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Representation of the Matrix for Conversion between Triangular Bezier Patches and Rectangular Bezier Patches

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Abstract. In this paper we studied Bezier surfaces that are very famous techniques and they are widely used in the area of Computer Aided Geometric Design. Mainly there are two kinds of Bezier surfaces which are classified as rectangular and triangular Bezier patches. In this paper we will give a simple representation for the conversion matrix which converts one type to another type in one step.

1 Introduction

The theory of Bezier curves has an important role and they are numerically the most stable among all polynomial bases currently used in CAD systems. On the other hand in these days Bezier surfaces are very famous techniques and widely used in Computer Aided Geometric Design (see [1] and [2]). Mainly there are two kinds of Bezier surfaces which are triangular and rectangular Bezier patches and their definition is given by the uni-variate Bernstein polynomials $B_i^n(s) = \binom{n}{i} s^i (1-s)^{n-i}$ and the bi-variate Bernstein polynomials $B_{i,j,k}^n(\alpha,\beta,\eta) = \binom{n}{i,j,k} \alpha^i \beta^j \eta^k$ where the sum $\alpha+\beta+\eta=1$. Definition of an n-th degree triangular Bezier patch with control points $D_{i,j,k}$ is given by

$$D(\alpha,\beta,\eta) = \sum_{i+j+k=n} D_{i,j,k} B_{i,j,k}^n(\alpha,\beta,\eta), \quad \alpha,\beta,\eta \ge 0, \quad \alpha+\beta+\eta=1.$$

On the other hand $n \times m$ degree rectangular Bezier patch with the control points $P_{i,j}$ is defined by

$$P(\alpha,\beta) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} B_i^n(\alpha) B_j^m(\beta),\ 0 \le \alpha,\beta \le 1, \quad \text{(see [3])}$$

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Since these two patches have different geometric properties, it is not easy to deal with both of them in one Computer Aided Design system (see [4] and [5]). Thus from this reason we have to convert one type to another.

2 Construction of the Conversion Matrices

In the following theorem (see [3]), it is given that a triangular Bezier patch can be converted to an equivalent degenerate rectangular Bezier patch, whose control points are computed by degree elevating some Bezier curves. Here a degenerate rectangular patch means a rectangular patch in which one of its edges collapsed into a point.

Theorem 1. (see [3]) *A degree n triangular Bézier patch $D(\alpha,\beta,\eta)$ can be represented as a degenerate rectangular Bézier patch of degree $n \times n$:*

$$P(\alpha,\beta)=\sum_{i=0}^{n}\sum_{j=0}^{n}P_{i,j}B_i^n(\alpha)B_j^n(\beta),\ 0\le\alpha,\beta\le 1$$

where $P_{i,j}$ are the control points and they are determined as

$$\begin{pmatrix} P_{i,0} \\ P_{i,1} \\ \vdots \\ P_{i,n} \end{pmatrix} = A_1A_2\ldots A_i \begin{pmatrix} D_{i,0} \\ D_{i,1} \\ \vdots \\ D_{i,n-i} \end{pmatrix}, \qquad i=0,1,\ldots,n.$$

and A_i where $i=0,1,\ldots n$, are matrices (or operators) of degree elevation which are represented in the following form:

$$A_i=\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{n+1-i} & \frac{n-i}{n+1-i} & 0 & \ldots & 0 & 0 \\ 0 & \frac{2}{n+1-i} & \frac{n-i-1}{n+1-i} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \frac{n-i}{n+1-i} & \frac{1}{n+1-i} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(n-i+2)\times(n-i+1)}$$

Here in Theorem 2 we give a simple representation for the matrix product $A_1 A_2 \ldots A_i$.

