Lacunary Statistical Convergence of Order alpha and Lacunary Strongly Summable Sequences of Order alpha of Generalized Difference Sequences

Mikail Et[1,*], *Hacer Şengül*[2], and *Muhammed Çınar*[3]

[1]Department of Mathematics, Fırat University, Elazığ, Turkey

[2]Faculty of Education, Harran University, Osmanbey Campus, Şanlıurfa, Turkey

[3]Department of Mathematic Education, Muş Alparslan University, Muş, Turkey

Abstract. The notion of the α th order $\Delta_i^m -$lacunary statistical convergence and α th order lacunary strongly $\left(\Delta_i^m, p\right)-$summable sequences was introduced by Altınok et al. [1]. They also gave significant inclusion correlation associated with the aforementioned sequence spces. In this paper, our aim is to exploit some other important relations between the given notions.

1 Introduction

The idea of statistical convergence was firstly determined by Fast [12] and redefined independently by Schoenberg [28]. The concept of the density of the set of positive integers $\mathbb{N}$ was used as the main motivation to create this idea. For any subset F of the $\mathbb{N}$, the density of F is defined as

$$\beta(F) = \lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{l=1}^{n} \omega_F(l)$$

where ω_F is the characteristic function of F. It is needless to say that this definition is valid only for the existence of the limit. Let $y = (y_l)$ be a given any sequence then it is statistically converged to R, if for every $\varepsilon > 0$, $\beta\{l \in N : |y_l - R| \geq \varepsilon\} = 0$. Furthermore, the relation and some application of this definition with the summabilitiy theory was presented by using the perspective of the sequence space by Çınar et al. ([3], [6], [30]), Connor [5], Fridy [14], Et et al. ([2], [11], [30]), Işık et al. ([17], [18], [19], [20], [21])). Gadjiev and Orhan [16] stated firstly the order of the sequence's statistical

* Corresponding author: mikailet68@gmail.com

convergence. Based on this research, Çolak [4] studied the α th order of the statistical convergence and α th order of the strong $p-$Cesàro summability.

The lacunary sequence is defined by $h_k=(l_k-l_{k-1})\to\infty$ as $k\to\infty$, where $\gamma=(l_k)$ is an increasing sequence of integer. The interval denoted by γ is determined by $I_k=(l_{k-1},l_k]$ and q_k is the abbreviation of the ratio of $\frac{l_k}{l_{k-1}}$ ([13], [15], [20], [29]). The introduction of the difference sequence space and its modification and extension were recently conducted by Kızmaz [23] and Et and Çolak [7]. Tripathy et al. [31] obtained the generalized shape of the difference sequences by considering the former difference sequences in the following form

$$Y(\Delta_i^m)=\left\{y=(y_l)\ :\ (\Delta_i^m y_l)\in Y\right\}$$

where Y is regarded as a sequence, i and m are any two arbitrary positive integers, and $\Delta_i^m y_l=\sum_{w=0}^{m}(-1)^w\binom{m}{w}y_{l+iw}$.

We rather choose to take $Y(\Delta^m)$ in lieu of $Y(\Delta_i^m)$ and $\Delta^m y$ in lieu of $\Delta_i^m y$ if we assume that $i=1$. Other well known studies can be found in the following papers ([8], [9], [10], [17], [22], [25]).

2 Main Results

This section is the core of the study since it contains main results ans theorems.

Definition 2.1. [1] Let $\gamma=(l_k)$ be a lacunary sequence such that m and i are choosen as arbitrarily non-negative integers and $0<\alpha\le 1$ be given. If a real number R is defined as follows

$$\lim_{k\to\infty}\frac{1}{h_k^\alpha}\left|\left\{l\in I_k\ :\ \left|\Delta_i^m y_l-R\right|\ge\varepsilon\right\}\right|=0,$$

then the sequence $y=(y_l)$ is said to be Δ_i^m- lacunary statistically convergent of order α (or $S_\gamma^\alpha(\Delta_i^m)-$ convergent).

