A truncation regularization method for a time fractional diffusion equation with an in-homogeneous source

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Abstract. In the present paper, we consider a time-fractional inverse diffusion problem with an in-homogeneous source, where data is given at $x = 1$ and the solution is required in the interval $0 < x < 1$. This problem is ill-posed, i.e. the solution (if it exists) does not depend continuously on the data. We propose a regularization method to solve it based on the solution given by the Fourier method.

Keywords: Ill-posed problem, time fractional diffusion equation, regularization, regularized truncation method.

1 Introduction

In recent years, fractional calculus and fractional differential equations have gained much special attention by many authors. They appear naturally in wide sciences area of physics, chemical engineering, biology, signal processing, electrical, control theory, finance, population dynamics, etc; refer to [1–3].

Time-fractional diffusion equation arises by replacing the standard time partial derivative in the diffusion equation with a time-fractional partial derivative. They have arisen from a continuous-time random walk model with temporal memory and source where anomalous diffusion such as occurs in spatial environments, or in slow heat flux diffusion in thermal conductions. In this paper, we study a more general ill-posed problem for the time-fractional diffusion equation with the in-homogeneous source in a one-dimensional semi-infinite domain as follows

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} + \frac{\partial u(x,t)}{\partial x} = g(x,t), \qquad x > 0, \quad t > 0, \tag{1}$$

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with the Cauchy condition and initial condition

$$u(1,t) = f(t), \quad t \geq 0, \tag{2}$$

$$\lim_{x \to +\infty} u(x,t) = u(x,0) = 0, \tag{3}$$

where b is a constant diffusivity coefficient. The fractional derivative $\frac{\partial^\beta}{\partial t^\beta}$ is the Caputo fractional derivative of order β defined by

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{1}{\beta(1-\beta)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\beta},$$

for $0 < \beta < 1$ whenever the right-hand-side exists and

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{\partial u(x,t)}{\partial t}$$

for $\beta = 1$ where Γ is the Gamma function.

In 2007, D.A. Murio considered the fractional inverse heat conduction problem in a quarter plane which includes a classical derivative of first-order respectively the spatial variable as follow

$$\begin{aligned} u_x(x,t) =& -a^{-\frac{1}{2}}\left({}_0^{RL}D_t^{\frac{1}{2}} u(x,t)\right) + u_\infty(\pi a t)^{-\frac{1}{2}}, x > 0, t > 0 \\ u(1,t) =& f(t) \end{aligned}$$

with the initial condition

$$\begin{aligned} u(0,t) &= g(t) \\ -u_x(0,t) &= h(t), \end{aligned}$$

for all $t \geq 0$ in which the time fractional derivative be understood in the sense of Riemann-Liouville of order 1/2, see [7]. In 2011, G.H. Zheng and T. Wei had considered a homogeneous time fractional diffusion problem, where the time fractional derivative be understood in sense of Dzerbayshan-Caputo also in a quarter plane, in form of

$$\begin{aligned} -au_x(x,t) =& {}_0D_t^\beta u(x,t), \quad x > 0, t > 0, 0 < \beta < 1, \\ u(1,t) =& f(t) \end{aligned}$$

corresponding to the initial condition

$$u(x,0) = \lim_{x \to +\infty} u(x,t) = 0.$$

However, in many phenomena, diffusion progresses occur in spatially inhomogeneous environments or the slow diffusion progress require problem models which contains the inhomogeneous or nonlinear source term. The authors couldn't find any results for TFIDE in that cases. To solve that problem, it is required difficult techniques and new ideas to deal with the fractional terms and an in-homogeneous term which can not be treated by using known ideas.

That is the reason why we need a regularization method to approximate the problem (4). In the present paper, we choose the truncation method. There are two new points that are devised when we overcame the difficult points as we had discussed in the head of this section. Firstly, we note that solution of the classical time diffusion equation, where the time derivative order is an integer number, has just formulated by a form of real computations while there arise complex computations in case of the fractional derivatives, see [5, 8, 9, 11]. Secondly, techniques of using Gronwall inequality are used very sophisticated providing that we apply the Gronwall inequality for a product of the error and some appropriate exponential functions but not only for the error. This technique plays a key role to solve TFIDEs. Moreover, a hypothesis is considered utmost flexible which will be assumed to holds on the exact data, so that our result, by the way, had improved strongly. This helps us in grasping the role of solving in-homogeneous TFIDEs, also recognize the difficult points as we have discussed and the worthy validity of our results.

