A Note on a Binary Relation Corresponding to a Bipartite Graph

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Abstract. In this paper, we firstly define a binary relation corresponding to the bipartite graph and study its properties. We also establish a relationship between the independent sets of the bipartite graph and the definable sets of binary relations corresponding to the bipartite graph.

1 Introduction

Rough set theory was initially defined in [9] for overcome the uncertainty in information systems. This theory enabled approximately characterization an subset of an universe by using operators called as lower approximation and upper approximation operator. Rough set theory has many application in fields, e.g. machine learning, knowledge discovery, decision analysis.

In rough set theory, the space that Pawlak mentions is the approximation space created by the equivalence relation R . But, such an equivalence relation is inadequate to using in some applications [5,15,19,20]. Because of this, many extensions of equivalence relations such as arbitrary binary relations [10,17,16], tolerance relations [7,8,11,13], similarity relations [1,12,14] and pre-order relations [4,6] have been proposed.

It is shown that a relation R can be represented by a graph [4]. In [3], authors say that an undirected simple graph can be represented a corresponding binary relation. Moreover, they show that the binary relation induced by a graph has irreflexive property. A graph is named as a bipartite graph if set of vertices of the graph can be divided into two independent sets. In this paper, we study the binary relation corresponding to the bipartite graph. We prove that this relation has irreflexive and nontransitive properties.

2 Preliminaries

We present in section, preliminary definitions that we shall use in our work related generalized approximation spaces and graph theory.

2.1. Approximation Spaces and Approximation Operators

Definition 1. [3] Consider a finite non-empty set U .

i) An irreflexive relation is the relation R on U satisfies condition “For all $x \in U, (x,x) \notin R$ ”

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ii) A serial relation is the relation R on U satisfies condition "For all $x \in U$, there exists $y \in U$ such that $(x, y) \in R$."

Definition 2. [18] Consider a finite non-empty set U and a binary relation R on U. The pair (U, R) is a generalized approximation space. Also the space is marked as GA-space. When R is an equivalence relation on U, GA-space (U, R) coincide with Pawlak approximation space [10,17].

Definition 3. [15] Given a (U, R) GA-space. For any $X \subseteq U, \underline{R}: P(U) \to P(U), \quad \overline{R}: P(U) \to P(U)$ the lower and the upper approximation operators of approximation space (U, R) respectively are defined as:

$\underline{R}X = \{x \in U : R_s(x) \subseteq X\}, \quad \overline{R}X = \{x \in U : R_s(x) \cap X \neq \varnothing\}$.

Here $R_s(x) = \{y \in U : (x, y) \in R\}$.

The subset X is named definable set iff $\underline{R}X = \overline{R}X$. Or else, X is named a rough set.

2.2. Bipartite Graphs

Because bipartite graphs frequently used in data modeling in computer science have an important place in graph theory, characterization of bipartite graphs is important [2].

Definition 4. [2] Consider a finite non-empty set U. The pair $G = (U, E)$ whose sets of vertices and edges repectively is U and $E \subseteq U \times U$ is a graph. A vertice is called isolated vertice if it is not an end of any edge. Arbitrary two vertices of G are called as adjacent if there is at least one edge linking them. The graph G whose every edge links different two vertices is called as a simple graph. A set of vertices whose elements are pairwise non-adjacent is called an independent set [2].

Definition 5. [2] A bipartite graph is a graph G whose vertex set U can be divided into non-empty subsets X and Y such that every one edge has one vertice in X and one vertice in Y. That is, a graph G is called bipartite iff the vertex set U of G can be divided into two independent sets.

Binary relations on U can be represented by graphs whose vertices are element of U, where there is an edge that vertex x is related vertex y iff $(x, y) \in R$ and $(y, x) \in R$. The graph induced by R is called a graph of R [4]. An undirected simple graph can also be represented a corresponding binary relation [3]. When such the graph $G = (U, E)$ is given corresponding approximation space can be defined (U, R) by: $(x, y) \in E$ iff $(x, y) \in R$ and $(y, x) \in R$. The relation R is called relation induced by G.

Now we give properties of the relation induced by G.

Definition 6. [3] Let R be a serial, irreflexive and symmetric relation on U. Then R is a serial preclusivity relation (SP-relation) on U. The approximation space (U, R) is a serial preclusivity approximation space (SPA-space).

Clearly, the relation R induced by the undirected simple graph G is SP-relation. SP-relations and undirected simple graphs can be represented by each other [3].

3 Main Result

We firstly shall give following theorem that allows to characterize the independent sets of a simple graph by the way of the upper approximation of R binary relation induced by this graph.

Theorem 1. [3] Given a simple graph $G=(U,E)$ and the relation R induced by G. For $X \subseteq U$, X is an independent set of G iff $\overline{R}X \cap X = \varnothing$.

Definition 7. Let U be a finite non-empty set and R be a relation on U. If there are not elements that provide transitive property with regard to R on U, then R is called nontransitive relation on U.

Definition 8. Let U be a finite nonempty set and R be a serial, irreflexive, symmetric relation. If R is nontransitive, then it is called R serial-nontransitive relation on U. (U,R) is called serial-nontransitive space.

Now we shall prove the following theorem.

