A proposal method for reducing the order of general heat conduction equation

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Abstract. In the context of investigating methods dedicated to identifying unknown parameters of the system described by partial differential equations, particularly in the field of heat transfer, it has been realized that the heat transfer process in particular three-dimensional features is really complex and takes longer to calculate. Therefore, an equivalent mathematical model which is simpler proposed to reduce the calculation time and the costs of experimental activities. We observe that the mathematical models of the diffusion equation can be minimized in three-dimensional space into a similar two-dimensional pattern within certain limits did not change the physical properties of heat transfer process. A mathematical model and the numerical results of simulation experiments in order to prove effectiveness the proposed method will be presented in detail in this article.

Keywords: heat transfer equation, mathematical model, minimization, multidimensional space, partial differential equation.

1 Introduction

Nowadays, it's necessary to model the physical phenomena mathematically dedicated to research activities and purpose decision or prediction about their acts on our life, environment and production activities [1][2]. Evidently, there are many physical problems could be described by partial differential equation (PDE), such as the spread of oil spills on the seawater surface, the movement of contaminated water from the plant or forest fires, the movement of polluted clouds, etc [3][4].

Physical processes often have mathematical models similar to the ones mentioned above. Instead of developing algorithms or solutions for specific phenomena, we can build similar problems. Especially, the heat transfer in the conductor, it can also be modeled by the second order partial differential equation. In addition, simulation and experimentation can be made easier and cheaper to reproduce real-world phenomena.

However, during the study of heat transfer in the conductor, it was found that solving the problem of heat transfer equation in three-dimensional (3D) is not simple and takes too long when the computer was calculating these equations [5]. Instead, this general equation can be minimized into a two-dimensional (2D) space in some specific applications but without losing the physical properties of the heat transfer. This proposed model makes the calculation simpler, so it helps to find solutions faster and more efficient. That is the main objective of this article. Accordingly, mathematical models will in turn be introduced, compared and

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concluded through the simulation results on the software. In addition, the results of this study will be useful for research and development of algorithms for problems described by the partial differential equation.

The content of this article consists of four following sections. The next section will briefly the problematic and mathematical models describing the heat transfer process in 3D space is shown in third part. The third section will focus on the proposed method for minimizing heat conduction equation in 3D into 2D. The numerical results obtained from the simulation and some discussions will be addressed in the fourth section. Finally, the concluding remarks will be represented in the last section.

2 Problematic and mathematical models

2.1 Problematic on physical phenomena

To construct an experimental model for verifying the method proposed in this paper, we assume that there are two heat sources, in case more general, $S_{j=1,2}$ moving on the surface of a square aluminum sheet $\Omega \in \mathbb{R}^3$ of lateral dimension L and thickness e described as shown in Figure 1. The boundary of studying domain is denoted by $\partial\Omega \in \mathbb{R}^2$ [6][7][8]. The variable space of the system $(x, y, z) \in \Omega = \left]-\frac{L}{2}, +\frac{L}{2}\right[\times \left]-\frac{L}{2}, +\frac{L}{2}\right[\times \left]-\frac{e}{2}, +\frac{e}{2}\right[$ are measured in meter and the variable time $t \in \mathrm{T} = \left[0, t_f\right]$ is measured in second. This metal sheet is heated by two heat sources having the thermal flux density functions $\phi_{j=1,2}(t)$ in Wm^{-2} which are assumed to be a homogeneous disk D_j with center $I_j\,(x(t), y(t), z(t))$ and radius r_j. The temperature distribution function of a metal plate $\theta(x, y, z, t)$ in Kelvin is a continuous function in space and time.