We divide the paper into four sections. The ill-posedness and the method of regularization for the problem (1) - (3) will be analyze in section 2. In section 3, the main results are speech. Some discussion and conclusion are given to explain the results more clearly. The proof part of our results will be given in section 4.

2 Regularized problem

2.1 Representation for solution

By extending all functions to the whole line $-\infty < t < \infty$ which are replaced by zero value for $t < 0$ and taking a Fourier transformation to the problem (1) with respect to t, we obtain

$$\begin{aligned}(i\xi)^\beta \mathcal{F}(u(x,.))(\xi) + \mathcal{F}(u_x(x,.))(\xi) &= \mathcal{F}(g(x,.))(\xi),\\ \mathcal{F}(u(1,.))(\xi) &= \mathcal{F}(f(.))(\xi),\end{aligned}$$

for $x > 0, \xi \in \mathbb{R}$ where we denote by $\mathcal{F}(v(.))$ the Fourier transforms of v defined by the following equation

$$\mathcal{F}(v(.))(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(t)e^{-i\xi t}dt.$$

Solving the above differential equation, the solution of problem (1) - (3) can be obtained in form of

$$\begin{aligned}\mathcal{F}(u(x,.))(\xi) &= \exp\Big((i\xi)^\beta(1-x)\Big)\mathcal{F}(f(.))(\xi)\\ &+ \int_x^1 \exp\Big((i\xi)^\beta(\eta - x)\Big)\mathcal{F}(g(\eta,.))(\xi)d\eta,\end{aligned}$$

or

$$\begin{aligned}u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp\Big((i\xi)^\beta(1-x)\Big)\mathcal{F}(f(.))(\xi)e^{i\xi t}d\xi\\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_x^1 \exp\Big((i\xi)^\beta(\eta - x)\Big)\mathcal{F}(g(\eta,.))(\xi)e^{i\xi t}d\eta d\xi,\end{aligned} \tag{4}$$

where the real part and im-real part of $(i\xi)^\beta$ be given by

$$(i\xi)^\beta = \Re((i\xi)^\beta) + i\Im((i\xi)^\beta), \tag{5}$$

$$\Re((i\xi)^\beta) = |\xi|^\beta \cos\frac{\beta\pi}{2}, \tag{6}$$

$$\Im((i\xi)^\beta) = |\xi|^\beta \mathrm{sign}(\xi)\sin\frac{\beta\pi}{2}. \tag{7}$$

2.2 Ill-posedness and truncation regularization method

We emphasize that $\exp\left((i\xi)^\beta(1-x)\right)$ and $\exp\left((i\xi)^\beta(\eta-x)\right)$ can be go to infinity module. So by applying the Parseval theorem, the exact solution u may be go to infinity $L^2(\mathbb{R})-$ norm value. For finding the solution $u(x,.)$ in $L^2(\mathbb{R})$ for every $0 \le x \le 1$, the exact data functions

$$\mathcal{F}\left(f(.)\right)(\xi), \quad \mathcal{F}\left(g(x,.)\right)(\xi)$$

must rapidly decay when $|\xi| \to \infty$. But we couldn't have the exact data, we only have the noisy data from the progress of real measurement which instead of the exact data.

Therefore, we can not expect the noisy data functions f_ϵ, g_ϵ have the same decay in high frequency f, g although

$$||f_\epsilon - f||_{L^2(\mathbb{R})} + ||g_\epsilon - g||_{L^2(0,1;L^2(\mathbb{R}))} \le \epsilon, \tag{8}$$

for small ϵ, i.e., small errors in high frequency domain can blow up and destroy the solution for $0 < x < 1$. Thus the time fractional inverse diffusion problem (1)-(3) is sererely ill-posed. That is the reason why it's necessary to propose a regularization approximation for the exact solution $u(x,t)$. In this section, we considere the following approximative problem of finding $u(0,t)$ for all time $t > 0$ as follows

$$\begin{aligned} u_\epsilon^\alpha(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left((i\xi)^\beta(1-x)\right)\mathcal{F}\left(f^\epsilon(.)\right)(\xi)\chi_{[-\alpha;\alpha]}(\xi)e^{i\xi t}d\xi \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_x^1\exp\left((i\xi)^\beta(\eta-x)\right)\mathcal{F}\left(g^\epsilon(\eta,.)\right)(\xi)d\eta\chi_{[-\alpha;\alpha]}(\xi)e^{i\xi t}d\xi, \end{aligned} \tag{9}$$

where the noisy data f^ϵ, g^ϵ satisfying the noisy assumption (8) and $\alpha := \alpha(\epsilon)$ is a positive constant which depends on ϵ and will be specified later as a parameter of the regularization method.

