A Soft Approach to Ring-Groupoids

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Abstract. This study introduces a soft approach to the concept of ring-groupoid which is the one of the structured groupoids. Some properties and characterizations of soft ring-groupoids are established. Also, the category of soft ring-groupoids constructed by the homomorphism between two soft ring-groupoids is presented.

1 Introduction

Several theories have been developed to model the uncertainties appearing in engineering, economics, medicine, and many other field. Soft set theory is one of the most powerful theories that modeling the uncertainty and vagueness [1]. It, which was initiated by Molodtsov in 1999, has been worked on many areas including mathematics. The boundaries of this theory, which attracted a great interest of mathematicians, have been expanded in algebraic, topological and categorical sense [2-9, 12-14].

The first algebraic studies on this theory were presented by Maji and et.al. [2]. They presented different operators on soft sets and obtained several of their fundamental features. After that, Aktas and Cagman described the concept of soft group, which is a parametrized collection of subgroups of a group, over a group [3]. In [7], Acar and et.al. defined the notion of soft ring over a ring. On the other hand, the topological studies on this theory were first made by Shabir and Naz [4]. They examined the separation axioms on a soft topological space by defining soft topological space. Later, Cagman and et.al., and Min established some basic results by studying on soft topological spaces and algebraic structures [6, 9].

Recently, categorical studies on soft theory have been carried out. We refer to [12-14] for these studies. The definition of soft category was first proposed by Sardar and Gupta [12]. Also, they introduced the main notions of soft category theory and explained the relationship between soft category and fuzzy category.

Also, another concept in the focus of this study is groupoid. Although the notion of groupoid was first mentioned by Brandt, pioneering work on it was done by Ehresmann and Pradines [10, 20-21]. Subsequently, some algebraic structures were added to the groupoid, which the composition of groupoid is compatible with operators of these added structures. One of these is the ring groupoid which is defined as a groupoid endowed with a ring structure [17]. Afterwards, Gursoy extended the notion of ring groupoid to the generalized ring-groupoid [18].

In this manuscript, we describe the notion of soft ring-groupoid and discuss its features. Subsequently, by the definition of soft ring-groupoid homomorphism , we establısh the cate-

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gory of soft ring-groupoids. We also present the definition of soft subring-groupoid. Lastly, we complete this manuscript with a summary for further research.

In summary, we contribute to this expanding literature by offering a different approach to soft set theory.

2 Preliminaries

Here, we give some of the main notions such as soft set, soft ring, groupoid and ring-groupoid that we will use later. For more about these concepts, please refer $[1-3, 7, 11, 15-17]$.

Let X be a set of parameters and let $P(U)$ denotes the power set of U for the initial universe set U.

Definition 2.1 [1] *Let A be a non-empty subset of X. A pair (F, A) is said to be a soft set over U,in which F is a mapping defined by*

$$F : A \longrightarrow P(U)$$

That is, a soft set over U can actually be considered as a parametrized collection of subsets of the universe U.

In this study, we will sometimes use the triplet (U, F, A) instead of soft set (F, A) over U.

Definition 2.2 [2] *A soft set (U, F, A) is called a null soft set, indicated by Φ, if $F(\alpha) = \emptyset$ for all $\alpha \in A$.*

Definition 2.3 [2] *The support of the soft set (F, A) over U is defined as the set $Supp(F, A) = \{\alpha \in A \ : \ F(\alpha) \neq \emptyset\}$.*

Definition 2.4 [2] *Let (U, F, A) and (U, F', B) be two soft sets. Then, (U, F', B) is said to be a soft subset of (U, F, A) if*
i. *$B \subset A$, and*
ii. *$F(\beta)$ and $F'(\beta)$ are identical approximations for all $\beta \in B$.*

We denote it as $(F', B)\widetilde{\subset}(F, A)$.

Now we proceed on to recall the concept of soft ring. Suppose that $\mathcal{R}$ is a commutative ring and A be a non-empty set. Then,

Definition 2.5 [7] *For the non-null soft set (F, A) over $\mathcal{R}$, (F, A) is said to be a soft ring over $\mathcal{R}$ if and only if $F(\alpha)$ is a subring of $\mathcal{R}$ for all $\alpha \in A$.*

From here, it is said that the soft ring (F, A) is a parametrized family of subrings of the ring $\mathcal{R}$. In what follows, the notation $(\mathcal{R}, F, A)$ stands for the soft ring (F, A) over $\mathcal{R}$.