Theorem 2. *The product of the degree elevation matrices that converts a degree n triangular Bezier patch to a degenerate rectangular Bezier patch of degree $n \times n$ is given as follows :*

$$A_1 A_2 \ldots A_k = \overline{A}_k = \left[\overline{a}_{i,j}^{(k)} \right]_{(n+1)\times(n-k+1)}$$

where

$$\overline{a}_{i,j}^{(k)} = \frac{\binom{i-1}{j-1} (k)_{i-j} (n-k)_{j-1}}{(n)_{i-1}},$$

$$(k)_n = k(k-1)\ldots(k-n+1) = \prod_{j=1}^{n} (k-(j-1)) \quad \text{and}$$

$$A_k = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{n+1-k} & \frac{n-k}{n+1-k} & 0 & \ldots & 0 & 0 \\ 0 & \frac{2}{n+1-k} & \frac{n-k-1}{n+1-k} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \frac{n-k}{n+1-k} & \frac{1}{n+1-k} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(n-k+2)\times(n-k+1)}$$

Now in the following theorem we will consider the inverse process which will convert a rectangular Bezier patch of degree $n \times n$ to a triangular Bezier patch of degree n :

Theorem 3 *A rectangular Bezier patch $P(\alpha, \beta)$ of degree $n \times n$ can be represented as a triangular Bezier patch $D(\alpha, \beta, \eta)$ of degree n :*

$$D(\alpha,\beta,\eta) = \sum_{i+j+k=n} D_{i,j,k} B_{i,j,k}^{n}(\alpha,\beta,\eta), \quad \alpha,\beta,\eta \geq 0, \quad \alpha+\beta+\eta = 1$$

where $D_{i,j,k}$ *are the control points and they are defined as*

$$\begin{pmatrix} D_{i,0} \\ D_{i,1} \\ \vdots \\ D_{i,n-i} \end{pmatrix} = B_i B_{i-1} \ldots B_1 \begin{pmatrix} P_{i,0} \\ P_{i,1} \\ \vdots \\ P_{i,n} \end{pmatrix} \quad i = 0,1,\ldots,n$$

and $B_i (i = 0,1,\ldots.,n)$ *are matrices (or operators) of degree reduction which are given in the following form*

$$B_k = \begin{bmatrix} 1-t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-t & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1-t & t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1-t & t & 0 \\ 0 & 0 & 0 & 0 & & 1-t & t \end{bmatrix}_{(n-k+1)\times(n-k+2)}$$

Theorem 4 *The product of the matrices in the above theorem* $B_k B_{k-1} \ldots B_1$ *that converts a rectangular Bezier patch of degree* $n \times n$ *to a triangular Bezier patch of degree* n *can be generalized as follows:*

$$Z^k = B_k B_{k-1} \ldots B_1 = \begin{bmatrix} b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,\mathrm{k}-1} & b_{k,\mathrm{k}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,\mathrm{k}-1} & b_{k,\mathrm{k}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,\mathrm{k}-1} & b_{k,\mathrm{k}} & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,\mathrm{k}-1} & b_{k,\mathrm{k}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,\mathrm{k}-1} & b_{k,\mathrm{k}} \end{bmatrix}_{(n-k+1)\times(\mathrm{n}+1)}$$

where $b_{k,j} = \begin{pmatrix} k \\ j \end{pmatrix} t^j (1-t)^{k-j}$.

Remark *The proofs of Theorems 2, 3 and 4 will be presented later in the complete paper.*

3 Conclusion

In this paper, we studied on the conversion matrix to convert triangular Bezier patch to a rectangular Bezier patch and a rectangular Bezier patch to a triangular Bezier patch. We found simple representations for these two matrices which will allow the conversion in one step.

References

1. Farin, G., Curve and Surface for CAGD : A Practical Guide. Academic Press, New York (1990)
2. Farin, G., Computer Aided Design, 21, 293-296 (1989)
3. Shi-Min Hu, Computer Aided Geometric Design, 18, 667-671 (2001)
4. Goldman, R., Filip, D., Computer Aided Design, 19, 25–27 (1987)
5. Shi-Min Hu, Computer Aided Geometric Design, 13, 219–226 (1996)