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On the Solutions of Fractional Cauchy Problem Featuring Conformable Derivative

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Abstract. In this study, we have obtained analytical solutions of fractional Cauchy problem by using q-Homotopy Analysis Method (q-HAM) featuring conformable derivative. We have considered different situations according to the homogeneity and linearity of the fractional Cauchy differential equation. A detailed analysis of the results obtained in the study has been reported. According to the results, we have found out that our obtained solutions approach very speedily to the exact solutions.

1 Introduction

In recent years, mathematical modelling with fractional PDEs in some special areas such like physics, mathematics, engineering, finance, biology and medicine [1-8] have been studied. Analytical solutions for fractional differential equations (FDEs) are not very popular in the literature, but the methods applied to avoid effective engineering analyzes, and therefore, numerical methods are frequently applied [9-13].

Recently, [14-16] recommended a new derivative operator namely "conformable" (CDO) and some applications of this operator in various fields have been improved. In this context, some researchers [5, 7, 17-20] applied the conformable operator to solve the problems in engineering, finance, biology, medicine, physics and applied mathematics. In this study, we have solved the linear/nonlinear fractional Cauchy problem with the proposed q-HAM described by using the conformable operator.

2 Fundamental Properties of the Conformable Operator

Definition 1. Let $u:[0,\infty)\to\mathbb{R}$ be a function. The conformable derivative of u order $\alpha\in(0,1]$ is defined by [14]:

$$CD_t^{\alpha}(u)(t)=\lim_{\varepsilon\to 0}\frac{u\left(t+\varepsilon t^{1-\alpha}\right)-u(t)}{\varepsilon} \tag{1}$$

for all $t>0$.

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Definition 2. [14, 16] The $\alpha-$ fractional integral of u function starting is defined by

$$I_a^{\alpha}(u)(t)=I_a^{1}\left(t^{\alpha-1}u\right)=\int_a^t \frac{u(x)}{x^{1-\alpha}}dx, \quad \alpha\in(0,1). \tag{2}$$

3 q-Homotopy Analysis Method in the Conformable Sense

Let us take the nonlinear FPDE:

$$\mathcal{N}\left(CD_t^{\alpha}u(x,t)\right)=f(x,t), \tag{3}$$

where CD_t^{α} shows the conformable derivative of order α. We show the nonlinear term with $\mathcal{N}$ the known function with $f(x,t)$, and the unknown function with $u(x,t)$. We construct the zero-order modified equation which is related with standard homotopy method as [21]:

$$(1-nq)L\left(\omega(x,t;q)-u_0(x,t)\right)=qh\mathcal{H}(x,t)\left(\mathcal{N}\left(CD_t^{\alpha}\omega(x,t;q)\right)-f(x,t)\right), \tag{4}$$

where $n\geq 1$, $q\in[0,1/n]$ represents the embedded parameter, L is a linear operator, $\mathcal{H}(x,t)$ is non-zero supportive function. Distinctly, since $q=0$ and $q=1/n$, we can see Eq.(4) as

$$\omega(x,t;0)=u_0(x,t), \qquad \omega\left(x,t;\frac{1}{n}\right)=u(x,t), \tag{5}$$

respectively. Therefore, q increases from zero to $1/n$, the solution $\omega(x,t;q)$ varies from the initial value $u_0(x,t)$ to the solution $u(x,t)$. If the parameters are chosen appropriately, solution of Eq.(5) is available. Now, we have the following expansion of $\omega(x,t;q)$

$$\omega(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{6}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\omega(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{7}$$