Definition 2.6 [7] *Let $(\mathcal{R}, F, A)$ and $(\mathcal{R}, F', B)$ be two soft rings. Then, $(\mathcal{R}, F', B)$ is said to be a soft subring of $(\mathcal{R}, F, A)$ if*
i. *$B \subset A$,*
ii. *$F'(\beta)$ is a subring of $F(\alpha)$ for all $\beta \in Supp(F', B)$.*

Definition 2.7 [7] *Let $(\mathcal{R}, F, A)$ and $(\mathcal{R}', F', B)$ be two soft rings and let $\theta : \mathcal{R} \longrightarrow \mathcal{R}'$ and $\psi : A \longrightarrow B$ be two mappings. Then, the pair (θ, ψ) is said to be a soft ring homomorphism if*
i. *θ is a ring epimorphism,*
ii. *ψ is a surjection, and*
iii. *$\theta(F(\alpha)) = F'(\psi(\alpha))$ for all $\alpha \in A$.*

In this case, (F, A) is called soft homomorphic to (F', B) and this situation is indicated by $(F, A) \sim (F', B)$. Further, (F, A) is called softly isomorphic to (F', B) if θ is a ring isomorphism and ψ is bijective.

Here, the definitions of groupoid and ring-groupoid will be presented assuming that the foundation of the category concept is known.

Definition 2.8 [15] *A groupoid $\mathcal{G}$ is a category $C = (Mor(C),\ Ob(C),\ s,\ t,\ m,\ i)$ where each morphism is an isomorphism.*

Definition 2.9 [15] *Let $\mathcal{G}$ and $\mathcal{G}'$ be two groupoids. Then, $\mathcal{G}'$ is called a subgroupoid of $\mathcal{G}$ if $\mathcal{G}'$ consists of the subcollections of $Mor(\mathcal{G})$ and $Ob(\mathcal{G})$, where it is closed under the composition and inversion in $\mathcal{G}$.*

The following definition is due to Mucuk [17].

Definition 2.10 [17] *A ring-groupoid $\mathcal{R}$ is a groupoid endowed with a ring structure such that the following maps are groupoid homomorphisms:*
i. $m : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$, $(x, y) \longmapsto x + y$, *group operation,*
ii. $u : \mathcal{R} \longrightarrow \mathcal{R}$, $x \longmapsto -x$, *group inversion map,*
iii. $n : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$, $(x, y) \longmapsto xy$, *ring operation, and*
iv. $e : \{\star\} \longrightarrow \mathcal{R}$, *where* $\{\star\}$ *is a singleton.*

A ring-groupoid homomorphism between two ring-groupoids is a homomorphism of underlying groupoids which preserves the ring structure. Therefore, the category of ring-groupoids is established with ring-groupoids and their homomorphisms.

Example 2.1 [17] *For a ring $\mathcal{R}$, the cartesian product $\mathcal{R} \times \mathcal{R}$ is a ring-groupoid over $\mathcal{R}$ with the ring operation described by $(x, y)(x', y') = (xx', yy')$ for all $x, y, x', y' \in \mathcal{R}$.*

Definition 2.11 [19] *Let $\mathcal{R}$ be a ring-groupoid and $\mathcal{P} \subset \mathcal{R}$. Then $\mathcal{P}$ is called a subring-groupoid of $\mathcal{R}$ If $\mathcal{P}$ is a ring-groupoid with the groupoid structure maps and ring maps induced from $\mathcal{R}$.*

3 Soft ring-groupoids

In this section, we will describe the notion of soft ring-groupoid.

Definition 3.1 *Let $\mathcal{R}$ be a ring-groupoid and let $P(\mathcal{R})$ denotes the set of all subring-groupoids of $\mathcal{R}$. A pair (F, A) is said to be a soft ring-grupoid over $\mathcal{R}$ if each $F(\alpha)$ is a subring-grupoid of $\mathcal{R}$ for all $\alpha \in A$, where F is a mapping given by*

$$F : A \longrightarrow P(\mathcal{R})$$

and A is a set of parameters.

In fact, a soft ring-groupoid can be considered as a parameterized collection of subring-groupoids of the ring-groupoid $\mathcal{R}$. After that, we will occasionally write $(\mathcal{R}, F, A)$ for the soft ring-groupoid (F, A) over the ring-groupoid $\mathcal{R}$.