Thus it is obvious that $S_\gamma^\alpha(\Delta_i^m)-\lim y_l=R$. The set of all Δ_i^m- lacunary statistically convergent sequences will be denoted by $S_\gamma^\alpha(\Delta_i^m)$.

Δ_i^m-lacunary statistical convergence is well defined for $0<\alpha\le 1$, but it is not well defined for $\alpha>1$ in general. It is evidently true that every Δ^m-convergent sequence is lacunary Δ_i^m- statistically convergent of order α $(0<\alpha\le 1)$, but the converse does not hold.

Definition 2.2. [1] Let $\gamma=(l_k)$ be a lacunary sequence such that m and i are choosen as arbitrarily non-negative integers and $\alpha,p>0$ be given. If a real number R is defined as follows

$$\lim_{k\to\infty}\frac{1}{h_k^{\alpha}}\sum_{l\in I_k}\left|\Delta_i^m y_l - R\right|^p = 0,$$

then a sequence $y=(y_l)$ is said to be strongly $w_\gamma^\alpha\left(\Delta_i^m, p\right)-$summable (or lacunary strongly $\left(\Delta_i^m, p\right)-$summable of order α).

Thus it is obvious that $w_\gamma^\alpha\left(\Delta_i^m, p\right)-\lim y_l = R$. The set of all strongly $w_\gamma^\alpha\left(\Delta_i^m, p\right)-$summable sequences of order α will be denoted by $w_\gamma^\alpha\left(\Delta_i^m, p\right)$. If we take $\gamma=\left(2^k\right)$ then we write $w^\alpha\left(\Delta_i^m, p\right)$ instead of $w_\gamma^\alpha\left(\Delta_i^m, p\right)$.

For the following theorems we omit to give detailed proof since it is easily proved by using standard methods.

Theorem 2.3. Let $0<\alpha\le 1$ and $\gamma=(l_k)$ be a lacunary sequence. If $\limsup_k \frac{l_k}{l_{k-1}^\alpha}<\infty,$ then $w_\gamma^\alpha\left(\Delta_i^m, p\right)\subset w^\alpha\left(\Delta_i^m, p\right)$.

Theorem 2.4. Let $\gamma=(l_k)$ and $\gamma'=(s_k)$ be two lacunary sequences such that $I_k\subset J_k$ for all $k\in\mathbb{N}$ and let α and c be fixed real numbers such that $0<\alpha\le c\le 1,$

(i) If

$$\liminf_{k\to\infty}\frac{g_k^\alpha}{\ell_k^c}>0 \tag{1}$$

then $S_{\gamma'}^{\beta}\left(\Delta_i^m\right)\subseteq S_\gamma^\alpha\left(\Delta_i^m\right),$

(ii) If

$$\lim_{k\to\infty}\frac{\ell_k}{g_k^c}=1 \tag{2}$$

then $S_\gamma^\alpha\left(\Delta_i^m\right)\subseteq S_{\gamma'}^{c}\left(\Delta_i^m\right),$ where $I_k=\left(l_{k-1},l_k\right],$ $J_k=\left(s_{k-1},s_k\right],$ $g_k=l_k-l_{k-1}$ and $\ell_k=s_k-s_{k-1}$.

Theorem 2.5. Let $\gamma=(l_k)$ and $\gamma'=(s_k)$ be two lacunary sequences such that $I_k\subseteq B_k$ for all $k\in\mathbb{N}$, α and c be fixed real numbers such that $0<\alpha\le c\le 1$ and $0<p<\infty.$ Then we get

(i) If (1) holds then $w_{\gamma'}^{\beta}\left(\Delta_i^m, p\right)\subset w_\gamma^\alpha\left(\Delta_i^m, p\right),$

(ii) If (2) holds and $y\in\ell_\infty\left(\Delta_i^m\right)$ then $w_\gamma^\alpha\left(\Delta_i^m, p\right)\subset w_{\gamma'}^{\beta}\left(\Delta_i^m, p\right).$

