|}\int\limits_B V(x)dx,$$

for all ball $B \subset \mathbb{R}^n$.

It is well-known that if $V \in RH_q$ for some $q > 1$, then there exists $\epsilon > 0$, such that $V \in RH_{q+\epsilon}$. We introduce the definition of the reverse Hölder index of V as $q_0 := \sup\{q : V \in RH_q\}$. Under assumption $V \in RH_m$ we may conclude $q_0 > m$.

We say $V \in RH_\infty$ if $V \in L^\infty_{loc}(\mathbb{R}^n)$ and

$$||V||_{L^\infty(B(x,r))} \leq \frac{C}{|B(x,r)|}\int\limits_{B(x,r)} V(y)dy,$$

hold for every $x \in \mathbb{R}^n$ and $0 < r < \infty$. For every $1 < p < \infty$, it is easy to see that $RH_\infty \subset RH_p$.

For a weight ω, we shall mean ω is a nonnegative measurable and locally integrable function on $\mathbb{R}^n$. For $p \in (1, \infty)$, we say that ω belongs to the Muckenhoupt class A_p if the following holds: there exists $C > 0$ such that

$$\Big(\int\limits_B \omega(x)dx\Big)\Big(\int\limits_B \omega^{-\frac{1}{p-1}}(x)dx\Big)^{p-1} \leq C|B|^p, \tag{1.1}$$

for all balls B in $\mathbb{R}^n$.

For $p = 1$, we say that $\omega \in A_1$ if there is a positive constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B \omega(y)dy \leq C\omega(x) \text{ for a.e. } x \in B.$$

We set $A_\infty = \cup_{p\geq 1} A_p$. For the ball $B = B(x, r)$ and $\lambda > 0$, we denote by λB the ball $B(x, \lambda r)$.

For $\alpha \geq 1$, we say that the weight ω satisfies the doubling property with the doubling order α if there exists a constant C such that

$$\omega(\lambda B) \leq C\lambda^{n\alpha}\omega(B),$$

for all balls $B \subset \mathbb{R}^n$ and all $\lambda > 1$, where $\omega(E) = \int_E \omega(x)dx$ for any measurable subset E of $\mathbb{R}^n$.

We then denote by D_α the set of all weights ω satisfying the doubling property with the doubling order α. It is important to note that if $\omega \in A_p$ for $p \in [1, \infty)$ then $\omega \in D_p$. Moreover, if $V \in RH_q$, then V belongs to a certain A_p class for some $p \in [1, \infty)$ and hence $V \in D_p$. See for example [20].

Associated to the potential V, the function of the critical radius is defined by

$$\rho(x) := \sup\Big\{r > 0 : \frac{1}{r^{n-2}}\int\limits_{B(x,r)} V(y)dy \leq 1\Big\},$$

for all $x \in \mathbb{R}^n$, see [16].

Let $\rho_1(x)$ be the function of the critical radius associated to $|\nabla V|$. The auxillary functions $\rho(x), \rho_1(x)$ play an important role in the problem of harmonic analysis related to Schrödinger operators.

Definition 1.1. *Let $0 \leq \beta < 1$ and w be a weight. The space $BMO^{\beta}_{\mathcal{L}}(\omega)$ is defined as the set of functions $f \in L^1_{loc}(\mathbb{R}^n)$ satisfying*

$$\int_B |f(y) - f_B|\, dy \leq C\omega(B)|B|^{\beta/n}, \tag{1.2}$$

for all the balls $B \subset \mathbb{R}^n$, with $r < \rho(x)$, where $f_B = \frac{1}{|B|}\int_B f(y)dy$, and

$$\int_B |f(y)|dy \leq C\omega(B)|B|^{\beta/n}, \tag{1.3}$$

for all balls $B = B(x, r)$, with $r \geq \rho(x)$, where C is a positive constant.

A norm in the space $BMO^{\beta}_{\mathcal{L}}(\omega)$ can be given by the maximum of the two infima of the constants in (1.2) and (1.3) respectively. This norm will be denoted by $\|\cdot\|_{BMO^{\beta}_{\mathcal{L}}(\omega)}$.

We now recall the definition of spaces $BMO^{\beta}(\omega)$ for $0 \leq \beta < 1$,

$$BMO^{\beta}(\omega) = \Big\{f \in L^1_{\text{loc}} : \sup_B \frac{1}{|B|^{\beta/n}\omega(B)} \int_B |f(x) - f_B|dx < \infty\Big\},$$

with the supremum taken over all balls B and $f_B = \frac{1}{|B|}\int_B f(y)dy$.

For brevity, if $\omega \equiv 1$, then we will denote $BMO^{\beta}_{\mathcal{L}}, BMO^{\beta}$ by $BMO^{\beta}_{\mathcal{L}}(\omega), BMO^{\beta}(\omega)$ respectively . When $\beta = 0$, we write $BMO_{\mathcal{L}}, BMO$ instead of $BMO^{\beta}_{\mathcal{L}}, BMO^{\beta}$ respectively.

Inspired by the recent studies on the boundedness of the Riesz transforms in $\mathbb{R}^n$, see for example [1–4, 7, 8] and the references therein. In the case $\omega \equiv 1$, it is well-known that if $V \in RH_q$ then the integral operators $V\mathcal{L}^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 \leq p \leq s$, and $\nabla^2\mathcal{L}^{-1}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \leq s'$, in [19], where s, s' depend only on q, n. The ranges of p in these results are optimal. This seminal results of Z. Shen play the crucial role in the boundedness of the Riesz transform on the other types functional space. When $\omega \equiv 1$, and $\beta = 0$, J. Dong and Y. Liu [8] proved $\mathcal{L}^{-1}\nabla^2$ is bounded on $BMO^{\beta}_{\mathcal{L}}$.

In this paper, we improve the results of J Dong and Y. Liu in [8] by studying the boundedness of the second order Riesz transform $R^* = \mathcal{L}^{-1}\nabla^2$ on the spaces $BMO^{\beta}_{\mathcal{L}}(\omega), BMO^{\beta}(\omega)$. The next theorems are the main results of this paper.

Theorem 1.2. *Suppose that $V \in RH_q$ for some $q > n, |\nabla V| \in RH_{n/2}$ such that $q_1^* \leq n, \rho \lesssim \rho_1 \lesssim 1$, where $q_1^* := \sup\{q : |\nabla V| \in RH_q\}$ is the reverse Hölder index of $|\nabla V|$. Let $w \in D_\alpha \cap \bigcup_{s>p'} \left(A_{p/s'} \cap RH_s\right)$ where $\frac{1}{p} = \frac{1}{q_1^*} - \frac{1}{n}$. For any $0 \leq \beta < \varpi := \min\{1 - n/q, 2 - n/q_1^*\}$ and $1 \leq \alpha < 1 + \frac{\varpi-\beta}{n}$, the operator $\mathcal{L}^{-1}\nabla^2$ is bounded from $BMO^{\beta}(\omega)$ into $BMO^{\beta}_{\mathcal{L}}(\omega)$.*

In [12], the authors gave some examples for the potential V which satisfy the assumption $\rho \lesssim \rho_1 \lesssim 1$. By Theorem 1.7 in [8], the authors proved that $\mathcal{L}^{-1}\nabla^2$ is bounded on the space $BMO_{\mathcal{L}}$. Note that, since $BMO^{\beta}_{\mathcal{L}}(\omega) \subset BMO^{\beta}(\omega)$, our results can be considered as improvements of those in [8], even in the case $\omega \equiv 1$.

The organization of this paper is as follows. In section 2, we give some preliminary results. Section 3 is devoted to the estimates on the size and smoothness of the kernels. Finally, the proof of the main theorems are also investigated in Section 4. Throughout this article, all the positive constants are signified as C although they may be different on the same line. Note that, ∇ means that we are taking all the partial derivaties with respect to the first variable. We write $A \sim B$ and $A \lesssim B$ if there exists some positive constants C, C' such that $C \leq \frac{A}{B} \leq C'$ and $A \leq C'B$, respectively.