Theorem 2. Let $G=(U,E)$ be a simple graph and R be the relation induced by

G. If G is bipartite then R is nontransitive relation on U.

Proof. We suppose that G is a bipartite graph. Then there are A and B independent sets such that $\overline{R}A=B$, $\overline{R}B=A$, $A\cap B=\varnothing$, $A\cup B=U$. We assume that there exist $x,y,z\in U$ which provide the transitive property with regard to R on U. Let $A=\{a_1,a_2,...,a_m\}, B=\{b_1,b_2,...,b_n\}$. Since $A\cup B=U$ and $A\cap B=\varnothing$, $x=a_i\in A$ or $x=b_j\in B$. Let $x=a_i\in A$. Since A is independent set, $(a_i,y)\in R$ iff $y\in B$. Thus, we have $y=b_j\in B$. Since B is independent set, $(b_j,z)\in R$ iff $z\in A$. Thus, we have $z=a_k\in A$. Since there exist $a_i,b_j,a_k\in U$ such that $(a_i,b_j)\in R$ and $(b_j,a_k)\in R$, we obtain $(a_i,a_k)\in R$. This contradicts with A being an independent set. Similarly, if we assume that $x=b_j\in B$ then we obtain contradict with B being independent set. Therefore, for any $x,y,z\in U$ such that $(x,y)\in R$ and $(y,z)\in R$, $(x,z)\notin R$. Hence R is nontransitive relation on U.

Example 1. Bipartite graph $G=(U,E)$ is given by Fig. 1. Here $U=\{x,y,z,t,u,v\}$, $E\subseteq U\times U$. The relation R induced by the bipartite graph G is

$R=\{(x,y),(y,x),(x,z),(z,x),(x,u),(u,x),(y,t),(t,y),(y,v),(v,y),(z,t),(t,z),(t,u),(u,t)\}$.
R is serial-nontransitive relation.

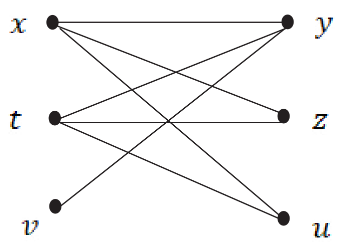

Fig. 1. Bipartite Graph G

Example 2. Let $U=\{x,y,z,t,u\}$, $R=\{(x,y),(y,x),(x,u),(u,x),(z,t),(t,z),(t,u),(u,t)\}$.
R is serial-nontransitive relation. The graph of R is shown as follows:

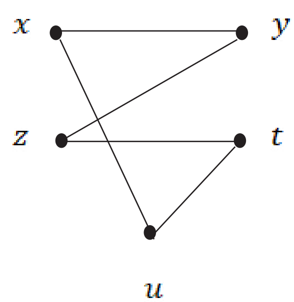

Fig. 2. Graph of Relation R

As seen above, the graph of serial-nontransitive relation R on U is not bipartite graph.

Example 3. Let's search definable sets of the above space (U,R). Since $\underline{R}\{x,t,v\}=\overline{R}\{x,t,v\}=\{y,z,u\}$ and $\underline{R}\{y,z,u\}=\overline{R}\{y,z,u\}=\{x,t,v\}$, the sets $X=\{x,t,v\}$ and $Y=\{y,z,u\}$ are definable sets of serial-nontransitive space (U,R).

It is seen that X and Y being disjoint independent sets such that $X\cup Y=U$ in a bipartite graph are definable sets with regard to binary relation induced by this bipartite graph in examples in above.
Now we give relationship between independent sets of a bipartite graph and definable sets of serial-nontransitive binary relation corresponding to this bipartite graph in the following theorem.

Theorem 3. Given a bipartite graph $G=(U,E)$ and the relation R induced by graph G. Then the disjoint independent sets X and Y such that $X\cup Y=U$ are definable sets of GA-approximation space (U,R).

Proof. Since subsets X and Y are independent sets, $\overline{R}X\cap X=\varnothing$ and $\overline{R}Y\cap Y=\varnothing$. We must show that $\overline{R}X=\underline{R}X$ and $\overline{R}Y=\underline{R}Y$ to show that sets X and Y are definable sets. Since $R_s(x)\subseteq X\Rightarrow R_s(x)\cap X\neq\varnothing$, then $\underline{R}X\subseteq\overline{R}X$. So we must show that $\overline{R}X\subseteq\underline{R}X$. Let $x\in\overline{R}X$. Since $\overline{R}X\cap X=\varnothing$, $X\cap Y=\varnothing$ and $X\cup Y=U$, then $x\in Y$. If $x\in\overline{R}X\Rightarrow R_s(x)\cap X\neq\varnothing$, there is $a\in U$ such that $a\in R_s(x)$ and $a\in X$, from which $(x,a)\in R$ and $a\in X$. Suppose that there is $b\in Y$ such that $(x,b)\in R$. $b\in R_s(x)$ and $b\in Y\Rightarrow R_s(x)\cap Y\neq\varnothing\Rightarrow x\in\overline{R}Y$ $\Rightarrow x\notin Y$ (since $\overline{R}Y\cap Y=\varnothing$) It is a contradiction. Hence we have $R_s(x)\cap Y=\varnothing$ which gives $R_s(x)\subseteq X$, that is, $x\in\underline{R}X$. Thus we obtain $\overline{R}X\subseteq\underline{R}X$. Similarly we obtain $\overline{R}Y=\underline{R}Y$. Then X and Y are definable sets of GA-approximation space (U,R).

Acknowledgment

The first author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during her PhD studies.

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