Theorem 2.6. Let $\gamma=(l_k)$ and $\gamma'=(s_k)$ be two lacunary sequences such that $I_k\subseteq B_k$ for all $k\in\mathbb{N},$ α and c be fixed real numbers such that $0<\alpha\le c\le 1$ and $0<p<\infty.$ Then

(i) Let (1) holds, if a sequence is strongly $w^{\beta}_{\gamma}\left(\Delta_i^m, p\right)-$ summable to R, then it is $S_{\gamma}^{\alpha}\left(\Delta_i^m\right)-$ convergent to R,

(ii) Let (2) holds, if a $\Delta_i^m -$ bounded sequence is $S_{\gamma}^{\alpha}\left(\Delta_i^m\right)-$ convergent to R then it is strongly $w^{\beta}_{\gamma}\left(\Delta_i^m, p\right)-$ summable to R.

Theorem 2.7. If $y\in w^{\alpha}\left(\Delta_i^m, p\right)\cap w_{\gamma}^{\alpha}\left(\Delta_i^m, p\right)$ and $\limsup_k \frac{l_k}{l_{k-1}^{\alpha}}<\infty,$ then $w_{\gamma}^{\alpha}\left(\Delta_i^m, p\right)-\lim y_l = w^{\alpha}\left(\Delta_i^m, p\right)-\lim y_l = R.$

Proof. Let $w_{\gamma}^{\alpha}\left(\Delta_i^m, p\right)-\lim y_l = R$ and $w^{\alpha}\left(\Delta_i^m, p\right)-\lim y_l = R^{'}$, and suppose that $R\neq R^{'}$. Since $\limsup_k \frac{l_k}{l_{k-1}^{\alpha}}<\infty,$ from Theorem 2.3 we already know that $w_{\gamma}^{\alpha}\left(\Delta_i^m, p\right)\subset w^{\alpha}\left(\Delta_i^m, p\right)$. Since $\left(\Delta_i^m y - R^{'}\right)\in w_{\gamma,0}^{\alpha}\left(\Delta_i^m, p\right)$, it follows that $\left(\Delta_i^m y - R^{'}\right)\in w_{0}^{\alpha}\left(\Delta_i^m, p\right)$ and therefore $\frac{1}{t^{\alpha}}\sum_{b=1}^{t}\left|\Delta_i^m y_b - R^{'}\right|\to 0.$ Then we have

$$\frac{1}{t^{\alpha}}\sum_{b=1}^{t}\left|\Delta_i^m y_b - R^{'}\right|+\frac{1}{t^{\alpha}}\sum_{b=1}^{t}\left|\Delta_i^m y_b - R\right|\geq\frac{1}{t^{\alpha}}\left|R-R^{'}\right|>0,$$

and hence $R=R^{'}$.

3 Results Related to Modulus Function

Nakano developed the mathematical background of the notion of the modulus [26]. Then this notion is used for the construction of some sequence spaces by Maddox [24] and Ruckle [27]. Now, we present the inclusion relations among the sets of lacunary strongly $\left(\Delta_i^m, p\right)-$summable sequences of order α and $\Delta_i^m -$lacunary statistically convergent sequences of order α with respect to the modulus function f.

Definition 3.1. Let f be a modulus function, $p=\left(p_l\right)$ be a sequence of strictly positive real numbers and $\alpha>0$. Now we define

$$w_{\gamma}^{\alpha}\left(\Delta_i^m,(p),f\right)=\left\{y=\left(y_l\right)\ :\ \lim_{k\to\infty}\frac{1}{g_k^{\alpha}}\sum_{l\in I_k}\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l}=0,\ \text{for some } R\right\}.$$

If $y\in w_{\gamma}^{\alpha}\left(\Delta_i^m,(p),f\right)$, then we say that y is strongly $w_{\gamma}^{\alpha}\left(\Delta_i^m,(p),f\right)-$summable with respect to the modulus function f. If $p_l = p$ for all $l\in\mathbb{N}$ and $f\left(y\right)=y$ we shall write $w_{\gamma}^{\alpha}\left(\Delta_i^m, p\right)$ instead of $w_{\gamma}^{\alpha}\left(\Delta_i^m,(p),f\right)$ and in the special case $f\left(y\right)=y$ we shall write $w_{\gamma}^{\alpha}\left(\Delta_i^m,(p)\right)$ instead of $w_{\gamma}^{\alpha}\left(\Delta_i^m,(p),f\right)$.