2 Some preliminary results

Let $V \in RH_q$ where $q > n/2$. We begin with the results concerning the estimates on the potential V which are taken from [4, Lemma 1] and [19, Lemma 1.2], respectively.

Lemma 2.1. *The measure $V(x)dx$ satisfies the doubling condition, that is, there exists $C > 0$ such that*

$$\int_{B(x,2r)} V(y)dy \leq C \int_{B(x,r)} V(y)dy,$$

for all balls $B(x, r)$ in $\mathbb{R}^n$.

Lemma 2.2. *There exists $C > 0$ such that, for $0 < r < R < +\infty$,*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C\Big(\frac{r}{R}\Big)^{2-\frac{n}{q}} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

Lemma 2.3. *If $r = \rho(x)$, then*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy = 1.$$

Moreover,

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \sim 1 \quad \text{if and only if} \quad r \sim \rho(x).$$

Lemma 2.4. *There exist $C > 0$ and $l_0 > 0$ such that, for any x and y in G,*

$$\frac{1}{C}\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{\frac{l_0}{l_0+1}}.$$

In particular, $\rho(x) \sim \rho(y)$ if $|x - y| \lesssim \rho(x)$.

Lemma 2.5. *There exist $C > 0$ and $l_1 > 0$ such that*

$$\int_{B(x,R)} \frac{V(y)}{|x-y|^{n-2}} dy \leq \frac{C}{R^{n-2}} \int_{B(x,R)} V(y)dy \leq C\Big(1 + \frac{R}{\rho(x)}\Big)^{l_1}.$$

See [19] for the proofs of Lemmas 2.1 - 2.5.

Lemma 2.6. *Let $V \in RH_q$ with $q > n/2$ and $\epsilon > \frac{n}{q}$. Then, for any constant C_1, there exists a constant C_2 such that*

$$\int_{B(x,C_1 r)} \frac{V(u)}{|u-x|^{n-\epsilon}} du \leq C_2 r^{\epsilon-2} \Big(\frac{r}{\rho(x)}\Big)^{2-n/q},$$

if $0 < r \leq C_1\rho(x)$, *and*

$$\int_{B(x,C_1 r)} \frac{V(u)}{|u-x|^{n-\epsilon}} du \leq C_2 r^{\epsilon-2} \Big(\frac{r}{\rho(x)}\Big)^{2+(\mu-1)n},$$

if $r > C_1\rho(x)$, where μ is such that $V \in D_\mu$.

The proof of this above lemma is similar to that of Lemma 1 in [4]. The following results give other characterization of the space $BMO^{\beta}_{\mathcal{L}}(\omega)$, the proof of this proposition is very similar with a small modification to that of [4, Proposition 2] and [18, Theorem 2.2], respectively.

Proposition 2.1. *Let $0 \leq \beta < 1$ and a weight $\omega \in D_\alpha$ for some $\alpha \geq 1$. A function f belongs to $BMO^{\beta}_{\mathcal{L}}(\omega)$ if and only if condition (1.2) is satisfied for every ball $B = B(x,r)$ with $r < \rho(x)$ and*

$$\int_{B(x,\rho(x))} |f(y)|dy \leq C\omega(B(x,\rho(x)))|\rho(x)|^{\beta}, \tag{2.4}$$

for all $x \in \mathbb{R}^n$.

It follows from [18, Theorem 2.2], we have the following lemma.

Lemma 2.7. ***(John-Nirenberg inequality)***
Let $1 \leq p < \infty, 0 \leq \beta < 1, \omega \in A_p, 1 \leq r \leq p'$, and $r < \infty$. Then $f \in BMO^{\beta}(\omega)$ if and only if

$$\sup_B \frac{1}{|B|^{\beta/n}} \Big(\frac{1}{\omega(B)} \int_B |f(x) - f_B|^r \omega^{1-r}(x)dx\Big)^{1/r} < \infty, \tag{2.5}$$

and, moreover, this quantity gives an equivalent norm.

In what follows we denote $\mathcal{I}_1 = (-\Delta)^{-1/2}$ the classical fractional integral of order 1. To prove the main theorems, we need the following technical lemma.

Lemma 2.8. *Let $V \in RH_q$ with $n/2 < q < n$ and $\omega \in RH_s \cap A_{p/s'}$ for some $s > p'$, where $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$. There exists $C > 0$ such that for any $k > 0, 0 \leq \beta < 1, f \in BMO^{\beta}(\omega)$, and any ball $B(x,r)$,*

$$\begin{aligned}\int_{2^k B} |f(u) - f_B| \mathcal{I}_1(V\chi_{2^{k+2}B})(u)du \leq C(k+1)2^{k(n\eta+\beta-1)} \\ \times \|f\|_{BMO^{\beta}(\omega)} \omega(B) r^{\beta-1} \Phi_{\beta,\eta}\Big(\frac{2^k r}{\rho(x)}\Big),\end{aligned} \tag{2.6}$$

where

$$\Phi_{\beta,\eta}(t) = \begin{cases} t^{2+\mu n-n} & \text{if } t \geq 1, \\ t^{n-n\eta+2-n/q} & \text{if } t < 1, \end{cases}$$

for η and μ being the exponent of the doubling property satisfied by ω and V respectively.

Proof. We first apply Hölder's inequality to estimate the right-hand side of (2.6) by

$$\|(f - f_B)\chi_{2^k B}\|_{p'} \|\mathcal{I}_1(V\chi_{2^{k+2}B})\|_p.$$

To bound the first factor we apply again Hölder's inequality with exponent σ such that $\sigma p' = (p/s')' = v$ to the functions $|f - f_B|^{p'}\omega^{\frac{1}{\sigma}-p'}$ and $\omega^{p'-\frac{1}{\sigma}}$. It is easy to check that $(p' - \frac{1}{\sigma})\sigma' = s$ and $\frac{1}{\sigma' p'} = \frac{s'}{sp}$. Therefore,

$$\begin{aligned}\|(f - f_B)\chi_{2^k B}\|_{p'} &\leq \Big(\int_{2^k B} \omega^s(z)dz\Big)^{s'/sp}\Big(\int_{2^k B} |f(z) - f_B|^v \omega^{1-v}(z)dz\Big)^{1/v} \\ &\lesssim \frac{\omega(2^k B)^{s'/p}}{|2^k B|^{1/p}}\Big(\int_{2^k B} |f(z) - f_B|^v \omega^{1-v}(z)dz\Big)^{1/v}.\end{aligned}$$

By Lemma 2.7, we come up to

$$\begin{aligned}&\Big(\int_{2^k B} |f(z) - f_B|^v \omega^{1-v}(z)dz\Big)^{1/v} \\ &\leq \Big(\int_{2^k B} |f(z) - f_{2^k B}|^v \omega^{1-v}(z)dz\Big)^{1/v} + (\omega^{1-v}(2^k B))^{1/v}\sum_{j=0}^{k-1}|f_{2^{j+1}B} - f_{2^j B}| \\ &\lesssim \omega(2^k B)^{1/v}|2^k B|^{\beta/n}\|f\|_{BMO^\beta(\omega)} + |2^k B|\omega(2^k B)^{-1/v'} \\ &\times \sum_{j=0}^{k-1}\frac{1}{|2^j B|}\int_{2^{j+1}B}|f(z) - f_{2^{j+1}B}|dz.\end{aligned}$$

We then obtain that

$$\begin{aligned}&\Big(\int_{2^k B} |f(z) - f_B|^v \omega^{1-v}(z)dz\Big)^{1/v} \\ &\lesssim \omega(2^k B)^{1/v}|2^k B|^{\beta/n}\|f\|_{BMO^\beta(\omega)} + |2^k B|\omega(2^k B)^{-1/v'}\sum_{j=0}^{k-1}\omega(2^{j+1}B) \\ &\times 2^{j+1}B|^{\beta/n-1}\|f\|_{BMO^\beta(\omega)}.\end{aligned}$$