After that we get

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{8}$$

Let the vector

$$\vec{u}_n=\left\{u_0(x,t),u_1(x,t),...,u_n(x,t)\right\}. \tag{9}$$

Then we have mth- order altered equation as [22]

$$L\left[u_m(x,t)-\chi_m^{*}u_{m-1}(x,t)\right]=h\mathcal{H}(x,t)\Re_m\left(\vec{u}_{m-1}(x,t)\right) \tag{10}$$

with initial conditions

$$u_m^{(k)}(x,0)=0, \quad k=0,1,2,...,m-1, \tag{11}$$

where

$$\Re_m\left(\vec{u}_{m-1}(x,t)\right)=\frac{1}{(m-1)!}\frac{\partial^{m-1}\left[\mathcal{N}\left(CD_t^{\alpha}\omega(x,t;q)\right)-f(x,t)\right]}{\partial q^{m-1}}\Bigg|_{q=0} \tag{12}$$

and

$$\chi_m^* = \begin{cases} 0, & m \le 1 \\ n, & m > 1. \end{cases} \tag{13}$$

Finally, we get the components of solution series as,

$$u_m(x,t) = \chi_m^* u_{m-1}(x,t) - \chi_m^* u_{m-1}(x,0^+) + hH(x,t)\Im_t^\alpha \Re_m(\vec{u}_{m-1}(x,t)). \tag{14}$$

4 Numerical Examples

Example 1. We consider a special case of fractional Cauchy equation [23, 24]

$$CD_t^\alpha u(x,t) + u_x(x,t) = x, \quad x \in R, \quad t > 0, \quad 0 < \alpha \le 1, \tag{15}$$

with initial condition

$$u(x,0) = e^x, \quad x \in R. \tag{16}$$

If we choose $H(x,t) = 1,$ we can construct the zeroth-order deformation equation as

$$L\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = h\Re_m(\vec{u}_{m-1}(x,t)), \tag{17}$$

with initial condition for $m \ge 1,\ u_m(x,0) = 0,\ \chi_m^*$ is as defined in (13) and

$$\Re_m(\vec{u}_{m-1}(x,t)) = CD_t^\alpha u_{m-1} + (u_{m-1})_x - x. \tag{18}$$

Therefore, the solution of Eq.(15) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m^* u_{m-1}(x,t) + h\Im_t^\alpha \Re_m(\vec{u}_{m-1}(x,t)). \tag{19}$$

Then using Eq.(19) we can obtain the q-HAM consequence as

$$\begin{aligned}
&u_0(x,t) = e^x, \\
&u_1(x,t) = h\left(e^x - x\right)\frac{t^\alpha}{\alpha}, \\
&u_2(x,t) = nh\left(e^x - x\right)\frac{t^\alpha}{\alpha} + h^2\left(e^x - x\right)\frac{t^\alpha}{\alpha} - hx\frac{t^\alpha}{\alpha} + h^2\left(e^x - 1\right)\frac{t^{2\alpha}}{2\alpha^2}, \\
&u_3(x,t) = \left(n^2h + 2nh^2 + h^3\right)\left(e^x - x\right)\frac{t^\alpha}{\alpha} - \left(h^2 + nh + h\right)x\frac{t^\alpha}{\alpha} \\
&\qquad + \left(2nh^2 + 2h^3\right)\left(e^x - 1\right)\frac{t^{2\alpha}}{2\alpha^2} - h^2\frac{t^{2\alpha}}{2\alpha^2} + h^3 e^x \frac{t^{3\alpha}}{6\alpha^3}, \\
&\vdots
\end{aligned} \tag{20}$$

The series solution of Eq.(15) by q-HAM can be considered as

$$u(x,t;n;h) = e^x + \sum_{m=1}^{\infty} u_m(x,t;n;h)\left(\frac{1}{n}\right)^m. \tag{21}$$

For special values of $h = -1,\ n = 1$ and $\alpha = 1,$ we get the exact solution of Eq.(15) as

$$u(x,t) = e^{x-t} + xt - \frac{t^2}{2}. \tag{22}$$

Example 2. Consider the following fractional nonlinear inviscid Burgers' equation [23, 24]

$$CD_t^\alpha u(x,t) + u(x,t)u_x(x,t) = 0, \quad x \in R, \quad t > 0, \quad 0 < \alpha \le 1, \tag{23}$$

subject to the initial condition

$$u(x,0) = x, \quad x \in R. \tag{24}$$

We can use the nonlinear operator for our problem as

$$\mathcal{N}\left[\omega(x,t;q)\right] = CD_t^\alpha \omega(x,t;q) + \omega(x,t;q)\,\omega_x(x,t;q). \tag{25}$$

