An Approximate Grid Solution of a Nonlocal Boundary Value Problem with Integral Boundary Condition for Laplace's Equation

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Abstract. A new method for the solution of a nonlocal boundary value problem with integral boundary condition for Laplace's equation on a rectangular domain is proposed and justified. The solution of the given problem is defined as a solution of the Dirichlet problem by constructing the approximate value of the unknown boundary function on the side of the rectangle where the integral boundary condition was given. Further, the five point approximation of the Laplace operator is used on the way of finding the uniform estimation of the error of the solution which is order of $O(h^2)$, where h is the mesh size. Numerical experiments are given to support the theoretical analysis made.

1 Introduction

In [1]-[5], a new constructive method that reduces the multilevel nonlocal problem for Poisson's equation to local Dirichlet problem, was given.

In this paper, the idea of the method in [5] is applied for the approximate solution of the Laplace's equation with integral boundary condition. By applying trapeizodal rule for the integral boundary condition, the problem is approximated by the multilevel nonlocal boundary value problem which is solved as the sum of two classical 5-point finite-difference Dirichlet problems. Finally, the numerical experiments are illustrated to support to obtain theoretical results.

On the convergence of difference schemes for problems with integral boundary conditions without reducing to the local problems see [6],[7] and references therein.

2 Nonlocal Boundary Value Problem

Let

$$R = \{(x,y): 0 < x < 1, 0 < y < 2\} \tag{1}$$

be an open rectangle, $\gamma^m, m = 1,2,3,4$, be its sides including the endpoints, numbered in the clockwise direction, begining with the side lying on the y-axis and let $\gamma = \cup_{m=1}^{4} \gamma^m$ be the boundary of R.

Let C^0 denote the linear space of continous functions of one variable x on the interval $[0,1]$ of x-axis, and vanish at the points $x = 0$ and $x = 1$. For the function $f \in C^0$ we define the norm

$$\|f\|_{C^o} = \max_{0 \leq x \leq 1} |f(x)|.$$

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Consider the following nonlocal boundary value problem on R :

$$\Delta u \; = \; 0 \; on \; R, \qquad u = 0 \; on \; \gamma^1 \cup \gamma^3, \qquad u = \tau \; on \; \gamma^2, \tag{2}$$

$$u(x,0) = \; \alpha \int_{\xi}^{2} u(x,y)dy + \mu(x), \quad 0 \le x \le 1, \qquad 0 < \xi < 2, \tag{3}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial x^2$ is the Laplacian, $\tau = \tau(x) \in C^0$ is a given function, and $|\alpha| < \frac{1}{2-\xi}$ is a given constant.

By replacing the integral in condition (3) with its approximation using trapezoidal rule, we have

$$u(x_i,0) = \alpha \sum_{k=1}^{M} \rho_k . u(x_i, \eta_k) + \mu_i, \qquad i = 1,2,\dots,N-1.$$

where $\rho_1 = \rho_M = \frac{h}{2}$, $\rho_j = h$ for $j = 2,3,\dots,M-1$, $\eta_j = \xi + (j-1)h$, $j = 1,2,\dots,M$, $h = \frac{1}{N}$,

$(M-1)h + \xi = 2$ and $\frac{\xi}{h}$ is a fixed integer number.

It follows that

$$|\alpha| \sum_{k=1}^{M} \rho_k = q_0 < 1. \tag{4}$$

We consider the following multilevel nonlocal boundary value problem

$$\Delta U \; = \; 0 \; on \; R, \qquad U = \tau \; on \; \gamma^2, \qquad U = 0 \; on \; \gamma^1 \cup \gamma^3, \tag{5}$$

$$U(x,0) \; = \; \alpha \sum_{k=1}^{M} \rho_k . U(x, \eta_k) + \mu(x), \; 0 \le x \le 1. \tag{6}$$

By the maximum principle and the error estimate of trapeziodal rule, we have

$$\max_{(x,y)\in \bar{R}} |u - U| \le c_1 h^2, \tag{7}$$

where c_1 is a constant independent of h.