This implies that

$$\begin{aligned}&\Big(\int_{2^k B} |f(z) - f_B|^v \omega^{1-v}(z)dz\Big)^{1/v} \\ &\lesssim \omega(2^k B)^{1/v}\|f\|_{BMO^\beta(\omega)}\Big[|2^k B|^{\beta/n} + |2^k B||B|^{\beta/n-1}\sum_{j=0}^{k-1}2^{j(\beta-n)}\Big] \\ &\lesssim \omega(2^k B)^{1/v}\|f\|_{BMO^\beta(\omega)}\Big[|2^k B|^{\beta/n} + k|2^k B||B|^{\beta/n-1}\Big] \\ &\lesssim (k+1)\omega(2^k B)^{1/v}2^{kn}|B|^{\beta/n}.\end{aligned}$$

By combining the above estimates, we obtain

$$\|(f - f_B)\chi_{2^k B}\|_{p'} \lesssim (k+1)2^{kn(1/p'+\eta)}r^{\beta-n/p}\omega(B)\|f\|_{BMO^\beta(\omega)}. \tag{2.7}$$

On the other hand, due to the boundedness of $\mathcal{I}_1$ and $V \in RH_q$, we have

$$\|\mathcal{I}_1(V\chi_{2^{k+2}B})\|_p \lesssim \|V\chi_{2^{k+2}B}\|_q \lesssim \frac{V(2^{k+2}B)}{|2^{k+2}B|^{1/q'}},$$

where $V(2^{k+2}B) = \int_{2^{k+2}B} V(y)dy.$

In the case $2^k r \geq \rho(x_0)$, we apply the second part of Lemma 2.6 to get

$$\|\mathcal{I}_1(V\chi_{2^{k+2}B})\|_p \lesssim (2^k r)^{n-2-n/q'}\Big(\frac{2^k r}{\rho(x)}\Big)^{2+(\mu-1)n}.$$

Combining the above estimates we arrive to (2.6). The case $2^k r < \rho(x)$, is handled similarly by using the first part of Lemma 2.6. □

We have the following simple result.

Lemma 2.9. *For $\alpha > 0$, we have* $\displaystyle\int_{B(x_0,r)} \frac{1}{|x_0 - z|^{n-\alpha}} dz \lesssim r^\alpha.$

Proof. Proof of this lemma is straightforward and we omit details here. □

3 Some estimates for the kernels

Let $\Gamma(g, h), \Gamma_0(g, h)$ denote the fundamental solution for the operator $-\Delta + V, -\Delta$ respectively. The following estimates of the fundamental solution for the Schrödinger operator on the nilpotent Lie group have been proved in [16].

Lemma 3.1. *Let $l > 0$ be an integer.*

1. *Suppose $V \in RH_{n/2}$. Then there exists $C_l > 0$ such that for $x \neq y$,*

$$|\Gamma(x, y)| \leq \frac{C_l}{(1 + |x - y|)^l (1 + |x - y|\rho(x)^{-1})^l} \frac{1}{|x - y|^{n-2}}.$$

2. *Suppose $V \in RH_n$. Then there exists $C_l > 0$ such that for $x \neq y$,*

   (a) $$|\nabla\Gamma(x, y)| + |\nabla_2\Gamma(x, y)| \leq \frac{C_l}{(1 + |x - y|)^l (1 + |x - y|\rho(x)^{-1})^l} \frac{1}{|x - y|^{n-1}},$$

   (b) $$|\nabla\nabla_2\Gamma(x, y)| \leq \frac{C_l}{(1 + |x - y|)^l (1 + |x - y|\rho(x)^{-1})^l} \frac{1}{|x - y|^n},$$

   where ∇, ∇_2 means that we are taking all the partial derivaties with respect to the first and second variable, respectively.

Next, we recall the imbedding theorem of Morrey.

Lemma 3.2. *(The imbedding theorem of Morrey)*
Let the ball $B = B(a, R) \subset \mathbb{R}^n$ and $u \in W_0^{1,p}(B)$ with $p > n$. Then

$$|u(x) - u(y)| \leq CR^\gamma \|(\nabla u)\chi_B\|_p, \qquad \textit{for all} \qquad x, y \in B,$$

where $C = C(n, p), \gamma = 1 - n/p$.

The following results give the estimate and the connection between the fundamental solutions of the Schrödinger operator $\mathcal{L}$ and the Laplacian $-\Delta$. The proofs of these lemmas are similar to that in [12, 13].

Lemma 3.3. *Suppose that $V \in RH_q$ for some $q > n$ and $|\nabla V| \in RH_{q_1}$ for some $q_1 > \frac{n}{2}$. There exists a constant $C_l > 0$ for each $l > 0$,*

$$|\nabla^2\Gamma(x,y)| \leq \frac{C_l}{\left(1+|x-y|\rho(x)^{-1}\right)^l}\frac{1}{|x-y|^{n-2}} \times\Big(\int_{B(x,2|x-y|)}\frac{|\nabla V(z)|}{|x-z|^{n-1}}dz+\frac{1}{|x-y|^2}\Big).$$

Lemma 3.4. *Suppose that $V \in RH_q$ for some $q > n$ and $|\nabla V| \in RH_{q_1}$ for some $q_1 > \frac{n}{2}$. Then*

$$|\nabla^2\Gamma(x,y) - \nabla^2\Gamma_0(x,y)| \lesssim \frac{1}{|x-y|^{n-2}}\int_{B(x,|x-y|)}\frac{|\nabla V(z)|}{|x-z|^{n-1}}dz + \frac{1}{|x-y|^{n-1}}\Big(\frac{|x-y|}{\rho_1(y)}\Big)^{\delta_1} + \frac{1}{|x-y|^n}\Big(\frac{|x-y|}{\rho(y)}\Big)^{\delta},$$

where $\delta = 2 - n/q$ and $\delta_1 = 2 - n/q_1$, if $|x-y| \leq A\rho_1(x)$ for some positive constant A.

Lemma 3.5. *Suppose that $V \in RH_q$ for some $q > n$ and $|\nabla V| \in RH_{q_1}$ for some $q_1 > \frac{n}{2}$. There exists a constant $C_l > 0$ for each $l > 0$,*

$$|\nabla^2\Gamma(x,y+h) - \nabla^2\Gamma(x,y)| \leq \frac{C_l}{\left(1+|x-y|\rho(x)^{-1}\right)^l}\frac{|h|^\delta}{|x-y|^{n-2+\delta}} \times\Big(\int_{B(x,|x-y|)}\frac{|\nabla V(z)|}{|x-z|^{n-1}}dz+\frac{1}{|x-y|^2}\Big),$$

where $\delta = 1 - n/t$ for $t > n$.

Lemma 3.6. *Suppose $V \in RH_q$ for some $q > n/2$. Then, for any integer $k > 0$,*

$$|\Gamma(x,u) - \Gamma(y,u)| \lesssim \frac{|x-y|^\delta}{|x-u|^{n-2+\delta}}\Big(1+\frac{|x-u|}{\rho(u)}\Big)^{-k},$$

for $|x-y| < \frac{2}{3}|x-u|$ and $0 < \delta < \min\{1, 2-\frac{n}{q}\}$.

Lemma 3.7. *If $V \in RH_q$ for some $q > n$ then*

$$\|\nabla^2\Gamma(x,y)\|_{L^q(B(x_0,R))} \leq \frac{C_k}{R^{n-n/q}}\Big(1+\frac{R}{\rho(x_0)}\Big)^{k_0-k},$$

for any integer $k > 0$.

We will prove the following lemma.

Lemma 3.8. *Suppose $V \in RH_q$ for $q > n$. Then, for any integer $k > 0$,*

$$|\nabla\Gamma(x,u) - \nabla\Gamma(y,u)| \lesssim \frac{|x-y|^\delta}{|x-u|^{n-1+\delta}}\Big(1+\frac{|x-u|}{\rho(x)}\Big)^{-k},$$

for $|x-y| < c|x-u|$, $0 < \delta < \min\{1, 1-\frac{n}{q}\}$, and $c \in (0,1)$.

