Reproducing kernel functions for linear tenth-order boundary value problems

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Abstract. Higher order differential equations have always been an onerous problem to investigate for the mathematicians and engineers. Different numerical methods were applied to get numerical approximations of such problems. This paper gives some reproducing kernel functions to find approximate solutions of the tenth-order boundary value problems (BVPs). These reproducing kernel functions are very important in the reproducing kernel Hilbert space method.

1 Introduction

A numerical approximation of tenth-order BVPs is rarely presented in the literature [1]. When heating an endless flat film of fluid from below, under the supposition that fluid is subject to the action of rotation and uniform magnetic field across the fluid is applied in the same direction as gravity, unsteadiness starts. When unsteadiness begins in as usual convection, then it can be modeled by tenth-order BVP [2]. Pervaiz et al. [3] introduced the numerical approximations of twelfth-order BVPs by applying non-polynomial cubic spline method. Omotayo et al. [4] presented non-polynomial spline technique for the fourth order BVPs. Usmani [5] introduced his work to approximate the fourth order BVPs by using the quartic spline method. Twizell and Boutayeb [6] enhanced and showed the numerical approximations for higher order value problems. The approximation of second order BVPs was introduced by Alberg and Ito [7]. Siraj-ul-Islam et al [8] introduced a non-polynomial spline method to approximate the sixth-order BVPs. Papamichael and Worsey [9] studied the cubic spline algorithm for solving linear fourth-order BVPs. Siddiqi and Twizell [10, 11] enhanced numerical approximations of tenth and twelfth-order BVPs.

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2 MATERIALS AND METHODS

The main aim of this study is to find some new functions for numerical approximations of linear tenth-order BVPs. We consider

$$\begin{aligned} h^{(x)}(t) + a_1(t)h^{(ix)}(t) + a_2(t)h^{(viii)}(t) + a_3(t)h^{(vii)}(t) + a_4(t)h^{(vi)}(t) \\ +a_5(t)h^{(v)}(t) + a_6(t)h^{(iv)}(t) + a_7(t)h'''(t) + a_8(t)h''(t) \\ +a_9(t)h'(t) + a_{10}(t)h(t) = M(t), \quad t \in [a, b], \end{aligned}$$

with boundary conditions:

$$\begin{aligned} h(a) = \alpha_0, \quad h(b) = \beta_0, \quad h''(a) = \alpha_1, \qquad h''(b) = \beta_1, \\ h^{(iv)}(a) = \alpha_2, \quad h^{(iv)}(b) = \beta_2, \quad h^{(vi)}(a) = \alpha_3, \qquad h^{(vi)}(b) = \beta_3, \\ h^{(viii)}(a) = \alpha_4, \qquad h^{(viii)}(b) = \beta_4, \end{aligned}$$

where, α_j, β_j, $j = 0, 1, 2, 3, 4$ are arbitrary fixed real constants, $a_j(t)$, $j = 1, 2, ..., 10$ and $M(t)$ are continuous functions described on $[a, b]$.

We obtain our reproducing kernel functions for following test problem.

Test Problem

We consider the following tenth-order equation as a second experiment:

$$h^{(10)}(t) = -(80 + 19t + t^2)\exp(t), \quad 0 \le t \le 1,$$

with boundary conditions:

$$\begin{aligned} h(0) &= 0, & h(1) &= 0, \\ h^2(0) &= 0, & h^2(1) &= -4\exp(1), \\ h^4(0) &= -8, & h^4(1) &= -16\exp(1), \\ h^6(0) &= -24, & h^6(1) &= -36\exp(1), \\ h^8(0) &= -48, & h^8(1) &= -64\exp(1). \end{aligned}$$

The exact solution to the above BVP is given by [1]:

$$h(t) = t(1 - t)\exp(t).$$

3 Reproducing kernel functions

We define $V_2^{11}[0, 1]$ as

$$V_2^{11}[0,1] = \left\{ \begin{array}{l} u, u', u'', u''', u^{(4)}, u^{(5)}, u^{(6)}, u^{(7)}, u^{(8)}, u^{(9)}, u^{(10)} \quad \text{are absolutely continuous functions,} \\ u^{(11)} \in L^2[0, 1], \quad u(0) = u''(0) = u^{(4)}(0) = u^{(6)}(0) = u^{(8)}(0) = 0 \\ \qquad u(1) = u''(1) = u^{(4)}(1) = u^{(6)}(1) = u^{(8)}(1) = 0. \end{array} \right\}$$

We give

$$\begin{aligned} \langle u, G_z \rangle_{V_2^{11}[0,1]} &= u(0)G_z(0) + u'(0)G_z'(0) + u''(0)G_z''(0) + u'''(0)G_z'''(0) + u^{(4)}(0)G_z^{(4)}(0) \\ &+ u^{(5)}(0)G_z^{(5)}(0) + u^{(6)}(0)G_z^{(6)}(0) + u^{(7)}(0)G_z^{(7)}(0) + u^{(8)}(0)G_z^{(8)}(0) \\ &+ u^{(9)}(0)G_z^{(9)}(0) + u^{(10)}(0)G_z^{(10)}(0) + \int_0^1 u^{(11)}(x)G_z^{(11)}(x)d(x). \end{aligned}$$