In order to obtain the convergence estimate of regularization method, we give the following a prior bound assumption

$$||u(0,.)||_{L^2(\mathbb{R})} \le E, \tag{10}$$

where E is a positive constant.

3 Main result

In this section, we present our results. For establishing the main theorem, some lemmas will be given. In addition, it is useful to introduce u^α defined by the following function

$$\begin{aligned} u^\alpha(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left((i\xi)^\beta(1-x)\right)\mathcal{F}\left(f(.)\right)(\xi)\chi_{[-\alpha;\alpha]}(\xi)e^{i\xi t}d\xi \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_x^1\exp\left((i\xi)^\beta(\eta-x)\right)\mathcal{F}\left(g(\eta,.)\right)(\xi)d\eta\chi_{[-\alpha;\alpha]}(\xi)e^{i\xi t}d\xi, \end{aligned} \tag{11}$$

where f, g are the exact data functions and α is the parameter of the regularization method.

Lemma 1 *Let u and u^α be the exact solution and regularized solution respectively defined by (4) and (11) which are corresponding to the exact data (f, g) and the noisy data (f^ϵ, g^ϵ). Assume the prior bound (10) hold. If there exists a positive constant M such that*

$$\int_{-\infty}^{+\infty} \exp\left(2\eta\Re((i\xi)^\beta)\right)|\mathcal{F}\left(g(\eta,.)\right)(\xi)|^2 d\xi \le M^2 \tag{12}$$

then

$$\|u(x,.) - u^\alpha(x,.)\|_{L^2(\mathbb{R})} \le (E+M)\exp\left(-x\alpha^\beta\cos\frac{\beta\pi}{2}\right), \tag{13}$$

for all $0 < x < 1$.

Proof. It follows from (4) and (11) that

$$\begin{aligned} u(x,t) - u^\alpha(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{|\xi|>\alpha}\exp\left((i\xi)^\beta(1-x)\right)\mathcal{F}\Big(f(.)\Big)(\xi)e^{i\xi t}d\xi \\ &+ \frac{1}{\sqrt{2\pi}}\int_{|\xi|>\alpha}\int_x^1 \exp\left((i\xi)^\beta(\eta-x)\right)\mathcal{F}\Big(g(\eta,.)\Big)(\xi)d\eta e^{i\xi t}d\xi. \end{aligned} \tag{14}$$

If we represent $u(x,t)$ as follows

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left((i\xi)^\beta(1-x)\right)Q(\xi)e^{i\xi t}d\xi \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_x^1 \exp\left((i\xi)^\beta(\eta-x)\right)\mathcal{F}\left(g(\eta,.)\right)(\xi)e^{i\xi t}d\eta d\xi, \end{aligned}$$

then

$$\begin{aligned} Q(\xi) &= \exp\left(-(i\xi)^\beta\right)\mathcal{F}(u(0,.))(\xi) \\ &- \exp\left(-(i\xi)^\beta\right)\int_0^1 \exp\left((i\xi)^\beta\eta\right)\mathcal{F}\left(g(\eta,.)\right)(\xi)d\eta \end{aligned}$$

by substituting 0 into x. This implies

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(-(i\xi)^\beta x\right)\mathcal{F}(u(0,.))(\xi)e^{i\xi t}d\xi \\ &- \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_0^x \exp\left((i\xi)^\beta(\eta-x)\right)\mathcal{F}\left(g(\eta,.)\right)(\xi)e^{i\xi t}d\eta d\xi. \end{aligned} \tag{15}$$