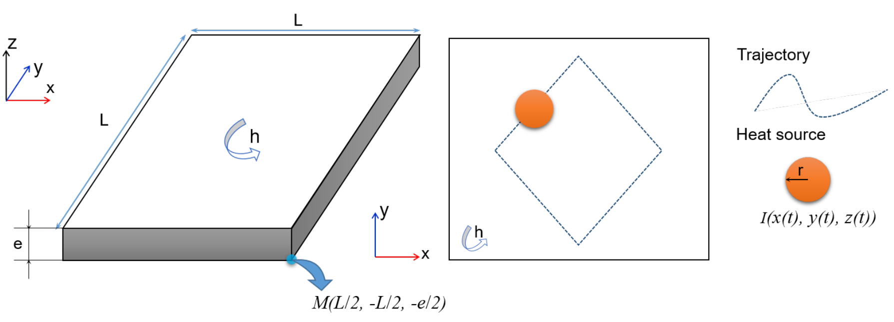


Figure 1. Representation of studied modeled system

Assuming that the values of parameters of a system used to construct the experimental model $\Psi\,(\Omega, \rho, c, \lambda, h, \phi(t), I(t), \theta_0)$ are known with the unit of measure of the quantities in the unit of measure in International System of Units (SI). In particular, the metal plate is heated by a heat source traveling on its surface (Cartesian coordinate system xOy) to allow us to investigate the heat transfer on the surface and inside the plate. The expression of the total thermal power density function of the sources $\Phi\,(x, y, z; t)$ is used to heat the experimental metal plate as follows:

$$\Phi\,(x, y, z; t) = \begin{cases} \phi_1(t) & if\ (x, y, z) \in D_1\,(I_1(t), r_1) \\ \phi_2(t) & if\ (x, y, z) \in D_2\,(I_2(t), r_2) \\ 0 & otherwise \end{cases} \quad (1)$$

In other words, this expression can be represented continuously and differentially as a function of the component density functions in time variables and in spatial coordinates as follows:

$$\Phi(x, y, z, t) = \sum_{j=1}^{2} \frac{\phi_j(t)}{\pi} arccotan\left(\eta \sqrt{\left(x - x_j(t)\right)^2 + \left(y - y_j(t)\right)^2 + \left(z - z_j(t)\right)^2} - r_j\right) \quad (2)$$

The regularization parameter $\eta \in \mathbb{R}^+$ has been chosen to describe with precision the heat flux discontinuity. The time interval $\left[0, t_f\right]$ can be divided into N_t segments and defined using piecewise continuous linear functions: $\left[0, t_f\right] = \bigcup_{i=0}^{N_t-1} [t_i, t_{i+1}]$ with $t_i = \tau i$ and a discretization step defined by $\tau = t_f/N_t$. In order to avoid losing of generality, the orbital equation of all positioning of heat source $I(x(t), y(t), z(t))$ were re-established as discrete functions linearly and rewritten using basic triangle function $s^i(t)$ with $i = 0, 1, \ldots, N_t$:

$$s^i(t) = \begin{cases} 1 + t/\tau - i & if \quad t \in [t_{i-1}, t_i] \\ 1 - t/\tau + i & if \quad t \in [t_i, t_{i+1}] \\ 0 & otherwise \end{cases} \quad (3)$$

Then, the density function of heat flow is expressed as follows $\phi_j(t) = \sum_{i=0}^{N_t} \phi_j^i s^i(t) = \left(\overline{\phi_j}\right)^{tr} \overline{s(t)}$ and the equation of motion of the heat source is expressed as follows $x_j(t) = \sum_{i=0}^{N_t} x_j^i s^i(t) = \left(\overline{x_j}\right)^{tr} \overline{s(t)}$, $y_j(t) = \sum_{i=0}^{N_t} y_j^i s^i(t) = \left(\overline{y_j}\right)^{tr} \overline{s(t)}$ and $z_j(t) = \sum_{i=0}^{N_t} z_j^i s^i(t) = \left(\overline{z_j}\right)^{tr} \overline{s(t)}$. It denotes that "*tr*" is the symbol of the transposition matrix.

To evaluate the reliability of the proposed mathematical model to simplify the heat transfer equation into two-dimensional space, 10 heat sensors are fixed on the metal plate as shown in Figure 3 in order to collect temperature data of the sensor locations during the experiment. Or, the temperature value on the sheet metal is heated by each heat source recorded by five sensors. Furthermore, to assess the effect of errors during the measurement process, it is assumed that the temperature collected from the sensors has been affected by the noise. These disturbances are followed by Gaussian probability distributions $\mathcal{N}(\mu, \sigma^2)$ with mean $\mu = 1$ and standard deviation $\sigma = 0$.