Following similar steps with Example 1, we construct the q-HAM iterations as below

$$
\begin{aligned}
&u_0(x,t) = x,\\
&u_1(x,t) = hx\frac{t^\alpha}{\alpha},\\
&u_2(x,t) = nhx\frac{t^\alpha}{\alpha} + h^2 x\frac{t^\alpha}{\alpha} + 2h^2 x\frac{t^{2\alpha}}{2\alpha^2},\\
&u_3(x,t) = \left(n^2h + 2nh^2 + h^3\right)x\frac{t^\alpha}{\alpha} + \left(4nh^2 + 4h^3\right)x\frac{t^{2\alpha}}{2\alpha^2} + h^3 x\frac{t^{3\alpha}}{\alpha^3},\\
&\vdots
\end{aligned}
\tag{26}
$$

An appropriate solution of Eq.(23) in terms of h and n is given

$$u(x,t;n;h) = x + \sum_{m=1}^{\infty} u_m(x,t;n;h)\left(\frac{1}{n}\right)^m. \tag{27}$$

For special values of $h=-1,\ n=1$ and $\alpha=1,$ we have the exact solution of Eq.(23) as

$$u(x,t) = \frac{x}{1-t}. \tag{28}$$

In Figures below, the plots of solution functions of Eq.(23) for different values of α, h, n, t and x are presented.

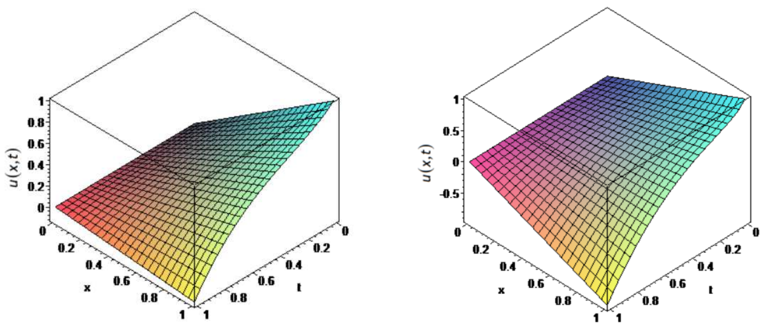

Fig.1. q-HAM Solutions of Eq.(23) with $\alpha = 0.75$ (left) $\alpha = 0.99$ (right)

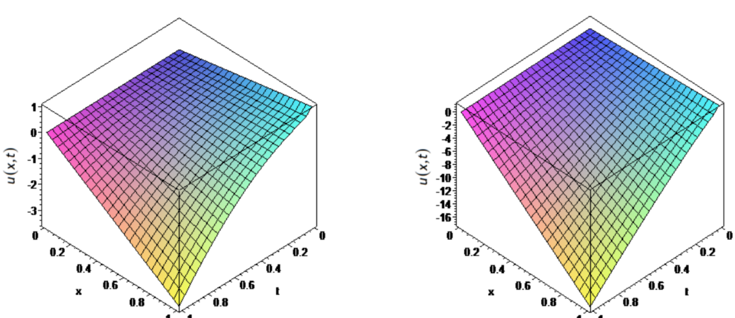

Fig.2. q-HAM Solutions of Eq.(23) with $\alpha = 0.55$ (left) $\alpha = 0.35$ (right)

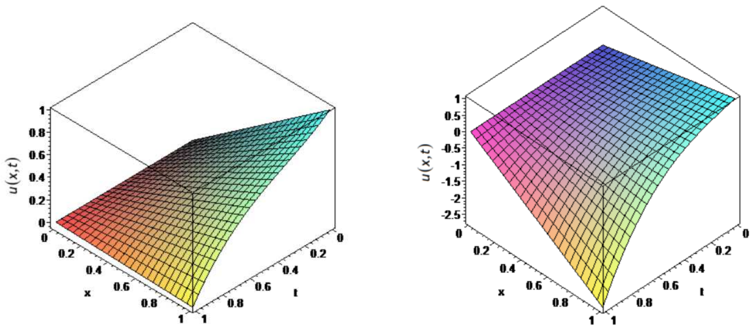

Fig.3. q-HAM Solutions of Eq.(23) for $n = 0.99$ (left) $n = 0.7$ (right)