We deduce from (15) that

$$\begin{aligned} u(x,t) - u^\alpha(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{|\xi|>\alpha}\exp\left(-(i\xi)^\beta x\right)\mathcal{F}(u(0,.))(\xi)e^{i\xi t}d\xi \\ &- \frac{1}{\sqrt{2\pi}}\int_{|\xi|>\alpha}\int_0^x \exp\left((i\xi)^\beta(\eta-x)\right)\mathcal{F}\left(g(\eta,.)\right)(\xi)e^{i\xi t}d\eta d\xi. \end{aligned}$$

By applying the Parseval identity, we have

$$\|u(x,.)-u^{\alpha}(x,.)\|^2_{L^2(\mathbb{R}} = \int_{|\xi|>\alpha} \exp\left(-2\Re((i\xi)^{\beta})x\right)|\mathcal{F}(u(0,.))(\xi)|^2 d\xi$$
$$+\int_{|\xi|>\alpha}\int_0^x \exp\left(2\Re((i\xi)^{\beta})(\eta-x)\right)|\mathcal{F}\left(g(\eta,.)\right)(\xi)|^2 d\eta d\xi$$
$$\leq \exp\left(-2x\alpha^{\beta}\cos\frac{\beta\pi}{2}\right)\|\mathcal{F}(u(0,.))\|^2_{L^2(\mathbb{R})}$$
$$+\exp\left(-2x\alpha^{\beta}\cos\frac{\beta\pi}{2}\right)\int_0^x\left(\int_{|\xi|>\alpha}\exp\left(2\eta\Re((i\xi)^{\beta})\right)|\mathcal{F}\left(g(\eta,.)\right)(\xi)|^2 d\xi\right)d\eta.$$

We note that

$$\|\mathcal{F}(u(0,.))\|_{L^2(\mathbb{R}} = \|u(0,.)\|_{L^2(\mathbb{R}},$$
$$\|\mathcal{F}\left(g(\eta,.)\right)\|_{L^2(\mathbb{R}} = \|g(\eta,.)\|_{L^2(\mathbb{R})},$$

by using Parserval identity. Hence we imply

$$\|u(x,.)-u^{\alpha}(x,.)\|^2_{L^2(\mathbb{R}} \leq \exp\left(-2x\alpha^{\beta}\cos\frac{\beta\pi}{2}\right)\|u(0,.)\|^2_{L^2(\mathbb{R})}$$
$$+\exp\left(-2x\alpha^{\beta}\cos\frac{\beta\pi}{2}\right)\int_0^x\int_{|\xi|>\alpha}\exp\left(2\eta\Re((i\xi)^{\beta})\right)|\mathcal{F}\left(g(\eta,.)\right)(\xi)|^2 d\xi d\eta$$
$$\leq (E+M)^2\exp\left(-2x\alpha^{\beta}\cos\frac{\beta\pi}{2}\right),$$

where the prior estimate (10) and the assumption (20) on exact solution have been used.

Lemma 2 *Let u^{α}_{ϵ} and u^{α} be respectively defined by (9) and (11) which are corresponding to the noisy data $(f^{\epsilon}, g^{\epsilon})$ and the exact data. Assume that the noisy assumption (8) holds then we have*

$$\left\|u^{\alpha}(x,.)-u^{\alpha}_{\epsilon}(x,.)\right\|_{L^2(\mathbb{R})} \leq 2\epsilon\exp\left((1-x)\alpha^{\beta}\cos\frac{\beta\pi}{2}\right), \tag{16}$$

for all $0<x<1$.

Proof. By denoting

$$I_1(x,t) := \frac{1}{\sqrt{2\pi}}\int_{|\xi|\leq\alpha}\exp\left((i\xi)^{\beta}(1-x)\right)\mathcal{F}\left(f(.)-f^{\epsilon}(.)\right)(\xi)\chi_{[-\alpha;\alpha]}(\xi)e^{i\xi t}d\xi,$$
$$I_2(x,t) := \frac{1}{\sqrt{2\pi}}\int_{|\xi|\leq\alpha}\int_x^1\exp\left((i\xi)^{\beta}(\eta-x)\right)\mathcal{F}\left(g(\eta,.)-g^{\epsilon}(\eta,.)\right)(\xi)d\eta\chi_{[-\alpha;\alpha]}(\xi)e^{i\xi t}d\xi.$$

