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On Majorization Problems Associated with the Subclass $Q(j,\lambda,\alpha,n)$ of Starlike Functions with Positive Coefficients

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Abstract. We consider the subclass $Q(j,\lambda,\alpha,n)$ of starlike functions by using thedifferential D^n operator and functions of the form $f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k$ which are analytic in the open unit disk. In this paper is to investigate anmajorizationproblem for the subclass $Q(j,\lambda,\alpha,n)$. Relevant connections of the main result obtained in this paper with those given by earlier workers on the subject are also pointed out.

1 Introduction

Let the functions $f(z)$ and $g(z)$ be analytic in the open unit disk

$$U = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}. \tag{1}$$

It is called that $f(z)$ is majorized by $g(z)$ in U and write

$$f(z) \ll g(z) (z \epsilon U) \tag{2}$$

if there exists a function $\vartheta(z)$, analytic in U, such that

$$\vartheta(z) \le 1 \text{ and } f(z) = \vartheta(z) g(z) (z \in U) \tag{3}$$

Let A denote the family of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{4}$$

that are analytic in the open unit disk U. A function $f \in A$ is said to be starlike of order $\alpha (0 \le \alpha < 1)$ if and only if

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$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z\epsilon U). \tag{5}$$

We denote by $\boldsymbol{S^*(\alpha)}$, the class of all such functions. On the other hand, a function $\boldsymbol{f \in A}$ is said to be convex of order $\boldsymbol{\alpha}$ $(\boldsymbol{0 \le \alpha < 1})$ if and only if

$$Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha \; (z\epsilon U). \tag{6}$$

Let $\boldsymbol{C(\alpha)}$ denote the class of all those functions which are convex of order $\boldsymbol{\alpha}$ in $\boldsymbol{U}$. Note that $\boldsymbol{S^*(0) = S^*}$ and $\boldsymbol{C(0) = C}$ are, respectively, the classes of starlike and convex functions in $\boldsymbol{U}$.

Let $\boldsymbol{A(j)}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \; (j \in \mathbb{N} = \{1,2,3,\dots\}) \tag{7}$$

which are analytic in the open unit disk $\boldsymbol{U}$.

For a function $\boldsymbol{f(z)}$ in $\boldsymbol{A(j)}$, we define

$$\boldsymbol{D^0 f(z) = f(z)}$$

$$\boldsymbol{D^1 f(z) = Df(z) = zf'(z) = z + \sum_{k=j+1}^{\infty} k a_k z^k}$$

$$\vdots$$

$$\boldsymbol{D^n f(z) = D\left(D^{n-1} f(z)\right) = z + \sum_{k=j+1}^{\infty} k^n a_k z^k} \qquad (\boldsymbol{n} \in \mathbb{N}).$$

The differential operator $\boldsymbol{D^n}$ was introduced by Sălăgean [1]. With the help of the differential operator $\boldsymbol{D^n}$, we say that a function $\boldsymbol{f(z)}$ belonging to $\boldsymbol{A(j)}$ is in the class $\boldsymbol{Q(j,\lambda,\alpha,n)}$ if and only if

$$Re\left\{\frac{(1-\lambda)z\left(D^n f(z)\right)' + \lambda z\left(D^{n+1} f(z)\right)'}{(1-\lambda)D^n f(z) + \lambda D^{n+1} f(z)}\right\} > \alpha \tag{8}$$

for some $\boldsymbol{\alpha}$ $(\boldsymbol{0 \le \alpha < 1})$ and $\boldsymbol{\lambda}$ $(\boldsymbol{0 \le \lambda \le 1})$, and for all $\boldsymbol{z\epsilon U}$. The subclass $\boldsymbol{Q(j,\lambda,\alpha,n)}$ is defined by [2].