Example 3.1 *Let (F, A) be a soft ring over the ring $\mathcal{R}$ with identity. Then, since (F, A) is a soft ring, each $F(\alpha)$ is a subring of $\mathcal{R}$ for all $\alpha \in A$. From the viewpoint of groupoid theory, every ring is a groupoid with only one object in which all of the arrows are the elements of the ring and the composition of arrows is the multiplication operation of the ring and the addition of* *arrows is the addition of elements of the ring. Furthermore, every ring is a ring-groupoid. For this reason , the ring $\mathcal{R}$ and $F(\alpha)$ are ring-groupoids for all $\alpha \in A$. From here, it is easy to verify that $F(\alpha)$ is a subring-groupoid of $\mathcal{R}$ for all $\alpha \in A$. Therefore, (F, A) is a soft ring-groupoid over $\mathcal{R}$.*

In this example, it is not hard to see that each soft ring is a soft ring-groupoid as above. As a more specific example, the following can be given.

Example 3.2 *Let (F, A) be a soft ring over $\mathcal{R}$. Hence, the mapping F can be written as*

$$\begin{array}{rcl} F : A & \longrightarrow & P(\mathcal{R}) \\ \alpha & \mapsto & F(\alpha) \end{array}$$

Using this mapping, we can define a mapping F' as follows:

$$\begin{array}{rcl} F' : A & \longrightarrow & P(\mathcal{R} \times \mathcal{R}) \\ \alpha & \mapsto & F'(\alpha) = F(\alpha) \times F(\alpha) \end{array}$$

we know from the groupoid theory that If $\mathcal{R}$ is a ring, $\mathcal{R}\times\mathcal{R}$ is a ring-groupoid over $\mathcal{R}$ with the ring operation defined by $(x, y)(x', y') = (xx', yy')$ for all $x, y, x', y' \in \mathcal{R}$. Furthermore, each $F'(\alpha) = F(\alpha) \times F(\alpha)$ is a subring-groupoid of $\mathcal{R} \times \mathcal{R}$ with the groupoid structure maps and ring maps induced from $\mathcal{R} \times \mathcal{R}$ for all $\alpha \in A$. So (F', A) is a soft ring-groupoid over $\mathcal{R} \times \mathcal{R}$.

Therefore, it is important to note that a soft ring-groupoid can be obtained from each soft ring.

Definition 3.2 *For two soft ring-groupoids $(\mathcal{R}, F, A)$ and $(\mathcal{P}, F', B)$, define the pair $(\mathcal{K}, \psi)$, where $\psi : A \longrightarrow B$ is a surjection and $\mathcal{K} : \mathcal{R} \longrightarrow \mathcal{P}$ is a functor. Then, the pair $(\mathcal{K}, \psi)$ which satisfies the following conditions is called a soft ring-groupoid homomorphism if it is a soft ring homomorphism:*
i. *$\mathcal{K}$ is full.*
ii. *$\mathcal{K}(F(a)) = F'(\psi(\alpha))$ for all $\alpha \in A$.*

In this manner, we construct the category of softring-groupoids whose objects are soft ring-groupoids and whose morphisms are soft ring-groupoid homomorphisms. This category is written as $SRgd$.

Example 3.3 *For two soft rings $(\mathcal{R}, F, A)$ and $(\mathcal{P}, F', B)$, let the pair (θ, ψ) be a soft ring homomorphism between them as defined in Definition 2.7. It follows from Example 3.1 that the soft rings $(\mathcal{R}, F, A)$ and $(\mathcal{P}, F', B)$ are soft ring-groupoids. Also, the conditions **i.** and **ii.** in Definition 2.7 imply that the soft ring homomorphism (θ, ψ) is a soft ring-groupoid homomorphism.*

We now present a concept related to the soft ring-groupoids as follows:

Definition 3.3 *Let $(\mathcal{P}, F', B)$ and $(\mathcal{R}, F, A)$ be two soft ring-groupoids. Then $(\mathcal{P}, F', B)$ is called a soft subring-groupoid of $(\mathcal{R}, F, A)$ if the following conditions are satisfied:*
i. *$B \subset A$.*
ii. *$F'(\beta)$ is a subgroupoid of $F(\beta)$ for all $\beta \in Supp(F', B)$.*
iii. *$Ob(F'(\beta))$ and $Mor(F'(\beta))$ are subrings of $Ob(F(\beta))$ and $Mor(F(\beta))$ for all $\beta \in Supp(F', B)$, respectively.*

Example 3.4 *For the soft ring-groupoid $(\mathcal{R}, F, A)$, let us constitute*

$$\mathcal{P} = \{Id_x : x \in Ob(F(\alpha)), \forall\alpha \in A\}$$

. It is straightforward to see that $(\mathcal{P}, F, A)$ is a soft subring-groupoid of $(\mathcal{R}, F, A)$.

4 Conclusion

In this manuscript, the notion of ring-groupoid is examined in the context of soft set theory and some fundamental features are presented. As the notion of ring-grupoid is softfied , it would be interesting to develop the other concept of groupoid theory with this setting.

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