2.2 Mathematical modelling of studied system

In the context of this research, it hypothesizes that characteristic parameters of the system $\Psi(\Omega, \rho, c, \lambda, h, \phi(t), I(t), \theta_0)$ are well-known and that the assumptions of parameters ρ, c, λ, h are constants. If the used material is heterogeneous that means thermal conductivity is a function of other parameters $\lambda(t) = \lambda(\zeta(t))$, then the solution of this problem becomes more complex even though they do not affect its results. In addition, the natural heat transfer coefficient is difficult to determine because it is highly dependent on the particular environment surrounding the heat conductor and has a great influence on the heat exchanger. However, these issues will be addressed in other articles.

The heat transfer equation in the aluminum sheet mentioned above is modeled by the mathematical equation system in the form of partial differential equations in a general way, as follows:

$$\begin{cases} \rho c \dfrac{\partial \theta(x,y,z,t)}{\partial t} - \lambda \Delta \theta(x,y,z,t) = 0 & \forall (x,y,z,t) \in \Omega \times \Gamma \\ \theta(x,y,z,0) = \theta_0 & \forall (x,y) \in \Omega \\ -\lambda \dfrac{\partial \theta(x,y,z,t)}{\partial \vec{n}} = h(\theta(x,y,z,t) - \theta_0) - \Phi(x,y,z,t) & \forall (x,y,z,t) \in \partial\Omega \times \Gamma \end{cases} \tag{4}$$

where $\Delta \theta(x,y,z,t) = \left(\dfrac{\partial^2 \theta(x,y,z,t)}{\partial x^2} + \dfrac{\partial^2 \theta(x,y,z,t)}{\partial y^2} + \dfrac{\partial^2 \theta(x,y,z,t)}{\partial z^2} \right)$ is the Laplace operator of the temperature variable $\theta(x,y,z,t)$ in space and time.

In the above equation, the initial condition of the differential equation is considered as the temperature of the surrounding environment. The boundary conditions of this equation depend on the natural heat convection coefficient h and the total heat flux density function $\Phi(x,y,z,t)$. In that, the natural heat convection coefficient is difficult to measure directly, it is usually determined by experience or model and chosen so that the sensitivity of the temperature distribution function changes at least compared to the variation of other parameters of system.

3 Minimization of heat conduction equation in 3D into 2D

3.1 Hypothesis

The idea of this article is to propose a thermal model in 2D geometry that has been considered and used as a good approximation of thermal transfers in a thin plate. The hypothesis of a 2D geometry (infinitely fine plate) was considered in order to reduce the computation time. This hypothesis is valid if we can neglect heat transfers in the thickness of the plate.

To do this, it is necessary to consider a thin plate of a metal of high thermal conductivity. For financial reasons, aluminum was retained. This square plate of 3 m is placed horizontally on a support providing insulation. The latter is composed of rock wool, thermal insulation supporting high temperatures. In order to the validate the model for the experimental prototype, a study aimed at studying the errors generated by an approximation in 2D geometry has been investigated and is presented in this section.

To reduce the general heat conduction equation in three-dimensional space into bi-dimensional, we fix $z(t) = e$ and this value is enough small. From the equation (4), we calculate boundary conditions of this system.

- *Boundary condition of lateral surface:*

$\partial\Omega_{lat}$ is the boundary of lateral surface of the aluminum plate and the thickness is too small.

$$-\lambda \frac{\partial \theta(x,y,z,t)}{\partial \vec{n}} = 0 \quad with \quad \forall (x,y,z;t) \in \partial\Omega_{lat} \times T \tag{5}$$

- *Boundary condition of superior face:*

$\partial\Omega_{sup}$ is the boundary of superior face of the aluminum plate.

$$-\lambda \frac{\partial \theta(x,y,z,t)}{\partial \vec{n}} = -e\Phi(x,y,z,t) \quad with \quad \forall (x,y,z;t) \in \partial\Omega_{sup} \times T \tag{6}$$

- *Boundary condition of inferior face:*

$\partial\Omega_{sup}$ is the boundary of inferior face of the aluminum plate.

$$-\lambda \frac{\partial \theta(x,y,z,t)}{\partial \vec{n}} = \frac{1}{R}(\theta(x,y,z,t) - \theta_0) \quad with \quad \forall (x,y,z;t) \in \partial\Omega_{inf} \times T \tag{7}$$