Proof. Fix $x, u \in \mathbb{R}^n$. Set $R = c|x-u|$. Then for every $h \in \mathbb{R}^n$ ($|h| < c|x-u|$), it follows from the imbedding theorem of Morrey and Lemma 3.7 that

$$\begin{aligned}|\nabla\Gamma(x+h,u) - \nabla\Gamma(x,u)| &\le C|h|^{1-n/q}\Big(\int_{B(x,R)} |\nabla^2\Gamma(z,u)|^q dz\Big)^{1/q}\\ &\le C_k\Big(\frac{|h|}{R}\Big)^{1-n/q}\frac{1}{R^{n-1}}\Big(1+\frac{R}{\rho(x)}\Big)^{-k_1}\\ &= \frac{C|h|^\delta}{|x-u|^{n-1+\delta}}\Big(1+\frac{|x-u|}{\rho(x)}\Big)^{-k}.\end{aligned}$$

This completes the our proof. □

We denote by $\mathcal{K}^*$ and $\mathbf{K}^*$ the associated kernels of $\mathcal{R}^* = \nabla^2(-\Delta)$ and R^*, respectively. The following result will play an important role in the sequel.

Lemma 3.9. *Let $V \in RH_q$ with $q > n$ such that $|\nabla V| \in RH_{q_1}$ with $q_1 > n/2$ and $0 < \delta < \min\{1-\frac{n}{q}, 2-\frac{n}{q_1}\}$. Suppose that $\rho \lesssim \rho_1 \lesssim 1$. Then, there exists a positive constant C such that*

$$\begin{aligned}&\Big|[\mathbf{K}^*(x,z) - \mathcal{K}^*(x,z)] - [\mathbf{K}^*(y,z) - \mathcal{K}^*(y,z)]\Big|\\ &\lesssim \frac{|x-y|^\delta}{|x-z|^{n+\delta}}\Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q} + \frac{|x-y|^\delta}{|x-z|^{n-1+\delta}}\Big(\frac{|x-z|}{\rho_1(x)}\Big)^{2-n/q_1}\\ &+ \frac{|x-y|^\delta}{|x-z|^{n-2+\delta}}\int_{B(z,3|x-z|)} \frac{|\nabla V(u)|}{|z-u|^{n-1}}du,\end{aligned} \tag{3.8}$$

whenever $|x-z| \ge 2|x-y|$ and $|x-y| \le A\rho(x)$ for some positive constant A.

Proof. We consider two cases.

Case 1. $|x-z| \ge \rho(x)/2$.

By Lemma 3.5 the lemma is proved .

Case 2. $|x-z| < \rho(x)/2$.

Note that $(-\Delta + V)\Gamma(x,y) = 0$. By standard arguments, we come up to

$$-\Delta(\nabla\Gamma) = -\nabla(V\Gamma).$$

Moreover, using the fact that $\Delta(\nabla\Gamma_0) = 0$, we have

$$\begin{aligned}(\nabla\Gamma - \nabla\Gamma_0)(x,z) &= -(-\Delta)^{-1}\nabla(V\Gamma)(x,z)\\ &= -\int_{\mathbb{R}}^{n} \Gamma_0(x,u)(\nabla V(u))\Gamma(u,z)du\\ &- \int_{\mathbb{R}}^{n} \Gamma_0(x,u)V(u)(\nabla\Gamma(u,z))du\end{aligned}$$

Since $\Gamma(u,x)=\Gamma(x,u)$, we obtain

$$\begin{aligned}
&[\mathbf{K}^*(x,z)-\mathcal{K}^*(x,z)]-[\mathbf{K}^*(y,z)-\mathcal{K}^*(y,z)]\\
&=-\int_{\mathbb{R}}^{n}\nabla\Gamma_0(z,u)(\nabla V(u))[\Gamma(x,u)-\Gamma(y,u)]du\\
&\quad-\int_{\mathbb{R}}^{n}\nabla\Gamma_0(z,u)V(u)[\nabla\Gamma(x,u)-\nabla\Gamma(y,u)]du\\
&=-(A_1+A_2).
\end{aligned}$$

We claim that

$$|A_2|\le\frac{C|x-y|^\delta}{|x-z|^{n+\delta}}\Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q}. \tag{3.9}$$

To show this, we split $\mathbb{R}^n$ into 4 regions:

$$E_1=\left\{u:|x-u|<\frac{3}{2}|x-y|\right\},$$

$$E_2=\left\{u:\frac{3}{2}|x-y|\le|x-u|<\frac{1}{2}|x-z|\right\},$$

$$E_3=\left\{u:\frac{1}{2}|x-z|\le|x-u|<2|x-z|\right\},$$

$$E_4=\{u:|x-u|\ge 2|x-z|\}.$$

Write $I_j=\int_{E_j}|\nabla\Gamma_0(z,u)|V(u)\,|\nabla\Gamma(x,u)-\nabla\Gamma(y,u)|\,du$ for $j=1,2,3,4$.
We have

$$\begin{aligned}
I_1&\le\int_{E_1}|\nabla\Gamma_0(z,u)\nabla\Gamma(x,u)|\,V(u)du+\int_{E_1}|\nabla\Gamma_0(z,u)\nabla\Gamma(y,u)|\,V(u)du\\
&:=J_1+J_2.
\end{aligned}$$

We only estimate J_1 here. The same proof works for J_2.

Since $|x-z|\ge 2|x-y|$ implies $|z-u|\ge\frac{1}{4}|x-z|$ for all $u\in E_1$, and since $\delta<1-n/q$, by using Lemma 2.6 with $\epsilon=1-\delta$ and $r=|x-z|<\frac{\rho(x)}{2}$, we have

$$\begin{aligned}
\int_{E_1}|\nabla\Gamma_0(z,u)\nabla\Gamma(x,u)|V(u)du&\le\frac{C}{|x-z|^{n-1}}\int_{B(x,2|x-y|)}\frac{V(u)}{|x-u|^{n-1}}du\\
&\le\frac{C|x-y|^\delta}{|x-z|^{n-1}}\int_{B(x,|x-z|)}\frac{V(u)}{|x-u|^{n-1+\delta}}du\\
&\le\frac{C|x-y|^\delta}{|x-z|^{n+\delta}}\Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q}.
\end{aligned}$$

Using Lemma 3.8 yields $I_j\lesssim\int_{E_j}\frac{V(u)}{|u-z|^{n-1}}\frac{|x-y|^\delta}{|x-u|^{n-1+\delta}}\Big(1+\frac{|x-u|}{\rho(x)}\Big)^{-k}du$ for $j=2,3,4$.

Note that $|x-z| \geq 2|x-y|$ implies $|u-z| \geq \frac{1}{2}|x-z|$ for $u \in E_2$. It follows from Lemma 2.6 with $\epsilon = 1-\delta$ and $r = \frac{1}{2}|x-z|$ that

$$I_2 \leq \frac{C|x-y|^\delta}{|x-z|^{n-1}} \int_{B(x,\frac{|x-z|}{2})} \frac{V(u)}{|x-u|^{n-1+\delta}} du \leq \frac{C|x-y|^\delta}{|x-z|^{n+\delta}} \Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q}.$$

Noticing that $E_3 \subset B(z, 3|x-z|)$, we observe that

$$I_3 \leq \frac{C|x-y|^\delta}{|x-z|^{n-1+\delta}} \int_{B(z,3|x-z|)} \frac{V(u)}{|u-z|^{n-1}} du \leq \frac{C|x-y|^\delta}{|x-z|^{n+\delta}} \Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q},$$

where we have used Lemma 2.6 in the last inequality with $\epsilon = 1$, $r = |x-z|$ and the fact that $\rho(z) \sim \rho(x)$.