Let V be a solution of the Dirichlet problem,

$$\Delta V = 0 \; on \; R, \qquad V = \tau \; on \; \gamma^2, \qquad V = 0 \; on \; \gamma/\gamma^2,$$

and denote by

$$\phi_k(x) = V(x, \eta_k) \; for \; k = 1,2,\dots,M, \tag{8}$$

$$\phi = \alpha \sum_{k=1}^{M} \rho_k \phi_k. \tag{9}$$

We consider the Dirichlet problem

$$\Delta W = 0 \; on \; R, \qquad W = 0 \; on \; \gamma/\gamma^4, \qquad W = f \; on \; \gamma^4,$$

where f be an unknown function from C^0.

We define the operator $B_i : C^0 \to C^0$ as

$$B_i f(x) = W(x, \eta_i) \in C^0, \qquad i = 1,2,\dots,M. \tag{10}$$

Since W is a harmonic function on R, by the maximum principle for the norm of operator B_i, we have

$$|B_i| < 1 - \frac{\xi + (i-1)h}{2}, \quad i = 1, 2, \ldots, M \tag{11}$$

and

$$0 < |B_M| < |B_{M-1}| < \ldots < |B_1| < 1.$$

Then the following inequality holds

$$|B_1| q_0 = q < 1, \tag{12}$$

where q_0 is defined in (4).

It is obivious that,

$$U(x, 0) = f(x), \ 0 \leq x \leq 1.$$

Relying on (8), (9) and (10), the funtion f satisfies relation

$$f = \phi + \mu + \alpha \sum_{k=1}^{M} \rho_k B_k f. \tag{13}$$

On the basis of (11), by analogy with the proof of Theorem 1 and 2 in [4] the following Theorem is proved.

Theorem 1. There is a unique function $f \in C^0$ for which relation (13) holds.

3 Finite Difference Approximation

We say that $F \in C^{k,\lambda}(E)$, if F has k-th derivatives on E satisfying the Hölder condition with exponent λ. We assume that $\tau(x) \in C^{2,\lambda}(\gamma^2)$ and $\mu(x) \in C^{2,\lambda}(\gamma^4), 0 < \lambda < 1$, in (2) and (3), respectively.

We define a square mesh with the mesh size $h = 1/N, N > 2$ is an integer, constructed with the lines $x, y = h, 2h, \ldots$. Let D_h be the set of nodes of this square grid, $R_h = R \cap D_h$, where R is the rectangle (1), and $\gamma_h^m = \gamma^m \cap D_h, \ m = 1,2,3,4$. We denote by $\gamma_h = \cup_{m=1}^{4} \gamma_h{}^m, \bar{R}_h = R_h \cup \gamma_h$.

Let

$$[0,1]_h = \left\{ x = x_i \colon x_i = ih, \ i = 0,1,\ldots,N, \ h = \frac{1}{N} \right\}$$

be the set of points divided by the step size h on $[0,1]$.

Let C_h^0 be the linear space of grid functions defined on $[0,1]_h$ that vanish at $x = 0$ and $x = 1$. The norm of a function $f_h \in C_h^0$ is defined as

$$\|f_h\|_{C_h^0} = \max_{x \in [0,1]_h} |f_h|.$$

Let A be the operator defined by

$$Au_h \equiv (u_h(x+h, y) + u_h(x-h, y) + u_h(x, y+h) + u_h(x, y-h))/4.$$

Consider the system of grid equations

$$v_h = Av_h \ on \ R_h, \quad v_h = \tau_h \ on \ \gamma_h^2, \quad v_h = 0 \ on \ \gamma_h/\gamma_h^2,$$

where τ_h is the trace of τ on γ_h^2 and for each $\eta_i = \xi + (i-1)h$, we define

$$\widetilde{\phi}_{i,h}(x) = v_h(x, \eta_i), \qquad i = 1, 2, \ldots, M. \tag{14}$$

Let w_h be a solution of the finite difference problem

$$w_h = Aw_h \ on \ R_h, \quad w_h = 0 \ on \ \gamma_h/\gamma_h^4, \quad w_h = \tilde{f}_h \ on \ \gamma_h^4,$$

where $\tilde{f}_h \in C_h^0$, is an arbitrary function.

Let B_i^h be a linear operator from C_h^0 to C_h^0 defined by

$$B_i^h f_h(x) = w_h(x, \eta_i), \quad i = 1,2,\ldots,M.$$

The following inequality holds

$$\left\| B_i^h f_h(x) \right\|_{C_h^0} \le \|f_h\|_{C_h^0} \left(1 - \frac{\xi + (i-1)h}{2} \right), \tag{15}$$

We define

$$\tilde{\phi}_h = \alpha \sum_{k=1}^{M} \rho_k \tilde{\phi}_{k,h}(x), \quad x \in [0,1]_h, \tag{16}$$

where $\tilde{\phi}_{k,h}(x)$ defined by (14).