From the equations (5) to (7), we can rewrite the formulation of the heat transfer equation in 3D with a combination of the boundary conditions of (4) in this case, as follow:

$$\begin{cases} \forall\,(x,y,z;t)\in\Omega\times T & \rho c\dfrac{\partial\theta(x,y,t)}{\partial t}-\lambda\Delta\theta(x,y,t)=0 \\ \forall\,(x,y,z)\in\Omega & \theta(x,y,t)\,(x,y;0)=\theta_0 \\ \forall\,(x,y,z;t)\in\partial\Omega_{lat}\times T & -\lambda\dfrac{\partial\theta(x,y,t)}{\partial\vec{n}}=0 \\ \forall\,(x,y,z;t)\in\partial\Omega_{sup}\times T & -\lambda\dfrac{\partial\theta(x,y,t)}{\partial\vec{n}}=-e\Phi\,(x,y,t) \\ \forall\,(x,y,z;t)\in\partial\Omega_{inf}\times T & -\lambda\dfrac{\partial\theta(x,y,t)_p}{\partial\vec{n}}=\dfrac{1}{R}\,(\theta(x,y,t)-\theta_0) \end{cases} \tag{8}$$

The coefficient R represents thermal resistance, also called thermal isolation coefficient of surface, is expresses itself in m^2KW^{-1}. We can also reuse the equation (8) to build another form of the partial differential equation system in order to obtain a new formulation. In the next section, a proposed PDE in second order which can replace this one in 3D will be represented.

3.2 Formulation of models

In this study, the hypothesis of a geometry 2D *(infinitely fine plate)* was considered in order to reduce the computing times. This assumption is valid only if we can neglect the heat transfer in the thickness of the plate. To do this, it is necessary to consider a fine plate of a metal having a high thermal conductivity. For financial reasons, aluminum was chosen (gold, silver and copper not being possible). This square plate 3m of the each side posed horizontally on a support providing insulation. This last is composed of Rockwool thermal insulation to high temperatures.

$$\begin{cases} \forall\,(x,y,t)\in\Omega\times T & \rho c\dfrac{\partial\theta\,(x,y,t)}{\partial t}-\lambda\Delta\theta\,(x,y,t)=\dfrac{\Phi\,(x,y,t)-2h\,(\theta\,(x,y,t)-\theta_0)}{e} \\ \forall\,(x,y)\in\Omega & \theta\,(x,y;0)=\theta_0 \\ \forall\,(x,y,t)\in\partial\Omega\times T & -\lambda\dfrac{\partial\theta\,(x,y,t)}{\partial\vec{n}}=0 \end{cases} \tag{9}$$

The heat flux density function can be rewritten as follows:

$$\Phi(x,y;t)=\begin{cases} \phi(t) \quad if \quad (x,y)\in D(I(t),r) \\ 0 \qquad\qquad otherwise \end{cases}$$

and the total flux can also be expressed continuously and differentially as: $\Phi\,(x,y,t) = \dfrac{\phi(t)}{\pi}\text{arcoctan}\left(\eta\left(\sqrt{(x-x_I\,(t))^2+(y-y_I\,(t))^2}-r\right)\right)$

4 Numerical results and discussion

4.1 Numerical results

In order to verify the effectiveness of proposed method or verify that the heat transfers are surfacing on the aluminum plate, two mathematical models were compared, a modeled simulation of a system was built which the parameters are in Table 1. The systems of partial differential equations (4) and (9) are solved using the finite element method [3] implemented by the Comsol Multiphysics code interfaced with Matlab software [9][10][11][12].

Table 1. Input parameters of proposed models

$\rho c = 2.4 \times 10^6$	$\lambda = 237$	$h = 15$	$R = 1.2$
$\phi_{heat}(x, y, t) = 10^5 e^{-10^3 d^2} sin\left(\frac{2\pi t}{1200}\right)$			$\theta_0 = 291$

In Table 2, $d(x, y)$ is the distance between the point (x, y) and the center of the top face of the aluminum plate. It can be noted that the 3D configuration studied is axisymmetric. The following table indicates at various points the mean absolute error between two models, as well as the absolute average deviation of temperature $emod[K]$ between 2 points located on both sides of the plate (information $ecart[K]$ obtained with the model (2)).