Finally, noticing $|x-u| \sim |u-z|$ if $u \in E_4$, we ensures that

$$I_4 \leq C|x-y|^\delta \int_{E_4} \frac{V(u)}{|x-u|^{2n-2+\delta}} \left[1 + \frac{|x-u|}{\rho(x)}\right]^{-k} du.$$

Set $F_1 = \{u \in \mathbb{R}^n : 2|x-z| \leq |x-u| < \rho(x)\}$ and $F_2 = \{u \in \mathbb{R}^n : |x-u| \geq \rho(x)\}$.

Then $E_4 = F_1 \cup F_2$. Hence, we can get estimate

$$\begin{aligned}
&\int_{F_1} \frac{V(u)}{|x-u|^{2n-2+\delta}} \left[1 + \frac{|x-u|}{\rho(x)}\right]^{-k} du \\
&\leq \Big(\int_{B(x,\rho(x))} V^q(u) du\Big)^{1/q} \Big(\int_{|x-u|\geq 2|x-z|} \frac{1}{|x-u|^{(2n-2+\delta)q'}} du\Big)^{1/q'} \\
&\leq \rho(x)^{n/q-n} \int_{B(x,\rho(x))} V(u) du \\
&\times \Big(\sum_{i=1}^{\infty} \int_{2^i|x-z|\leq|x-u|\leq 2^{i+1}|x-z|} \frac{1}{|x-u|^{(2n-2+\delta)q'}} du\Big)^{1/q'} \\
&\leq \frac{C}{|x-z|^{n+\delta}} \Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q},
\end{aligned}$$

where we have used the definition of $\rho(x)$ in the last inequality.

Taking into account of $n + \delta \geq 2 - n/q$, for sufficiently large k leads us to

$$\begin{aligned}
&\int_{F_2} \frac{V(u)}{|x-u|^{2n-2+\delta}}\Big[1 + \frac{|x-u|}{\rho(x)}\Big]^{-k} du \\
&\leq \int_{|x-u|\geq\rho(x)} \frac{V(u)}{|x-u|^{2n-2+\delta}}\Big(\frac{\rho(x)}{|x-u|}\Big)^k du \\
&\leq \rho(x)^{-2n+2-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-2+\delta+k)} \int_{|x-u|<2^j\rho(x)} V(u)du \\
&\lesssim \rho(x)^{-2n+2-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-2+\delta+k-Q\mu)} \int_{|x-u|<\rho(x)} V(u)du \\
&\lesssim \rho(x)^{-n-\delta}\Big(\frac{\rho(x)}{|x-z|}\Big)^{n+\delta-2+n/q} \\
&\lesssim \frac{1}{|x-z|^{n+\delta}}\Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q},
\end{aligned}$$

where we have used that V belongs to D_μ for some $\mu \geq 1$ in the third inequality.

Therefore, we imply that

$$I_4 \leq \frac{C|x-y|^\delta}{|x-z|^{n+\delta}}\Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q}.$$

Write $J_i = \int_{E_i} |\nabla\Gamma_0(z,u)|(\nabla V(u))[\Gamma(x,u) - \Gamma(y,u)]du$ for $i = 1, 2, 3, 4$. For J_1, we majorize the difference related to Γ by the sum of the absolute values of each term and estimate each integral separately. Since both are similar we work out one of them. First we notice that $|x-z| > 2|x-y|$ implies $|z-u| > \frac{1}{4}|x-z|$ for $u \in E_1$. Then, using Lemma 3.1, we deduce that

$$\begin{aligned}
\int_{E_1} |\nabla\Gamma_0(z,u)\Gamma(x,u)||\nabla V(u)|du &\lesssim \frac{1}{|x-z|^{n-1}} \int_{B(x,2|x-y|)} \frac{|\nabla V(u)|}{|x-u|^{n-2}} du \\
&\lesssim \frac{|x-y|^\delta}{|x-z|^{n-1+\delta}}\Big(\frac{|x-z|}{\rho_1(x)}\Big)^{2-n/q_1},
\end{aligned}$$

where in the last inequality we have used Lemma 2.6 with $\epsilon = 2 - \delta$ and $r = |x-y| < 2\rho(x)$.

For the remaining regions we will use Lemma 3.6. To estimate J_2 we use Lemma 3.6 to get

$$J_2 \lesssim \frac{|x-y|^\delta}{|x-z|^{n-1}} \int_{B(x,\frac{1}{2}|x-z|)} \frac{|\nabla V(u)|}{|x-u|^{n-2+\delta}} du \lesssim \frac{|x-y|^\delta}{|x-z|^{n-1+\delta}}\Big(\frac{|x-z|}{\rho_1(x)}\Big)^{2-n/q_1},$$

where in the last inequality we have used Lemma 2.6 with $\epsilon = 2 - \delta$ and $r = \frac{1}{2}|x-z|$.

To deal with J_3 we notice $E_3 \subset B(z, 3|x-z|)$. Using again Lemma 3.6 we arrive to

$$J_3 \lesssim \frac{|x-y|^\delta}{|x-z|^{n-2+\delta}} \int_{B(z,3|x-z|)} \frac{|\nabla V(u)|}{|u-z|^{n-1}} du.$$

Finally, for $u \in E_4$ we have $|x-u| \sim |u-z|$ and hence, using Lemma 3.6

$$J_4 \leq |x-y|^{\delta} \int_{E_4} \frac{|\nabla V(u)|}{|x-u|^{2n-3+\delta}} \Big[1 + \Big(\frac{|x-u|}{\rho(u)}\Big)\Big]^{-k} du.$$

We set $E_4 = E_4^1 \cup E_4^2$, where $E_4^1 = \{u : 2|x-z| \leq |x-u| \leq \rho_1(x)\}$. Applying Hölder's inequality the above integral over E_4^1 is bounded by

$$\Big(\int_{B(x,\rho_1(x))} |\nabla V|^{q_1}\Big)^{1/q_1} \Big(\int_{|x-u|>2|x-z|} \frac{1}{|x-u|^{(2n-3+\delta)q_1'}} du\Big)^{1/q_1'}$$
$$\leq \frac{C}{|x-z|^{n-1+\delta}} \Big(\frac{|x-z|}{\rho_1(z)}\Big)^{2-n/q_1},$$

where in the last inequality we have used the reverse Hölder condition on $|\nabla V|$ and the definition of ρ_1.

To estimate the integral on E_4^2, we have

$$\rho(u) \leq C\rho(x)^{1-\sigma}|x-u|^{\sigma},$$

with $0 < \sigma < 1$. Therefore, we set $N = k(1-\sigma)$ to get

$$\int_{|x-u|>\rho_1(x)} \frac{|\nabla V(u)|}{|x-u|^{2n-3+\delta}} \Big(\frac{\rho_1(x)}{|x-u|}\Big)^{N} du$$
$$\leq \rho_1(x)^{-n+3-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-3+\delta+N)} \int_{|x-u|<2^j\rho_1(x)} |\nabla V|.$$

Since $|\nabla V|$ satisfies a doubling condition and we can choose k large enough, we have

$$\int_{|x-u|>\rho_1(x)} \frac{|\nabla V(u)|}{|x-u|^{2n-3+\delta}} \Big(\frac{\rho_1(x)}{|x-u|}\Big)^{N} du$$
$$\leq \rho_1(x)^{-n+3-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-3+\delta+N)} \int_{|x-u|<2^j\rho_1(x)} |\nabla V|$$
$$\lesssim \rho_1(x)^{-2n+3-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-3+\delta+N-n\mu)} \int_{|x-u|<\rho_1(x)} |\nabla V(u)| du$$
$$\lesssim \rho_1(x)^{-n+1-\delta} \lesssim \frac{1}{|x-z|^{n-1+\delta}} \Big(\frac{|x-z|}{\rho_1(z)}\Big)^{2-n/q_1},$$

where we have use that $|\nabla V|$ belongs to D_{μ} for some $\mu \geq 1$ and $n-1+\delta \geq 2-n/q_1$.