2 Majorization Problems ForThe Class $\boldsymbol{Q(j,\lambda,\alpha,n)}$

Theorem Let the function $\boldsymbol{f(z)}$ be in the $\boldsymbol{A(j)}$ and suppose that $\boldsymbol{g(z) \in Q(j,\lambda,\alpha,n)}$. If

$\boldsymbol{D^n f(z)}$ is majorized by $\boldsymbol{D^n g(z)}$ in $\boldsymbol{U}$ then

$$|(D^n f(z))'| \le |(D^n g(z))'| \text{ or } |D^{n+1} f(z)| \le |D^{n+1} g(z)| \quad (|z| < r), \tag{9}$$

where

$$r = r(\lambda, \alpha) = \frac{2(1+\lambda\alpha) - \sqrt{4(1+\lambda\alpha)^2 - (1-\lambda)^2}}{1-\lambda} \tag{10}$$

$(j \in \mathbb{N}; n \in \mathbb{N}_0; 0 \le \alpha < 1; 0 \le \lambda < 1;\ z \in U)$.

The result is sharp.

Proof Since $g(z) \in Q(j, \lambda, \alpha, n)$, we readily find from (8), if

$$k(z) = \frac{(1-\lambda)z(D^n g(z))' + \lambda z(D^{n+1} g(z))'}{(1-\lambda)D^n g(z) + \lambda z D^{n+1} g(z)} \text{ and } p(z) = \frac{k(z)-\alpha}{1-\alpha} \tag{11}$$

Then

$$p(z) = 1 + \cdots \text{ and } \Re\{p(z)\} > 0 \tag{12}$$

and

$$p(z) = \frac{1-w(z)}{1+w(z)},\ (w \in \Omega) \tag{13}$$

where $\boldsymbol{\Omega}$ denotes the well-known class of bounded analytic functions in $\boldsymbol{U}$, which satisfy the conditions [[3], p.58]

$$w(0) = 0 \text{and } |w(z)| \le |z| (z \in U). \tag{14}$$

Making use of (11) and (13), as readily obtain

$$\begin{aligned}
\lambda(1+w(z))z^2(D^n g(z))'' + [(1-\lambda\alpha)(1+w(z)) - (1-\alpha)(1-w(z))\lambda]z(D^n g(z))' \\
= [(1-\alpha)(1-\lambda)(1-w(z)) + \alpha(1-\lambda)(1+w(z))]D^n g(z) \\
\Rightarrow (1+|w(z)|)(1+\lambda+2\lambda\alpha)|D^n g(z)| \\
\ge (1-|w(z)|)(1-\lambda)|D^n g(z)|
\end{aligned}$$

or

$$(1-|w(z)|)(1-\lambda)|D^n g(z)| \le (1+|w(z)|)(1+\lambda+2\lambda\alpha)|D^{n+1} g(z)| \tag{15}$$

which, in view of (14), immediately yields the inequality

$$|D^n g(z)| \le \frac{(1+|z|)(1+\lambda+2\lambda\alpha)}{(1-|z|)(1-\lambda)} |D^{n+1} g(z)| (z \in U). \tag{16}$$

Next, since $\boldsymbol{D^n f(z)}$ is majorized by $\boldsymbol{D^n g(z)}$ in $\boldsymbol{U}$, from (3) we have

$$\boldsymbol{D(D^n f(z)) = D(\vartheta(z) D^n g(z))}$$

$$\boldsymbol{D^{n+1} f(z) = \vartheta'(z) z D^n g(z) + \vartheta(z) D^{n+1} g(z)}$$

$$\boldsymbol{|D^{n+1} f(z)| = |\vartheta'(z)||z||D^n g(z)| + |\vartheta(z)||D^{n+1} g(z)|.} \tag{17}$$

