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# ON CHARACTERIZATION OF OPEN SETS IN EUCLIDEAN SPACES

Firudin Kh. Muradov*
*Department of Mathematics, Near East University,Nicosia, North Cyprus, Mersin-10, Turkey

**Abstract**

A ternary semigroup is a nonempty set $T$ together with a ternary operation $[abc]$ satisfying the associative law $[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]]$ for all $a, b, c, d, e \in T$. A map $f$ between topological spaces $X$ and $Y$ is called open if the image of each set open in $X$ is open in $Y$. The purpose of this paper is to give an abstract characterization of the ternary semigroups of open maps defined on open sets in Euclidean $n$-spaces.

## 1 Introduction

A ternary semigroup is a nonempty set $T$ together with a ternary operation $[abc]$ satisfying the associative law $[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]]\,, \forall a, b, c, d, e \in T$. The theory of ternary algebraic systems was introduced by Lehmer in [1]. Sioson [2] studied ternary semigroups with special reference to ideals and radicals . Dutta [4], Santiago [5] developed regular and completely regular ternary semigroups.

## 2 An Abstract Charcterization of Ternary Semigroups of Open Maps

Let $R^n$ be the Euclidean $n$-space. The collection of all open $n$-cubes in $R^n$ forms a basis for a topology on $R^n$ called the standard topology on $R^n$. Let $X_1$ and $X_2$ be two open subsets of $R^n$ , $(n > 1)$ and $\varphi : X_1 \to X_2$, $\phi : X_2 \to X_1$. Denote the set of all pairs of $(\varphi, \phi)$ of open maps between $X_1$ and $X_2$ by $O\,(X_1, X_2)$. The set $O\,(X_1, X_2)$ is a ternary semigroup with respect to the ternary operation $[(\varphi_1, \phi_1)\,(\varphi_2, \phi_2)\,(\varphi_3, \phi_3)] = (\varphi_1\phi_2\varphi_3, \phi_1\varphi_2\phi_3)$.

**Theorem 1** *Let $X_1$ and $X_2$ be open subsets of $R^n$ and let $Y_1$ and $Y_2$ be open subsets of $R^m$ , $(n, m > 1)$. The ternary semigroups $O\,(X_1, X_2)$ and $O\,(Y_1, Y_2)$ are isomorphic if and only if the spaces $X_i$ and $Y_i$ are homeomorphic $(i = 1, 2)$.*

*firudin.muradov@neu.edu.tr

**Proof.** It is obvious that if $X_i$ and $Y_i$ are homeomorphic $(i = 1, 2)$ then the ternary semigroups $O(X_1, X_2)$ and $O(Y_1, Y_2)$ are isomorphic. Specifically, if $h_i$ is a homeomorphism from $X_i$ to $Y_i$ $(i = 1, 2)$ and $(\varphi, \phi) \in O(X_1, X_2)$ then the map $\theta(\varphi, \phi) \in O(Y_1, Y_2)$ defined by $\theta(\varphi, \phi) = \left(h_2 \circ \varphi \circ h_1^{-1}, h_1 \circ \phi \circ h_2^{-1}\right)$ is an isomorphism from $O(X_1, X_2)$ to $O(Y_1, Y_2)$. The proof of the necessary condition follows from Lemmas 2-6. Throughout this paper, the symbol $\theta$ denotes the isomorphism from $O(X_1, X_2)$ onto $O(Y_1, Y_2)$. ■

**Lemma 2** *Let $X_1$ and $X_2$ be two open subsets of $R^n$ and let $z$ be any point in $X_1$. Then, there exist open maps $f, g : X_1 \to X_2$ such that $f(z) \neq g(z)$ but $f(x) = g(x)$ for all $x \in X_1 \setminus \{z\}$.*