Now using the estimates in E_4^1 and E_4^2 reminding that $|x-z| \leq \rho_1(x)$, we obtain

$$J_4 \leq \frac{|x-y|^{\delta}}{|x-z|^{n-1+\delta}} \Big(\frac{|x-z|}{\rho_1(x)}\Big)^{2-n/q_1}$$

The proof is completed. □

4 Proofs of the main results

First, we prove the following theorem.

Theorem 4.1. *For any* $0 \leq \beta < 1, 1 \leq \alpha < 1 + \frac{1-\beta}{n}$, *and* $\omega \in A_\infty \cap D_\alpha$, *the operator* $\mathcal{R}^* = (-\Delta)^{-1}\nabla^2$ *is bounded on* $BMO^\beta(\omega)$.

Proof. For $B := B(x_0, r)$, set $g := f - f_{5B}$. We write $g = g_1 + g_2$, with $g_1 = g\chi_{5B}$.

Since $\mathcal{R}^*(f_{5B}) = 0$, we have $\mathcal{R}^*(f) = \mathcal{R}^*(g)$. Since $\omega \in A_\infty$, we have $\omega \in A_p$ for some $1 < p < \infty$ and hence $\omega^{1-p'} \in A_{p'}$. Since $\mathcal{R}^*$ is a Calderón-Zygmund operator, we have

$$\int_B |\mathcal{R}^* g(x) - (\mathcal{R}^* g)_B| dx$$
$$\leq \int_B |\mathcal{R}^* g_1(x) - (\mathcal{R}^* g_1)_B| dx + \int_B |\mathcal{R}^* g_2(x) - (\mathcal{R}^* g_2)_B| dx = I_1 + I_2.$$

By the boundedness of $\mathcal{R}^*$ and the John-Nirenberg's inequality, we have

$$I_1 \leq 2\int_B |\mathcal{R}^* g_1| \leq \omega(B)^{1/p}\Big(\int_B |\mathcal{R}^* g_1|^{p'} \omega^{1-p'}\Big)^{1/p'}$$
$$\leq C\omega(B)^{1/p}\Big(\int_{5B} |f - f_{5B}|^{p'} \omega^{1-p'}\Big)^{1/p'} \lesssim \|f\|_{BMO^\beta(\omega)}\omega(B)|B|^{\beta/n}.$$

On the other hand, observe that

$$I_2 \leq \frac{1}{|B|}\int_B\int_B\int_{(5B)^c} |\mathcal{K}^*(x-z) - \mathcal{K}^*(y-z)||f(z) - f_{5B}| dz dx dy.$$

Applying Proposition 1.7 in [9], we come up to

$$|\mathcal{K}^*(x-y) - \mathcal{K}^*(x)| \leq \frac{C|y|}{|x|^{n+1}},$$

for $|y| \leq \frac{|x|}{2}$.

Note that $|x - z| \geq 2|x - y|$ for $x, y \in B$ and $z \in (5B)^c$. Therefore, we have

$$I_2 \leq \frac{C}{|B|}\int_B\int_B\int_{(5B)^c} \frac{|x-y|}{|x-z|^{n+1}}|f(z) - f_{5B}| dz dx dy$$
$$\leq Cr^{n+1}\int_{(5B)^c} \frac{|f(z) - f_{5B}|}{|x_0 - z|^{n+1}} dz$$
$$\leq C\sum_{j=2}^{\infty} \frac{1}{2^{j(n+1)}}\int_{2^{j+1}B} |f(z) - f_{5B}| dz$$
$$\leq Cr^\beta \|f\|_{BMO^\beta(\omega)}\omega(B)\sum_{j=2}^{\infty}(j+1)2^{-j(n+1-\beta-n\alpha)}.$$

The last series converges provided that $\alpha < 1 + \dfrac{1-\beta}{n}$. Combining these, we obtain the desired result. □

Proof of Theorem 1.2. Let $s > p'$ such that $\omega \in A_{p/s'} \cap RH_s$. We choose q_1 satisfying

$$n/2 < q_1 < q_1^* \leq n, 0 \leq \beta < \delta < \min\{1 - \frac{n}{q}, 2 - \frac{n}{q_1}\}, 1 \leq \alpha < 1 + \frac{\delta - \beta}{n}$$

and such that $\omega \in A_{p_1/s'}$ for $\frac{1}{p_1} = \frac{1}{q_1} - \frac{1}{n}$.

Let $f \in BMO^\beta(\omega)$. By Proposition 2.1, we need to prove that

$$\int_B |R^* f| \lesssim \|f\|_{BMO^\beta(\omega)} \omega(B)|B|^{\beta/n}, \tag{4.10}$$

for all $B = B(x_0, \rho(x_0)), x_0 \in \mathbb{R}^n$, and

$$\int_B |R^* f - (R^* f)_B| \lesssim \|f\|_{BMO^\beta(\omega)} \omega(B)|B|^{\beta/n}, \tag{4.11}$$

with $B = B(x_0, r), r < \rho(x_0)$.

First, we prove (4.10). Let $B = B(x_0, \rho(x_0))$, set $g = f - f_B$. We write $g = g_1 + g_2$, with $g_1 = g\chi_{2B}$. Since $R^*(f_B) = 0, R^*(f) = R^*(g)$. Since R^* is bounded in $L^{p'}$, (2.7) yields that

$$\begin{aligned}\int_B |R^* g_1| &\leq |B|^{1/p}\Big(\int_B |R^* g_1|^{p'}\Big)^{1/p'} \leq C|B|^{1/p}\Big(\int_{2B} |f - f_B|^{p'}\Big)^{1/p'} \\ &\lesssim \|f\|_{BMO^\beta(\omega)}|B|^{\beta/n}\omega(B).\end{aligned}$$

For g_2 we estimate the size of $\mathbf{K}^*$ using Lemma 3.3. Now, note that for $x \in B$ and $z \in \mathbb{R}^n \setminus 2B, \rho_1(x) \sim \rho_1(x_0), |x_0 - z| \sim |x - z|$, we have

$$\begin{aligned}\int_B |R^* g_2| &\leq \int_B \int_{(2B)^c} |\mathbf{K}^*(x,z)||f(z) - f_B| dz dx \\ &\leq C_k \int_B \int_{(2B)^c} \Big(\frac{\rho(x)}{|x-z|}\Big)^k \frac{1}{|x-z|^n} |f(z) - f_B| dz dx \\ &+ C_k \int_B \int_{(2B)^c} \rho(x)^k \Big(\int_{B(z,2|x-z|)} \frac{|\nabla V(u)|}{|u-z|^{n-1}} du\Big) \frac{|f(z) - f_B|}{|x-z|^{k+n-2}} dz dx \\ &= I_1 + I_2.\end{aligned}$$

We have

$$I_1 \leq C_k \rho(x_0)^{k+n} \int_{(2B)^c} \frac{|f(z) - f_B|}{|x_0 - z|^{k+n}} dz.$$

Note that

$$\begin{aligned}\int_{2^jB}|f-f_B| &\le \int_{2^jB}|f-f_{2^jB}| + |2^jB|\sum_{l=0}^{j-1}|f_{2^{l+1}B-f_{2^lB}}| \\ &\le \omega(2^jB)|2^jB|^{\beta/n}\|f\|_{BMO^\beta(\omega)} \\ &+ |2^jB|\sum_{l=0}^{j-1}|2^lB|^{\beta/n-1}\omega(2^lB)\|f\|_{BMO^\beta(\omega)} \\ &\le 2^{j(n\alpha+\beta)}\omega(B)|B|^{\beta/n}\|f\|_{BMO^\beta(\omega)} \\ &+ 2^{jn}|B|^{\beta/n}\omega(B)\sum_{l=0}^{j-1}2^{l(\beta-n+n\alpha)}\|f\|_{BMO^\beta(\omega)} \\ &\le 2^{j(n\alpha+\beta)}\omega(B)|B|^{\beta/n}\|f\|_{BMO^\beta(\omega)}.\end{aligned}$$

Splitting the integral into dyadic annuli and using the doubling property, I_1 is bounded by

$$C_k\sum_{j=2}^{\infty}\frac{1}{2^{j(k+n)}}\int_{2^jB}|f(z)-f_B|dz \le C_k\|f\|_{BMO^\beta(\omega)}\omega(B)|B|^{\beta/n}\sum_{j=2}^{\infty}2^{-j(k+n-n\alpha-\beta)},$$

and the last sum is finite choosing k big enough.