Now, we introduce the function

$$\tilde{f}_h = \tilde{\phi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \tilde{\psi}_{k,h},$$

where μ_h is the trace of μ on $[0,1]_h$ and $\tilde{\psi}_{k,h} \in C_h^0, k = 1,2,\ldots M$, are the solution of the system of the equations

$$\tilde{\psi}_{i,h} = B_i^h (\tilde{\phi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \tilde{\psi}_{k,h}), \quad i = 1,2,\ldots,M. \tag{17}$$

We investigate the solution of system (17) by the the following fixed point iteration

$$\tilde{\psi}_{i,h}^0 = 0, \quad \tilde{\psi}_{i,h}^n = B_i^h (\tilde{\phi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \tilde{\psi}_{k,h}^{n-1}), i = 1,2,\ldots,M; \quad n = 1,2,\ldots. \tag{18}$$

Define

$$\tilde{f}_h^n = \tilde{\phi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \tilde{\psi}_{k,h}^n, \tag{19}$$

where $\tilde{\psi}_{k,h}^n$ is the n-th iteration of (18).

Theorem 2. The following inequality is true.

$$\left\| \tilde{f}_h^n - f_h \right\|_{C_h^0} \le c_2 h^2 + q_0 \left(\frac{q_1^{n+1}}{1-q_1} \right) (\|\phi\|_{C^0} + \|\mu\|_{C^0}), \tag{20}$$

where f_h is the trace of f on $[0,1]_h$, q_0 is a number defined by (4), $q_1 = max\{1 - \frac{\xi}{2}, q\}$, q is a number given in (12), c_2 is a constant independent of h.

Consider the actual finite difference problem for the approximate solution of problem (7) and (8),

$$\tilde{u}_h^n = A\tilde{u}_h^n \ on \ R_h, \quad \tilde{u}_h^n = \tau_h \ on \ \gamma_h^2, \quad \tilde{u}_h^n = 0 \ on \ \gamma_h^1 \cup \gamma_h^3, \tag{21}$$

$$\tilde{u}_h^n = \tilde{f}_h^n \ on \ \gamma_h^4, \tag{22}$$

where $\tilde{f}_h^n$ is computed function which approximates to f.

On the basis of (7) and Theorem 2, the following Theorem is proved:

Theorem 3. Let the boundary function $\tau(x)$ and $\mu(x)$ in (5), (6) be from the class $C^{2,\lambda}(\gamma^2)$. The estimation holds

$$\max_{(x,y)\in \bar{R}_h} |\tilde{u}_h^n - u| \le c_3 h^2 + q_0 \left(\frac{q_1^{n+1}}{1-q_1} \right) (\|\phi\|_{C^0} + \|\mu\|_{C^0}), \qquad (23)$$

where $\tilde{u}_h^n$ is a solution of the problem (21), (22), u is the solution of problem (2), (3) and c_3 is a constant independent of h.

In the estimation (23)

$$n = max\left\{ \left[\left(\frac{\ln h^{-2}(1-q^1)^{-1}q_0}{\ln q_1^{-1}} \right) \right], 1 \right\}.$$

4 Numerical Experiments

The following problem is solved:

$$\triangle u = 0 \ on \ R, \qquad u(0,y) = u(1,y) = 0, \qquad 0 \le y \le 2,$$

$$u(x,2) = e^{2\pi} sin(\pi x), \qquad 0 \le x \le 1,$$

$$u(x,0) = \alpha \int_{\xi}^{2} u(x,y)dy + \mu(x), \ 0 \le x \le 1,$$

where $u(x,y) = e^{\pi x} sin(\pi x)$ is the exact solution, $\mu(x) = sin(\pi x)\left[1 + \left(\frac{\alpha}{\pi}\right)(1 - e^{2\pi})\right]$ and $\alpha = \frac{1}{100}$ and $\xi = \frac{1}{16}$.

By using the present method, the approximate solutions found and showed on line $y = 0$ according to the decreasing mesh sizes h. This shows that the error of approximation has order of $O(h^2)$. To attain this accuracy just 4 iterations in (19) are applied. The comparisons of the CPU times of the proposed method and the method without reducing to the local problem show the efficiency of our approach.

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