We have

$$G_z(0) = G_z''(0) = G_z^{(4)}(0) = G_z^{(6)}(0) = G_z^{(8)}(0) = 0$$

$$G_z(1) = G_z''(1) = G_z^{(4)}(1) = G_z^{(6)}(1) = G_z^{(8)}(1) = 0,$$

by boundary conditions. Therefore, we get

$$\begin{aligned}
\langle u, G_z\rangle_{V_2^{11}[0,1]} &= u'(0)G_z'(0) + u'''(0)G_z'''(0) \\
&+ u^{(5)}(0)G_z^{(5)}(0) + u^{(7)}(0)G_z^{(7)}(0) \\
&+ u^{(9)}(0)G_z^{(9)}(0) + u^{(10)}(0)G_z^{(10)}(0) + \int_0^1 u^{(11)}(x)G_z^{(11)}(x)d(x).
\end{aligned}$$

We obtain

$$\begin{aligned}
\langle u, G_z\rangle_{V_2^{11}[0,1]} &= u'(0)G_z'(0) + u'''(0)G_z'''(0) + u^{(5)}(0)G_z^{(5)}(0) + u^{(7)}(0)G_z^{(7)}(0) \\
&+ u^{(9)}(0)G_z^{(9)}(0) + u^{(10)}(0)G_z^{(10)}(0) + u^{(10)}(1)G_z^{(11)}(1) - u^{(10)}(0)G_z^{(11)}(0) \\
&- u^{(9)}(1)G_z^{(12)}(1) + u^{(9)}(0)G_z^{(12)}(0) + u^{(8)}(1)G_z^{(13)}(1) - u^{(8)}(0)G_z^{(13)}(0) \\
&- u^{(7)}(1)G_z^{(14)}(1) + u^{(7)}(0)G_z^{(14)}(0) + u^{(6)}(1)G_z^{(15)}(1) - u^{(6)}(0)G_z^{(15)}(0) \\
&- u^{(5)}(1)G_z^{(16)}(1) + u^{(5)}(0)G_z^{(16)}(0) + u^{(4)}(1)G_z^{(17)}(1) - u^{(4)}(0)G_z^{(17)}(0) \\
&- u'''(1)G_z^{(18)}(1) + u'''(0)G_z^{(18)}(0) + u''(1)G_z^{(19)}(1) - u''(0)G_z^{(19)}(0) \\
&- u'(1)G_z^{(20)}(1) + u'(0)G_z^{(20)}(0) + u(1)G_z^{(21)}(1) - u(0)G_z^{(21)}(0) \\
&- \int_0^1 u(x)G_z^{(22)}(x)d(x)
\end{aligned}$$

by integration by parts. Then, we have

$$\begin{aligned}
\langle u, G_z\rangle_{V_2^{11}[0,1]} &= u'(0)G_z'(0) + u'''(0)G_z'''(0) + u^{(5)}(0)G_z^{(5)}(0) + u^{(7)}(0)G_z^{(7)}(0) \\
&+ u^{(9)}(0)G_z^{(9)}(0) + u^{(10)}(0)G_z^{(10)}(0) + u^{(10)}(1)G_z^{(11)}(1) - u^{(10)}(0)G_z^{(11)}(0) \\
&- u^{(9)}(1)G_z^{(12)}(1) + u^{(9)}(0)G_z^{(12)}(0) - u^{(7)}(1)G_z^{(14)}(1) + u^{(7)}(0)G_z^{(14)}(0) \\
&- u^{(5)}(1)G_z^{(16)}(1) + u^{(5)}(0)G_z^{(16)}(0) - u'''(1)G_z^{(18)}(1) + u'''(0)G_z^{(18)}(0) \\
&- u'(1)G_z^{(20)}(1) + u'(0)G_z^{(20)}(0) - \int_0^1 u(x)G_z^{(22)}(x)d(x).
\end{aligned}$$

We have

$$G_z'(0) + G_z^{(20)}(0) = 0, \quad G_z'''(0) + G_z^{18}(0) = 0$$

$$G_z^{(5)}(0) + G_z^{16}(0) = 0, \quad G_z^{(7)}(0) + G_z^{14}(0) = 0$$

$$G_z^{(9)}(0) + G_z^{12}(0) = 0, \quad G_z^{10}(0) - G_z^{11}(0) = 0$$

$$G_z^{(11)}(1) = 0, \quad G_z^{(12)}(1) = 0, \quad G_z^{(14)}(1) = 0$$

$$G_z^{(16)}(1) = 0, \quad G_z^{(18)}(1) = 0, \quad G_z^{(20)}(1) = 0$$

Thus,we acquire

$$\langle u, G_z\rangle_{V_2^{11}[0,1]} = -\int_0^1 u(x)G_z^{(22)}(x)d(x) = u(z).$$

Therefore, we obtain

$$G_z(x) = \begin{cases} \sum_{i=1}^{11} c_i x^{i-1}, & x \le z \\ \sum_{i=1}^{11} d_i x^{i-1}, & x > z \end{cases}$$

by

$$G_z^{(22)}(x) = -\delta(x-z).$$

4 Conclusion

We obtained some reproducing kernel functions to find approximate solutions of the tenth-order boundary value problems in this paper. We need these reproducing kernel functions for applications of the reproducing kernel Hilbert space method.

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