Now, using that for $x \in B$ and $z \in \mathbb{R}^n \setminus 2B$,

$$\rho_1(x) \sim \rho_1(x_0), |x-z| \sim |x_0-z|, B(z, 2|x-z|) \subset B(x_0, 4|x_0-z|),$$

we have that

$$I_2 \lesssim \rho^2(x_0)\sum_{j=1}^{\infty}\frac{1}{2^{j(k+n-2)}}\int_{2^{j+1}B\setminus 2^jB}\Big(\int_{2^{j+3}}\frac{|\nabla V(u)|}{|x-z|^{n-1}}du\Big)|f(z)-f_B|dz.$$

Noticing that

$$\int_{2^{j+3}}\frac{|\nabla V(u)|}{|x-z|^{n-1}}du = \mathcal{I}_1(|\nabla V|\chi_{2^{j+3}B}).$$

We may use Lemma 2.8 and $\omega \in D_\alpha, |\nabla V| \in D_\mu$ to obtain the bound

$$C\|f\|_{BMO^\beta(\omega)}\omega(B)\rho(x_0)^{\beta+1}\sum_{j=1}^{\infty}\frac{j+2}{2^{j(k+2n-n\alpha-n\mu-3)}}.$$

Choosing k large enough to make the series convergent and note $\rho \lesssim \rho_1 \lesssim 1$ we arrive to the desired estimate. Now we take care of the oscillation of R^* on a ball $B = B(x_0, r)$ with $r < \rho(x_0)$. Then, we deduce that

$$\begin{aligned}\int_B |R^*f(x)-(R^*f)_B|dx &\le \int_B |(R^*-\mathcal{R}^*)f(x) - [(R^*-\mathcal{R}^*)f]_B|dx \\ &+ \int_B |\mathcal{R}^*f(x)-(\mathcal{R}^*f)_B|dx = I + J,\end{aligned}$$

where $\mathcal{R}^* = (-\Delta)^{-1}\nabla^2$.

Note that $BMO^{\beta}_{\mathcal{L}}(\omega) \subset BMO^{\beta}(\omega)$. Using Theorem 4.1, we arrive as

$$J \le C\|f\|_{BMO^{\beta}(\omega)}\omega(B)|B|^{\beta/n}.$$

To take care of I, we set $g = f - f_B$. We write $g = g_1 + g_2 + g_3$, with $g_1 = g\chi_{5B}$, $g_2 = g\chi_{(B_0\setminus 5B)}$ and $g_3 = g\chi_{B_0^c}$, where $B_0 = B(x_0, 5\rho(x_0))$.

Since $(R^* - \mathcal{R}^*)(f_B) = 0, (R^* - \mathcal{R}^*)(f) = (R^* - \mathcal{R}^*)(g)$. Then, we get that $I \le I_1 + I_2 + I_3$, where

$$I_j = \int_B \Big|(R^* - \mathcal{R}^*)g_j(x) - [(R^* - \mathcal{R}^*)g_j]_B\Big|dx$$

for $j = 1, 2, 3$.

Now we proceed to estimate I_2. We apply Lemma 3.9 with $\beta < \delta < \min\{1 - n/q, 2 - n/q_1\}$

$$\begin{aligned}
I_2 \le \frac{1}{|B|}\int_B\int_B\int_{B_0\setminus 5B} & |[\mathbf{K}^*(x,z) - \mathcal{K}^*(x,z)] - [\mathbf{K}^*(y,z) - \mathcal{K}^*(y,z)]|\times \\
&\times |f(z) - f_B|dzdxdy \\
&\le \frac{1}{|B|}\int_B\int_B\int_{B_0\setminus 5B} \frac{|x-y|^{\delta}}{|x-z|^{n+\delta}}\Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q}|f(z) - f_B|dzdxdy \\
&+ \frac{1}{|B|}\int_B\int_B\int_{B_0\setminus 5B}\Big(\frac{|x-y|^{\delta}}{|x-z|^{n-2+\delta}}\int_{B(z,3|x-z|)}\frac{|\nabla V(u)|}{|u-z|^{n-1}}du\Big)|f(z) - f_B|dzdxdy \\
&+ \frac{1}{|B|}\int_B\int_B\int_{B_0\setminus 5B} \frac{|x-y|^{\delta}}{|x-z|^{n-1+\delta}}\Big(\frac{|x-z|}{\rho_1(x)}\Big)^{2-n/q_1}|f(z) - f_B|dzdxdy \\
&:= I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned}$$

For every $x, y \in B$ and for every $z \in (5B)^c$, noting that $|x-z| \ge 2|x-y|, \rho(x_0) \sim \rho(x), |x-z| \sim |x_0 - z|$ and $|x_0 - z| < 5\rho(x_0) \lesssim 1$,

$$\begin{aligned}
I_{2,1} &\le \frac{C}{|B|}\int_B\int_B\int_{B_0\setminus 5B} \frac{|x-y|^{\delta}}{|x-z|^{n+\delta}}\Big(\frac{|x-z|}{\rho(x)}\Big)^{2-n/q}|f(z) - f_B|dzdxdy \\
&\lesssim \frac{r^{n+\delta}}{\rho(x_0)^{2-n/q}}\int_{B_0\setminus 5B}\frac{|f(z) - f_B|}{|x_0 - z|^{n+\delta-2+n/q}}dz.
\end{aligned}$$

Let j_0 be the integer part of $\log_2(5\rho(x_0)/r)$. Then, we have

$$\begin{aligned}
I_{2,1} &\lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{2-n/q}\sum_{j=2}^{j_0}\frac{1}{2^{j(n+\delta-2+n/q)}}\int_{2^{j+1}B\setminus 2^jB}|f(z) - f_B|dz \\
&\lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{2-n/q}\omega(B)|B|^{\beta/n}\|f\|_{BMO^{\beta}(\omega)}\sum_{j=2}^{j_0} j2^{j(2+n\alpha+\beta-n-\delta-n/q)} \\
&\lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{n+\delta-n\alpha-\beta}\log_2(5\rho(x_0)/r)|B|^{\beta/n}\omega(B)\|f\|_{BMO^{\beta}(\omega)}.
\end{aligned}$$

Using the inequality $1 + \log_2(t) \le Ct^{\varepsilon/2}$ for $t > 1/8$, with ε small enough, we get

$$I_{2,1} \lesssim |B|^{\beta/n}\omega(B)\|f\|_{BMO^\beta(\omega)}.$$

We only have to take care of $I_{2,2}$ by $I_{2,3}$ is the same as $I_{2,1}$. Now, using that for $x \in B$ and $z \in \mathbb{R}^n \setminus 5B, \rho(x) \sim \rho(x_0), |x-z| \sim |x_0 - z|, B(z, 3|x-z|) \subset B(x_0, 5|x_0 - z|)$, we imply that

$$I_{2,2} \lesssim r^{\delta+n} \int\limits_{B_0\setminus 5B} \frac{|f(z) - f_B|}{|x_0 - z|^{n+\delta-2}} \int\limits_{B(x_0,5|x_0-z|)} \frac{|\nabla V(u)|}{|u-z|^{n-1}} dudz.$$

Breaking the integral in z dyadically and setting j_0 such that $2^{j_0-1}r \le \rho(x_0) \le 2^{j_0}r$

$$I_{2,2} \lesssim r^2 \sum_{j=3}^{j_0} \frac{1}{2^{j(n+\delta-2)}} \int\limits_{2^jB} |f(z) - f_B| \mathcal{I}_1(\chi_{2^{j+3}B}|\nabla V|)(z)dz.$$