**Proof.** Let $Z_1$ and $Z_2$ be open $n$-cubes in $R^n$. The open $n$-cube is the set of all points $(z_1, ..., z_n) \in R^n$ such that $a_i < z_i < b_i$ for $i = 1, 2, ..., n$, i.e. $Z_1$ has the form $B_1 \times B_2 \times ... \times B_n$, where $B_1, B_2, ..., B_n$ are open intervals in $R$. For $z_1 \in B_1, z_2 \in B_2, ..., z_n \in B_n$ the sets of the form $z_1 \times B_2 \times ... \times B_n$, $B_1 \times z_2 \times ... \times B_n$,..., $B_1 \times B_2 \times ... \times z_n$ are called coordinate planes through the point $(z_1, ..., z_n)$. Analogously to the one dimensional case we use the procedure of constructing an open map $f$ from $n$-cube to $n$-cube. Divide each coordinate interval $B_i, (i = 1, 2, ..., n)$ into 3 equal parts and the open $n$-cube bounded by the middle thirds map homeomorphically onto $Z_2$. Then divide the coordinate intervals of the remaining $n$-cubes into 3 equal parts and map the open n-cubes bounded by the middle thirds homeomorphically onto $Z_2$. Continuing this process infinitely we map countable number of open $n$-cubes onto $Z_2$. The constructed map is open.

Now let $X_1$ be an open subset in $R^n$and let $z \in X_1$. Assume without loss of generality that $X_1$ and $X_2$ are bounded. Let $Z_1$ be an open $n$-cube containing $X_1$ and let $Z_2$ be an open $n$-cube that is contained in $X_2$. Next we define the maps $f$ and $g$ at the point $z = (z_1, ..., z_n) \in X_1$ arbitrarily such that $f(z) \neq g(z)$. Determine the values of $f$ and $g$ on the coordinate planes $z_1 \times B_2 \times ... \times B_n$, $B_1 \times z_2 \times ... \times B_n$,..., $B_1 \times B_2 \times ... \times z_n$ through the point $z$ that divide $Z_1$ at the point $z$ into $2^n$ $n$-subcubes, to be a fixed point in $Z_2$. If we determine the values of $f$ and $g$ on the open $n$-subcubes applying the above procedure, then the maps $f$ and $g$ are open and we have $f(z) \neq g(z)$ but $f(x) = g(x)$ for all $x \in Z_1 \setminus \{z\}$. The restrictions of the maps $f$ and $g$ to $X_1$ are open and we have $f(z) \neq g(z)$ but $f(x) = g(x)$ for all $x \in X_1 \setminus \{z\}$. ■

**Lemma 3** *Let $(\varphi_1, \phi_1), (\varphi_2, \phi_2)$ be arbitrary elements of $O(X_1, X_2)$. For every $f, g : X_1 \to X_2$ and $h_1, h_2 : X_2 \to X_1$ the condition*

$$[(f, h_1)(f, h_2)(\varphi_1, \phi_1)] = [(g, h_2)(g, h_1)(\varphi_1, \phi_1)] \tag{1}$$

*implies*

$$[(f, h_1)(f, h_2)(\varphi_2, \phi_2)] = [(g, h_2)(g, h_1)(\varphi_2, \phi_2)]$$

*is necessary and sufficient for $\varphi_2(X_1) \subseteq \varphi_1(X_1)$ and $\phi_2(X_2) \subseteq \phi_1(X_2)$.*

**Proof.** If the condition $\varphi_2(X_1) \subseteq \varphi_1(X_1)$ is satisfied, then for every $x \in X_1$ there exists a point $z \in X_1$ such that $\varphi_2(x) = \varphi_1(z)$. Then

$$fh_2\varphi_2(x) = fh_2\varphi_1(z) = gh_1\varphi_1(z) = gh_1\varphi_2(x).$$

If the condition $\phi_2(X_2) \subseteq \phi_1(X_2)$ is satisfied, then for every $x \in X_2$ there exists a point $z \in X_2$ such that $\phi_2(x) = \phi_1(z)$. Then

$$h_1 f\phi_2(x) = h_1 f\phi_1(z) = h_2 g\phi_1(z) = h_2 g\phi_2(x).$$

So the condition 1 holds.