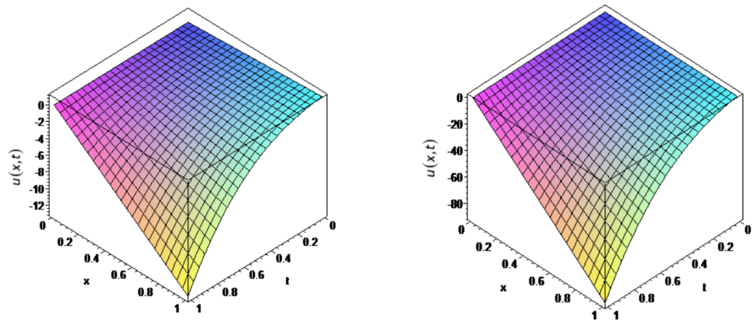

Fig.4. q-HAM Solutions of Eq.(23) for $n = 0.5$ (left) $n = 0.3$ (right)

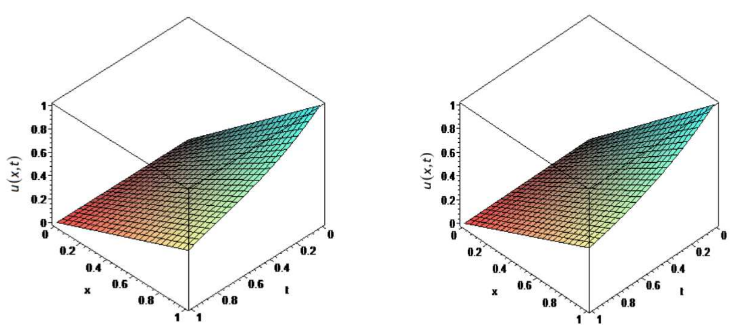

Fig.5. q-HAM Solutions of Eq.(23) for $h = -0.3$ (left) $h = -0.5$ (right)

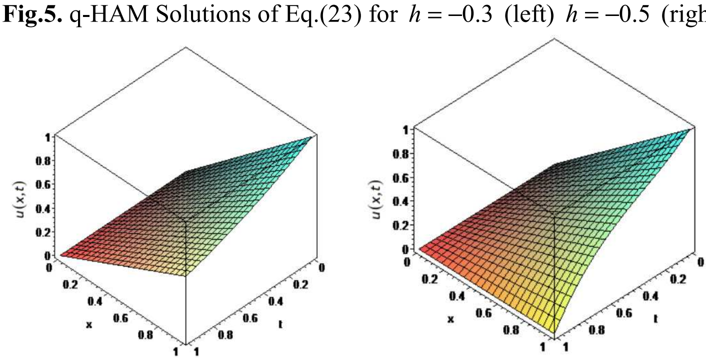

Fig.6. q-HAM Solutions of Eq.(23) for $h = -0.7$ (left) $h = -1$ (right)

4 Conclusions

In this study, we have considered the solutions of linear/nonlinear fractional PDEs by using the q-HAM that is a generalized form of HAM. We have redefined the suggested method with CDO to get the solutions of the fractional Cauchy problems and have applied it to the mentioned problems easily. Then we have verified the efficiencies and correctness of the recommended method by drawing the figures for different values of α, n, h, x and t. Moreover, the results show that the q-HAM solutions when $n=1$, give HAM solutions and when the auxiliary parameter is equal to -1, give the classical HPM solutions. The successful results of the mentioned method show that the method is in complete coherent with the exact solutions even for considering a nonlinear fractional partial differential equation.

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