Then it deduces from (9) and (11) that

$$u^{\alpha}(x,t)-u^{\alpha}_{\epsilon}(x,t) = I_1(x,t)+I_2(x,t). \tag{17}$$

We have

$$\|I_1(x,.)\|^2_{L^2(\mathbb{R})} = \int_{|\xi|\leq\alpha}\exp\left(2\Re((i\xi)^{\beta})(1-x)\right)|\mathcal{F}(f^{\epsilon}-f)(\xi)|^2 d\xi$$
$$\leq \exp\left(2(1-x)\alpha^{\beta}\cos\frac{\beta\pi}{2}\right)\|\mathcal{F}(f_{\epsilon}-f)\|^2_{L^2(\mathbb{R})}$$
$$\leq \exp\left(2(1-x)\alpha^{\beta}\cos\frac{\beta\pi}{2}\right)\|f_{\epsilon}-f\|^2_{L^2(\mathbb{R})},$$

where we have used Parseval identity. Consequently,

$$\|I_1(x,.)\|^2_{L^2(\mathbb{R})} \leq \epsilon^2 \exp\left(2(1-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right), \tag{18}$$

where we have used the noisy assumption (8). In addition, we have

$$\begin{aligned}\|I_2(x,.)\|^2_{L^2(\mathbb{R})} &= \int_{|\xi|\leq\alpha} \left|\int_x^1 \exp\left((i\xi)^\beta(\eta-x)\right)[\mathcal{F}(g^\epsilon(\eta,.)-g(\eta,.))(\xi)]\, d\eta\right|^2 d\xi \\ &\leq \int_x^1 \int_{|\xi|\leq\alpha} \exp\left(2\Re((i\xi)^\beta)(\eta-x)\right)|\mathcal{F}(g^\epsilon(\eta,.)-g(\eta,.))(\xi)|^2 d\xi d\eta.\end{aligned}$$

Thus

$$\begin{aligned}\|I_2(x,.)\|^2_{L^2(\mathbb{R})} &\leq \int_x^1 \exp\left(2(\eta-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right)\|\mathcal{F}(g^\epsilon(\eta,.)-g(\eta,.))\|^2_{L^2(\mathbb{R})}\, d\eta \\ &\leq \exp\left(2(1-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right)\int_x^1 \|g^\epsilon(\eta,.)-g(\eta,.)\|^2_{L^2(\mathbb{R})}\, d\eta \\ &\leq \exp\left(2(1-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right)\|g^\epsilon - g\|^2_{L^2(0,1;L^2(\mathbb{R}))},\end{aligned}$$

where we have used the Parseval identity

$$\|\mathcal{F}(g^\epsilon(\eta,.)-g(\eta,.))\|_{L^2(\mathbb{R})} = \|g^\epsilon(\eta,.)-g(\eta,.)\|_{L^2(\mathbb{R})}.$$

By using the noisy assumption (8), this follows

$$\|I_2(x,.)\|^2_{L^2(\mathbb{R})} \leq \epsilon^2 \exp\left(2(1-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right). \tag{19}$$

By combining (18) with (19), we derive

$$\begin{aligned}\left\|u^\alpha(x,.)-u^\alpha_\epsilon(x,.)\right\|_{L^2(\mathbb{R})} &\leq \|I_1(x,.)\|_{L^2(\mathbb{R})} + \|I_2(x,.)\|_{L^2(\mathbb{R})} \\ &\leq 2\epsilon \exp\left((1-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right).\end{aligned}$$

This completes the proof of Lemma.

Theorem 1 *Let u^ϵ_α and u be the regularized solution and exact solution defined by (9) and (4) which are corresponding to the noisy data (f^ϵ, g^ϵ) and the exact data (f, g). Assume that the noisy assumption (8) and (10) hold. If*

$$\int_{-\infty}^{+\infty} \exp\left(2\eta\Re((i\xi)^\beta)\right)|\mathcal{F}(g(\eta,.))(\xi)|^2 d\xi \leq M^2, \tag{20}$$

then we have

$$\left\|u^\alpha_\epsilon(x,.)-u(x,.)\right\|_{L^2(\mathbb{R})} \leq (E+M)\exp\left(-x\alpha^\beta \cos\frac{\beta\pi}{2}\right) + 2\epsilon \exp\left((1-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right), \tag{21}$$

for all $0 < x < 1$.