Conversely, let the condition 1 hold for some $(\varphi_1, \phi_1), (\varphi_2, \phi_2) \in O(X_1, X_2)$. Suppose that $\varphi_2(X_1) \setminus \varphi_1(X_1)$ is not empty. It follows from Lemma 2 that for any point $y = \varphi_2(x_1)$ in $\varphi_2(X_1) \setminus \varphi_1(X_1)$ there exist open maps $h_1, h_2 : X_2 \to X_1$ such that $h_1(y) \neq h_2(y)$ but $h_1(x) = h_2(x)$ for all $x \in X_2 \setminus \{y\}$. Then for every $x \in X_1$ we have $h_1\varphi_1(x) = h_2\varphi_1(x)$ and $fh_1\varphi_1(x) = fh_2\varphi_1(x)$ for each open map $f : X_1 \to X_2$. But for $y = \varphi_2(x_1)$ in $\varphi_2(X_1) \setminus \varphi_1(X_1)$ we have $h_1\varphi_2(x_1) = h_1(y) \neq h_2(y) = h_2\varphi_2(x_1)$ and $fh_1\varphi_2(x_1) \neq fh_2\varphi_2(x_1)$ for some open map $f : X_1 \to X_2$ which contradicts to 1. ∎

**Lemma 4** *Let $X_1$ and $X_2$ be open subsets of $R^n$ and let $Y_1$ and $Y_2$ be open subsets of $R^m$ , $(n, m > 1)$. Suppose that $\theta$ is an isomorphism from $O(X_1, X_2)$ to $O(Y_1, Y_2)$ and $(\varphi_1, \phi_1), (\varphi_2, \phi_2) \in O(X_1, X_2)$. If $\varphi_1(X_1) = \varphi_2(X_1)$ and $\phi_1(X_2) = \phi_2(X_2)$ then $(\theta\varphi_1)(Y_1) = (\theta\varphi_2)(Y_1)$ and $(\theta\phi_1)(Y_2) = (\theta\phi_2)(Y_2)$.*

**Proof.** Suppose that $\varphi_2(X_1) \subseteq \varphi_1(X_1)$. If $f'h_2'(\theta\varphi_1)) = g'h_2'(\theta\varphi_1)$ for some elements $f', g', h_1', h_2' \in O(Y_1, Y_2)$ then there exist $f, g, h_1, h_2 \in O(X_1, X_2)$ such that $f' = \theta f$, $g' = \theta g$, $h_1' = \theta h_1$ and $h_2' = \theta h_2$. Then $(\theta f)(\theta h_2)(\theta\varphi_1) = (\theta g)(\theta h_2)(\theta\varphi_1)$ and since $\theta$ is an isomorphism, $\theta(fh_2\varphi_1) = \theta(gh_2\varphi_1)$ and $fh_2\varphi_1 = gh_2\varphi_1$. We have $fh_2\varphi_2 = gh_2\varphi_2$, by Lemma 3. Again, since $\theta$ is an isomorphism, then $(\theta f)(\theta h_2)(\theta\varphi_2) = (\theta g)(\theta h_2)(\theta\varphi_2)$ and therefore $f'h_2'(\theta\varphi_2)) = g'h_2'(\theta\varphi_2)$. Because $f'h_2'(\theta\varphi_2)) = g'h_2'(\theta\varphi_2)$ is true for every $f', g', h_1', h_2' \in O(Y_1, Y_2)$ satisfying the condition $f'h_2'(\theta\varphi_1)) = g'h_2'(\theta\varphi_1)$ it follows from Lemma 3 that $(\theta\varphi_2)(Y_1) \subseteq (\theta\varphi_1)(Y_1)$. In the same way, we could show that if $\varphi_1(X_1) \subseteq \varphi_2(X_1)$ then $(\theta\varphi_1)(Y_1) \subseteq (\theta\varphi_2)(Y_1)$. ∎

**Lemma 5** *Let $X_1$ and $X_2$ be open subsets of $R^n$ and let $A$ and $B$ be open subsets of $X_2$ and $X_1$, respectively. Then there exist open maps $(\varphi, \phi) \in O(X_1, X_2)$ such that $\varphi(X_1) = A$ and $\phi(X_2) = B$.*