We deduce from Lemma 2.8 that

$$\begin{aligned}
I_{2,2} &\lesssim r^{\beta+1}\|f\|_{BMO^\beta(\omega)}\omega(B)\Big(\frac{r}{\rho(x_0)}\Big)^{n-n\alpha-\beta+2-n/q_1} \sum_{j=3}^{j_0} j2^{j(3-\delta-n/q_1)} \\
&\lesssim r^\beta \rho(x_0)\|f\|_{BMO^\beta(\omega)}\omega(B)\Big(\frac{r}{\rho(x_0)}\Big)^{n-n\alpha-\beta+3-n/q_1} \sum_{j=3}^{j_0} j2^{j(3-\delta-n/q_1)} \\
&\lesssim \|f\|_{BMO^\beta(\omega)}\omega(B)|B|^{\beta/n} \log_2(5\rho(x_0)/r)\Big(\frac{r}{\rho(x_0)}\Big)^{n+\delta-n\alpha-\beta}.
\end{aligned}$$

Using the inequality $1 + \log_2(t) \le Ct^{\varepsilon/2}$ for $t > 1/8$, with ε small enough, we obtain

$$I_{2,2} \lesssim \|f\|_{BMO^\beta(\omega)}\omega(B)|B|^{\beta/n}.$$

Now we take care of I_3. We use the smoothness of each kernel separately. For $\mathcal{K}^*$ we use Calderón-Zygmund condition and for $\mathbf{K}^*$ we use Lemma 3.5. We have

$$\begin{aligned}
I_3 &\lesssim r^{n+\delta} \int\limits_{B_0^c} \frac{|f(z) - f_B|}{|x_0 - z|^{n+\delta}} dz \\
&+ \rho(x_0)^k r^{\delta+n} \int\limits_{\mathbb{R}^n\setminus B(x_0,\rho(x_0))} \frac{|f(z) - f_B|}{|x_0 - z|^{k+n+\delta-2}} \int\limits_{B(x_0,4|x_0-z|)} \frac{|\nabla V(u)|}{|u-z|^{n-1}} dudz \\
&:= B_1 + B_2.
\end{aligned}$$

We estimate B_1

$$\begin{aligned}
B_1 &\lesssim r^{n+\delta} \int\limits_{B_0^c} \frac{|f(z) - f_B|}{|x_0 - z|^{n+\delta}} dz \lesssim \sum_{j=j_0}^{\infty} \frac{1}{2^{j(n+\delta)}} \int\limits_{2^{j+1}B} |f(z) - f_B| dz \\
&\lesssim \|f\|_{BMO^\beta(\omega)}|B|^{\beta/n}\omega(B) \sum_{j=j_0}^{\infty} j\frac{1}{2^{j(n+\delta-n\alpha-\beta)}} \lesssim \|f\|_{BMO^\beta(\omega)}|B|^{\beta/n}\omega(B).
\end{aligned}$$

Moreover

$$B_2 \le C_k \rho(x_0)^k r^{2-k} \sum_{j=j_0}^{\infty} \frac{1}{2^{j(k+n+\delta-2)}} \int_{2^j B} |f(z) - f_B| \mathcal{I}_1(\chi_{2^{j+3}B}|\nabla V|)(z) dz$$

$$\le C_k \|f\|_{BMO^{\beta}(\omega)} r^{\beta} \omega(B) \rho_1(x_0) \Big(\frac{r}{\rho_1(x_0)}\Big)^{3+(\mu-1)n-k} \sum_{j=j_0}^{\infty} j \frac{1}{2^{j(k+2n+\delta-3-n\mu-n\alpha)}}.$$

Using the inequality $j \le \frac{2^{j\epsilon}}{\epsilon ln2}$ with ϵ small enough, we arrive at

$$B_2 \le C_k \|f\|_{BMO^{\beta}(\omega)} r^{\beta} \omega(B) \rho_1(x_0) \Big(\frac{r}{\rho_1(x_0)}\Big)^{3+(\mu-1)n-k} \sum_{j=j_0}^{\infty} \frac{1}{2^{j(k+2n+\delta-3-n\mu-n\alpha-\epsilon)}}.$$

Choosing k large enough to make the series convergent we get

$$C_k \|f\|_{BMO^{\beta}(\omega)} r^{\beta} \Big(\frac{r}{\rho_1(x_0)}\Big)^{n+\delta-n\alpha-\epsilon},$$

and the last factor is bounded since $r < \rho_1(x_0)$.

Finally,

$$I_1 \le 2 \int_B |(R^* - \mathcal{R}^*) g_1(x)| dx \lesssim \int_B \int_{5B} |\mathbf{K}^*(x,y) - \mathcal{K}^*(x,y)| |f(y) - f_B| dy dx.$$

Then, Lemma 3.4 now implies

$$\begin{aligned} I_1 \lesssim & \int_B \int_{5B} \Big[\frac{1}{|x-y|^{n-2}} \int_{B(y,|x-y|)} \frac{|\nabla V(z)| dz}{|y-z|^{n-1}} + \frac{1}{|x-y|^{n-1}} \Big(\frac{|x-y|}{\rho_1(x)}\Big)^{\delta_1} \\ & + \frac{1}{|x-y|^n} \Big(\frac{|x-y|}{\rho(x)}\Big)^{\delta}\Big] |f(y) - f_B| dy dx \\ & := I_{1,1} + I_{1,2} + I_{1,3}. \end{aligned}$$

We have

$$\begin{aligned} I_{1,3} &\lesssim \int_B \int_{5B} \frac{1}{|x-y|^n} \Big(\frac{|x-y|}{\rho(x)}\Big)^{\delta} |f(y) - f_B| dy dx \\ &\lesssim \rho(x_0)^{-\delta} \int_{5B} \int_B \frac{1}{|x-y|^{n-\delta}} dx |f(y) - f_B| dy \\ &\lesssim \Big(\frac{r}{\rho(x_0)}\Big)^{\delta} \int_{5B} |f(y) - f_B| dy \lesssim \|f\|_{BMO^{\beta}(\omega)} |B|^{\beta/n} \omega(B). \end{aligned}$$

Since $r < \rho(x_0) \lesssim \rho_1(x_0) \lesssim 1$, we also have

$$\begin{aligned} I_{1,2} &\lesssim \int_B \int_{5B} \frac{1}{|x-y|^{n-1}} \Big(\frac{|x-y|}{\rho_1(x)}\Big)^{\delta_1} |f(y) - f_B| dy dx \\ &\lesssim \rho(x_0)^{-\delta_1} \int_{5B} \int_B \frac{1}{|x-y|^{n-1-\delta_1}} dx |f(y) - f_B| dy \\ &\lesssim r\Big(\frac{r}{\rho(x_0)}\Big)^{\delta_1} \int_{5B} |f(y) - f_B| dy \lesssim \|f\|_{BMO^\beta(\omega)} |B|^{\beta/n} \omega(B). \end{aligned}$$

To estimate $I_{1,1}$ we notice that $B(y, |x-y|) \subset 6B$ for $x \in B$ and $y \in 5B$. Applying Lemma 2.8, we imply that

$$\begin{aligned} I_{1,1} &\lesssim \int_{5B} \int_B \frac{1}{|x-y|^{n-2}} dx |f(y) - f_B| \mathcal{I}_1(|\nabla V| \chi_{6B})(y) dy \\ &\lesssim r^2 \int_{2^3 B} |f(y) - f_B| \mathcal{I}_1(|\nabla V| \chi_{2^5 B})(y) dy \\ &\lesssim r^{\beta+1} \omega(B) \|f\|_{BMO^\beta(\omega)} \Big(\frac{r}{\rho_1(x)}\Big)^{n-n\alpha+2-n/q_1} \lesssim r^\beta \omega(B) \|f\|_{BMO^\beta(\omega)}, \end{aligned}$$

where we have used $r < \rho(x_0) \lesssim \rho_1(x_0) \lesssim 1$ in the last inequality. The proof is completed. □

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