Table 2. Numerical results of comparison of two models 2D and 3D

d [m]	0	0.05	0.10	0.15	0.20	0.25
emod [K]	1.4	1.2	0.7	0.6 K	0.4	0.3
ecart [K]	0.3	0.02	0.003	0.002	0.001	0.0008

In Figure 2, the temperature changes at different points of the upper face of the plate are presented for a plate thickness $e = 2mm$. It shows that the 2D model is very satisfactory for points near the heating source. For points far away from the source, weak errors of model will be present.

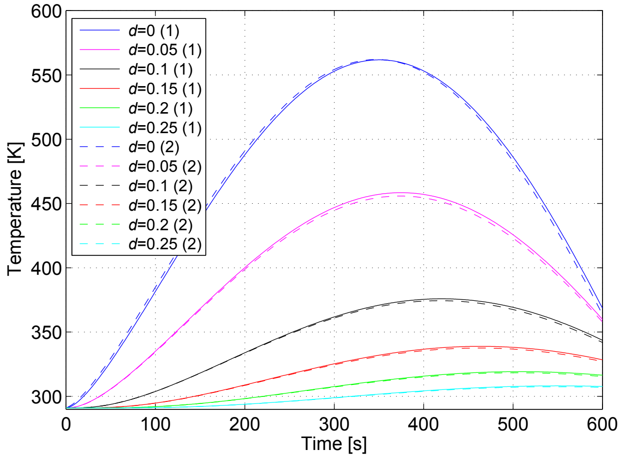
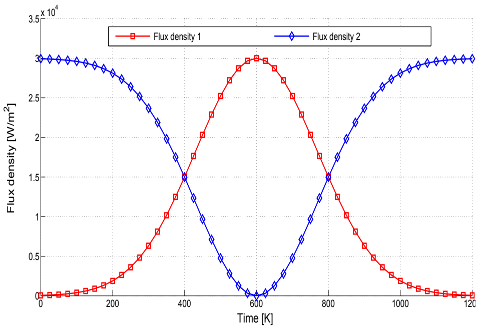


Figure 2. Comparison of the 2D and 3D models

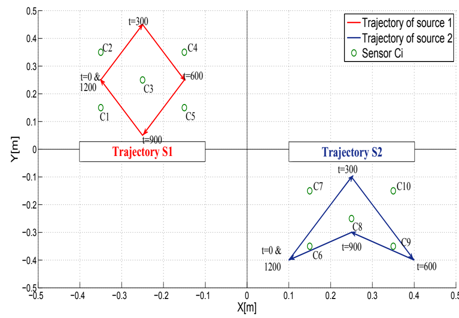
Moreover, when we focus on the computational time, the response time of model 2D is $t_{2D} = 5.32(s)$. While it can find the results of model 3D after $t_{3D} = 11.04(s)$, it means t_{3D}/t_{2D} is 2.08 times. It's shown that the goal of this research to reduce the mathematical complexity and the computational time is satisfied.

4.2 Application in modelling of mobile heat sources by using proposed method

In this section, the proposed method is applied to model a heat conduction problem generated by two mobile heat sources. The trajectory of these moving heat sources is described in Figure 3. At the same time, the heat density densities of the sources are given by the function $\phi_1(t) = \phi_{\max} \exp\left(-(t-\alpha)^2/\beta^2\right)$ and $\phi_2(t) = \phi_{\max} - \phi_1(t)$ in (W/m^2) with $\alpha = 600$ and $\beta = 175$, the graph of these functions is shown in Figure 2 (b).



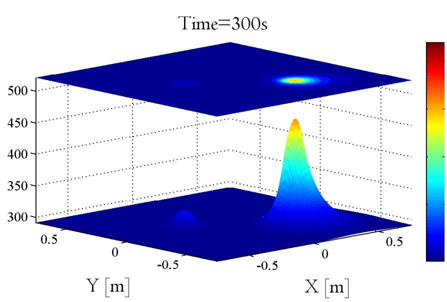
(a)