**Proof.** Let $X_1$ and $X_2$ be open subsets of $R^n$ and let $A$ be an open subset of $X_2$. Assume without loss of generality that $X_1$ and $X_2$ are bounded. Let $R_2$ be a closed $n$-cube contained in $A$ and let $R_1$ be a closed $n$-cube contained in $X_1$. If $E_1$ is a closed $n$-ball containing $X_1 \setminus R_1$ and if $E_2$ is a closed $n$-ball that is contained in $A \setminus R_2$, then there exists a homeomorphism from $E_1$onto $E_2$. Denote by $g_1$ the restriction of this homeomorphism to $X_1 \setminus R_1$. Clearly the map $g_1$is open. Then construct an open map from $IntR_1$ onto $IntR_2$ using the same procedure as in Lemma 2. Denote by $g_2$ the extension of this map from $IntR_1$ to $R_1$, obtained by mapping the boundary of $R_1$ to a fixed point in $IntR_2$. We next define the map $f : X_1 \to A$ by

$$f(x) = \begin{cases} g_2(x), & \text{if} \quad x \in R_1 \\ g_1(x), & \text{if} \ \ x \in X_1 \setminus R_1 \end{cases}$$

Suppose that $V$ is an open set in $X_1$. If $V$ is in $X_1 \setminus R_1$ then $f(V) = g_1(V)$ and therefore $f(V)$ is an open set of $X_2$. If $V \subset R_1$ then $V \subset IntR_1$ and then $f(V) = g_2(V)$ and therefore $f(V)$ is the open set of $X_2$. Now suppose that $V$ has nonempty intersections with $R_1$ and $X_1 \setminus R_1$, i.e., $V \cap R_1 \neq \varnothing$ and $V \cap (X_1 \setminus R_1) \neq \varnothing$. Since the set $V \cap (X_1 \setminus R_1)$ is open in $X_1$ and is contained in $X_1 \setminus R_1$, the set $f(V \cap (X_1 \setminus R_1))$ is the open set of $X_2$. If the set $V \cap R_1$ does not contain boundary points of $R_1$, then $V \cap R_1 \subset IntR_1$. By the definition of $f$ we have $f(V \cap R_1) = g_2(V \cap R_1)$ and therefore $f(V \cap R_1)$ is the open set of $X_2$. If the set $V \cap R_1$ contains boundary points of $R_1$, then at each neighborhood of these points there is an open $n$-cube that is homeomorphically mapped onto $IntR_2$. So, $f(V \cap R_1) = IntR_2$, that is $f(V \cap R_1)$ is open and therefore the map $f$ is open.

Now, let us change $f$ such that the image of the new constructed map $\varphi$ was $A$, i.e. $\varphi(X_1) = A$. Denote by $\widetilde{R_1}$ the sequence of the decreasing $n$-cubes in $R_1$ converging to some vertex of $R$ that are mapped homeomorphically onto $IntR_2$. Since $A$ has a countable base, we can map these $n$-cubes homeomorpically onto base sets of $A$. Define $\varphi : X_1 \to A$ as follows. On the sets $X_1 \setminus R_1$ and $R_1 \setminus \widetilde{R_1}$, the values of $\varphi$ coincide with the values of $f$. On the set $\widetilde{R_1}$ the image of $\varphi$ is $A$, that is we map the $n$-cubes of $\widetilde{R_1}$ homeomorpically onto base sets of $A$ . Show that the map $\varphi$ is an open map. Suppose that $V$ is an open set in $X_1$. If $V$ is in $X_1 \setminus R_1$ then $\varphi(V) = f(V)$ and therefore $\varphi(V)$ is an open set of $X_2$. If $V \subset R_1 \setminus \widetilde{R_1}$ then $\varphi(V) = f(V)$ and therefore $\varphi(V)$ is the open set of $X_2$. Now suppose that $V$ contains a point from boundary of $\widetilde{R_1}$. Then at each neighborhood of this point there is an open $n$-cube that is homeomorphically mapped onto $IntR_2$. Since the set $V$ is open, there exists a neighborhood of this boundary point contained in $V$ and therefore this neighborhood contains an $n$-cube from $R_1 \setminus \widetilde{R_1}$ that is homeomorphically mapped onto $IntR_2$. In this case $\varphi(V) = IntR_2 \cup \widetilde{A}$, where $\widetilde{A}$ is an open subset of $A$. Finally, if $V$ is in $\widetilde{R_1}$, then because the $n$-cubes of $\widetilde{R_1}$ are mapped homeomorpically onto base sets of $A$ it follows that $\varphi(V)$ is the open set of $X_2$. Consequently the map $\varphi$ is open and $\varphi(X_1) = A$. In the same way we can show that there is an open map $\phi : X_2 \to B$ such that $\varphi(X_2) = B$. ■