Thus, observing that $\boldsymbol{\vartheta \in \Omega}$ satisfiees the inequality

$$\boldsymbol{|\vartheta'(z)| \leq \frac{1 - |\vartheta(z)|^2}{1 - |z|^2}} \tag{18}$$

and applying (16) and (18) in (15), we get

$$\boldsymbol{|D^{n+1} f(z)| \leq \left[\frac{1 - |\vartheta(z)|^2}{1 - |z|^2} \frac{(1 + |z|)(1 + \lambda + 2\lambda\alpha)}{(1 - |z|)(1 - \lambda)} |z| + |\vartheta(z)| \right] |D^{n+1} g(z)|} \tag{19}$$

which upon setting

$\boldsymbol{|z| = r}$ and $\boldsymbol{|\vartheta(z)| = \rho (0 \leq \rho \leq 1)}$

leads us to the inequality

$$\boldsymbol{|D^{n+1} f(z)| \leq \frac{\theta(\rho)}{(1 - r^2)(1 - \lambda)} |D^{n+1} g(z)|} \tag{20}$$

where the function $\boldsymbol{\theta(\rho)}$ defined by

$$\boldsymbol{\theta(\rho) = (1 - \rho^2)(1 + \lambda + 2\lambda\alpha) + (1 - r)^2 (1 - \lambda)\rho} \tag{21}$$

takes on its maximum value at $\boldsymbol{\rho = 1}$ with

$$\boldsymbol{r = r(\lambda, \alpha)}$$

given by (10). Furthermore, if

$$\boldsymbol{0 \leq \sigma \leq r(\lambda, \alpha)}$$

where $\boldsymbol{r(\lambda, \alpha)}$ is given by (10) then the function $\boldsymbol{A(\rho)}$ defined

$$\boldsymbol{A(\rho) = (1 - \rho^2)(1 + \lambda + 2\lambda\alpha) + (1 - \sigma)^2 (1 - \lambda)\rho} \tag{22}$$

is seen to be in increasing function on the interval $\boldsymbol{0 \leq \rho \leq 1}$, so that

$$A(\rho) \leq A(1) = (1-\sigma)^2(1-\lambda)$$
$$0 \leq \rho \leq 1 \,;\, 0 \leq \sigma \leq r(\lambda, \alpha)$$

Hence, by setting $\rho = 1$ in (21), we conclude that (9) of Theorem holds true for $|z| \leq r(\lambda, \alpha)$, where $r(\lambda, \alpha)$ is given by (10). This completes the proof of Theorem.

3 Application

From the expression of the Theorem means that the radius is independent of n. If we set $n = 0$ in Theorem, we find the same radius.The following application refers to this.

Application Let the function $f(z)$ be in the $A(j)$ and suppose that $g(z) \in Q(j, \lambda, \alpha, 0)$. If $f(z)$ is majorized by $g(z)$ in U then

$$|Df(z)| \leq |Dg(z)| \text{or} |f'(z)| \leq |g'(z)| \quad (|z| < r),$$

where

$$r = r(\lambda, \alpha) = \frac{2(1+\lambda\alpha) - \sqrt{4(1+\lambda\alpha)^2 - (1-\lambda)^2}}{1-\lambda}$$

$(j \in \mathbb{N}; n = 0; 0 \leq \alpha < 1; 0 \leq \lambda < 1;\ z \in U)$.

Corollary If we set $\lambda = 0$ in Theorem, we immediately obtain. (MacGregor [[4], p.296, Therom 1]). Let the functions $f(z)$ be analytic in U and suppose that $g \in S^* = S^*(0)$, where S denotes the class of (normalized) analytic and univalent functions in U . If $f(z)$is majorized $g(z)$ in, then

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq 2 - \sqrt{3}).$$

4 Conclusion

Let S denote the family of functions $f(z)$ which are analytic and univalent in the open unit disc U and which are normalized so that $f(0) = 0$, $f'(0) = 1$.Let $F(z)$ be an arbitrary function in S. If $f(z) \ll F(z)$in U, determine the largest r for which $f'(z) \ll F'(z)$ in $|z| < r$ [5].

In this paper, we obtain solution of majorization problems for $Q(j, \lambda, \alpha, n)$ class. So if $D^n f(z) \ll D^n g(z)$ in U, determine the largest r for which $D^{n+1} f(z) \ll D^{n+1} g(z)$ in $|z| < r$. Relevant connections of the main result obtained in this paper with those given by earlier workers on the subject are also pointed out.

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