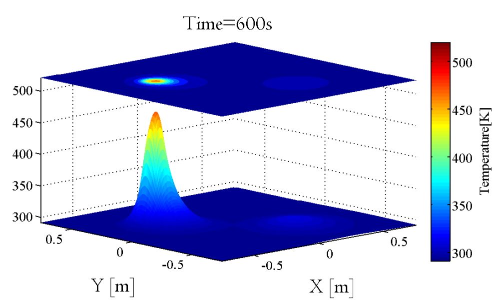
(b)

Figure 3. Heat flux density (a) & Trajectory (b) of two heat sources

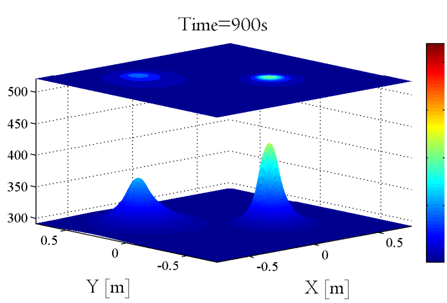
Solving the direct problem (9) with PDE in the second order, can be expressed the problem in 3D, yields the spatiotemporal distribution of temperature within the domain, see Figure 4 a to d, respectively correspond with the moments $t = \{300, 600, 900, 1200\}$ (s).



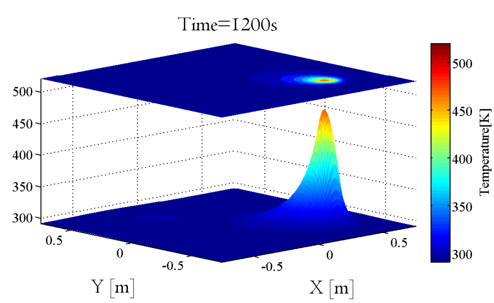
(a)



(b)



(c)



(d)

Figure 4. Spatial evolution of temperature on the plate generated by two heat sources

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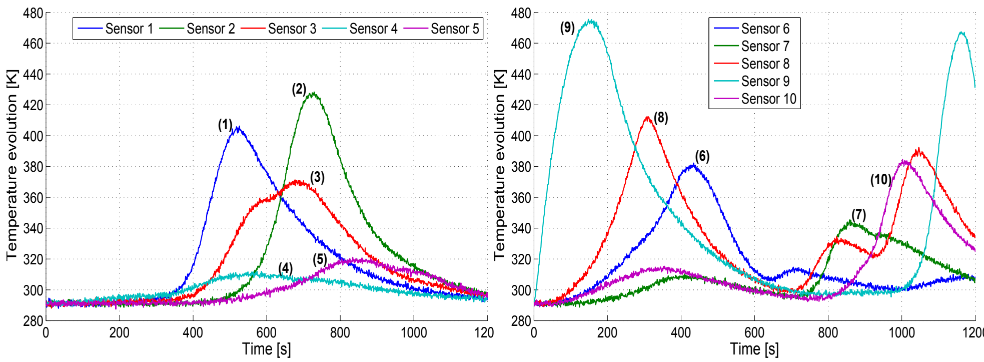

(a) Temperature evoltion of sensors C_{1-5} (b) Temperature evoltion of sensors C_{6-10}

Figure 5. Heat flux density (a) and Trajectory (b) of two heat sources

These numerical data are considered as "measures" of temperatures to test if the identification procedure makes it possible to determine what is the density of flow which is the cause, for an example. A network of fixed "temperature sensors" is considered (see Figure 3 (b)). The numerical results are noisy with a realistic measurement noise: disturbances followed by Gaussian probability distributions $\mathcal{N}(\mu, \sigma^2)$ with mean $\mu = 1$ and standard deviation $\sigma = 0$. The temperature evolution at these sensors is shown in Figure 5.

5 Conclusion and perspective

This paper proposes a mathematical model of heat transfer in two-dimensional space that can replace heat transfer in three-dimensional space but does not change the physical properties of the system. The purpose of the proposed alternative model is to simplify the mathematical model to make it easier to solve mathematical problems associated with this mathematical model. From there, it makes the response time of the system built from this model faster and makes the associated algorithms more robust. Through the statistics of the results from the simulation, it was shown that the mathematical model describing the heat transfer process in the proposed two-dimensional space could replace the heat equation of the reaction in complex three-dimensional space. more than The results of this study will be of great help to other studies related to the use of mathematical models describing heat transfer in later thermal conductivity.

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