Let $X$ be a topological space. Let $C(X)$ denote the family of all open sets of $X$. The family $C(X)$ is a complete distributive lattice if set inclusion is taken as the ordering. The supremum is given by the union of open sets and the infimum by the interior of the intersection. Two topological spaces $X$ and $Y$ are said to be lattice-equivalent if there is a bijective map from $C(X)$ to $C(Y)$ which together with its inverse is order-preserving. In his paper [3], Thron proved that for $T_D-$spaces any lattice-isomorphism can be induced by a homeomorphism.

**Lemma 6** *Let $X_1$ and $X_2$ be open sets of $R^n$ and let $Y_1$ and $Y_2$ be open sets of $R^m$ , $(n, m > 1)$. The lattices $C(X_i)$ and $C(Y_i)$ are lattice-isomorphic, $(i = 1, 2)$.*

**Proof.** Let $U_1$ and $U_2$ be open subsets of $X_1$ and $X_2$, respectively. By Lemma 5 there exists $(\varphi, \phi) \in O(X_1, X_2)$ such that $\varphi(X_1) = U_2$ and $\phi(X_2) = U_1$. Since the ternary semigroups $O(X_1, X_2)$ and $O(Y_1, Y_2)$ are isomorphic we can find $(\varphi', \phi') \in O(Y_1, Y_2)$ such that $\theta\varphi = \varphi'$ and $\theta\phi = \phi'$. Suppose that $\varphi'(Y_1) = U_2'$ and $\phi'(Y_2) = U_1'$. Let us define maps $\xi_i : C(X_i) \to C(Y_i)$, $(i = 1, 2)$ by mapping to each open set $U_i \subset X_i$ the set $U_i' \subset Y_i$. The maps $\xi_i$ do not depend on the choice of $(\varphi, \phi) \in O(X_1, X_2)$. Indeed , if $\varphi_1(X_1) = U_2$ and $\phi_1(X_2) = U_1$ for some $(\varphi_1, \phi_1) \in O(X_1, X_2)$ then Lemma 4 says that $(\theta\varphi)(Y_1) = (\theta\varphi_1)(Y_1) = U_2'$ and $(\theta\phi)(Y_2) = (\theta\phi_1)(Y_2) = U_1'$. Let $U$ and $V$ be any two different open subsets of $X_2$. By Lemma 5 there are two open maps $\varphi_1$ and $\varphi_2$ from $X_1$ to $X_2$ such that $\varphi_1(X_2) = U$ and $\varphi_2(X_2) = V$. Since the ternary semigroups $O(X_1, X_2)$ and $O(Y_1, Y_2)$ are isomorphic the maps $\theta\varphi_1$ and $\theta\varphi_2$ are different and it follows from Lemma 4 that $(\xi_2\varphi_1)Y_2 \neq (\xi_2\varphi_2)Y_2$. Hence $\xi_2$ is a one-to-one map between $C(X_2)$ and $C(Y_2)$. In the same way we can show that the map $\xi_1$ is a one-to-one map between $C(X_1)$ and $C(Y_1)$.Now suppose that $U_i'$ is an arbitrary open set in $Y_i$ for $i = 1, 2$. From isomorphism between $O(X_1, X_2)$ and $O(Y_1, Y_2)$ and Lemma 4 there exists an open set $U_i \subset X_i$ such that $\xi_i(U_i) = U_i'$. Again using Lemma 4 we can conclude $\xi_i$ saves order, that is, if $U_i \subseteq V_i$ then $\xi_i(U_i) \subseteq \xi_i(V_i)$ for $i = 1, 2$. ■

From Theorem 2.1 of [3] it follows that the open sets $X_i$ and $Y_i$, $(i = 1, 2)$ are homeomorphic.

# 3 Conclusion

In this paper we consider ternary semigroups and give an abstract characterization of ternary semigroups of open maps defined on open sets of Euclidean $n$-spaces.

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