Proof. It follows from

$$u(x,t) - u_\epsilon^\alpha(x,t) = u(x,t) - u^\alpha(x,t) + u^\alpha(x,t) - u_\epsilon^\alpha(x,t)$$

that

$$\left\|u_\epsilon^\alpha(x,.) - u(x,.)\right\|_{L^2(\mathbb{R})} \leq \|u(x,.) - u^\alpha(x,.)\|_{L^2(\mathbb{R})} + \left\|u^\alpha(x,.) - u_\epsilon^\alpha(x,.)\right\|_{L^2(\mathbb{R})},$$

where u^α is defined by (11). We note that the assumptions of Lemma 1 and Lemma 2 hold. Therefore, by applying Lemma 1 and Lemma 2, we obtain

$$\left\|u_\epsilon^\alpha(x,.) - u(x,.)\right\|_{L^2(\mathbb{R})} \leq (E+M)\exp\left(-x\alpha^\beta \cos\frac{\beta\pi}{2}\right) + 2\epsilon \exp\left((1-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right),$$

This implies (21).

Remark 2 (convergence) In the above theorem, the error

$$\left\|u_\epsilon^\alpha(x,.) - u(x,.)\right\|_{L^2(\mathbb{R})}$$

converges to 0 if the parameter of regularization method satisfies the following condition

$$\alpha \to \infty \quad \text{as} \quad \epsilon \to 0, \text{ and} \tag{22}$$

$$\epsilon \exp\left((1-x)\alpha^\beta \cos\frac{\beta\pi}{2}\right) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{23}$$

Remark 3 (convergent rate) Let us choose

$$\alpha = \left(\frac{\ln\left(\ln\left(\frac{1}{\epsilon}\right)\right)}{\cos\frac{\beta\pi}{2}}\right)^{1/\beta}$$

then (22) and (23) hold. Moreover, we derive

$$\left\|u_\alpha^\epsilon(x,.) - u(x,.)\right\|_{L^2(\mathbb{R})} \leq (E+M)\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-x} + 2\epsilon\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{1-x}, \tag{24}$$

for $0 < x < 1$.

References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional integrals and derivatives* (Theory and Applications, Gordon and Breach Science, Naukai Tekhnika, Minsk, 1987).

[2] I. Podlubny, *Fractional differential equations* (Academic Press, London, 1999).

[3] K. Diethelm, *The analysis of fractional differential equation* (Springer, Berlin, 2010).

[4] J.A.T. Machado, V. Kiryakova, F. Mainardi, A poster about the recent history of fractional calculus, Fract. Calc. Appl. Anal., **Vol. 13**, 329-334 (2010).

[5] D.A. Murio, Time fractional IHCP with Caputo fractional derivatives, Comput. Math. with Appl., **Vol. 56**, 2371- 2381 (2008).

[6] K.B. Oldham, J. Spanier, A general solution of the diffusion equation for semifinite geometries, J. Math. Anal. Appl., **Vol. 39**, 655 - 669 (1972).

[7] D.A. Murio, Stable numerical solution of a fractional-diffusion inverse heat conduction problem, Comput. Math. Appl., **Vol. 53**, 1492 - 1501 (2007).

[8] W. Cheng, C.L. Fu, Z. Qian, Two regularization methods for a spherically symmetric inverse heat conduction problem, Comput. Math. Appl., **Vol. 56**, 1138 - 1145 (2008).

[9] G.H. Zheng, T. Wei, Spectral regularization method for a Cauchy problem of the time fractional advection-dispersion equation, J. Comput. Appl. Math., **Vol. 233**, 2631 - 2640 (2010).

[10] G.H. Zheng, T. Wei, Spectral regularization method for the time fractional inverse advection-dispersion equation, Math. Comput. Simul., **Vol 81**, 37 - 51 (2010).

[11] X. Xiong, H. Guo, X. Liu, An inverse problem for a fractional diffusion equation, J. Comput. Appl. Math., **Vol. 236**, 4474 - 4484 (2012).

[12] X. Xiong, Q. Zhou, Y.C. Hon, An inverse problem for fractional diffusion equation in 2-dimensional case: Stability analysis and regularization, J. Math. Anal. Appl., **Vol. 393**, 185 - 199 (2012).