In the following theorems we shall assume that the sequence $(p_l) = p$ is bounded and $0 < g = \inf_l p_l \le p_l \le \sup_l p_l = G < \infty$.

Theorem 3.2. Let m and i be two non-negative integers, $\gamma = (l_k)$ be a lacunary sequence, α and c be fixed real numbers such that $0 < \alpha \le c \le 1$ and f be a modulus function, then $w_\gamma^\alpha\left(\Delta_i^m,(p),f\right) \subset S_\gamma^c\left(\Delta_i^m\right)$.

Proof. Let $y \in w_\gamma^\alpha\left(\Delta_i^m,(p),f\right)$ and let $\varepsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $l \in I_k$, $\left|\Delta_i^m y_l - R\right| \ge \varepsilon$ and $l \in I_k$, $\left|\Delta_i^m y_l - R\right| < \varepsilon$, respectively. Since $g_k^\alpha \le g_k^c$ for each k we may write

$$\begin{aligned}\frac{1}{g_k^\alpha}\sum_{l\in I_k}\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l} &= \frac{1}{g_k^\alpha}\left[\sum\nolimits_1\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l} + \sum\nolimits_2\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l}\right]\\ &\ge \frac{1}{g_k^c}\left[\sum\nolimits_1\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l} + \sum\nolimits_2\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l}\right]\\ &\ge \frac{1}{g_k^c}\left|\left\{l \in I_k \ : \ \left|\Delta_i^m y_l - R\right| \ge \varepsilon\right\}\right| \ \min(\left[f(\varepsilon)\right]^g, \left[f(\varepsilon)\right]^G).\end{aligned}$$

Hence $y \in S_\gamma^c\left(\Delta_i^m\right)$.

Theorem 3.3. Let $\gamma = (l_k)$ be a lacunary sequence, m and i be two non-negative integers and $0 < \alpha \le 1$. If $\lim_{k\to\infty}\frac{g_k}{g_k^\alpha} = 1$ and the modulus f is bounded, then $S_\gamma^\alpha\left(\Delta_i^m\right) \subset w_\gamma^\alpha\left(\Delta_i^m,(p),f\right)$.

Proof. Let $y \in S_\gamma^\alpha\left(\Delta_i^m\right)$ and suppose that f is bounded and $\varepsilon > 0$ be given. Since f is bounded there exists an integer K such that $f(y) \le K$, for all $y \ge 0$. Then we may write

$$\begin{aligned}\frac{1}{g_k^\alpha}\sum_{l\in I_k}\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l} &= \frac{1}{g_k^\alpha}\left(\sum\nolimits_1\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l} + \sum\nolimits_2\left[f\left(\left|\Delta_i^m y_l - R\right|\right)\right]^{p_l}\right)\\ &\le \max\left(K^g, K^G\right)\frac{1}{g_k^\alpha}\left|\left\{l \in I_n \ : \ \left|\Delta_i^m y_l - R\right| \ge \varepsilon\right\}\right|\\ &\quad + \frac{g_k}{g_k^\alpha}\max\left(f(\varepsilon)^g, f(\varepsilon)^G\right).\end{aligned}$$

and so $w_\gamma^\alpha\left(\Delta_i^m,(p),f\right)$.

Theorem 3.4. Let $\gamma=(l_k)$ be a lacunary sequence, m and i be two non-negative integers and $0<\alpha\leq 1$. If $y=(y_l)$ is strongly $w_\gamma^\alpha\left(\Delta_i^m,(p),f\right)-$ summable to R with respect to the modulus function f and $\lim p_l>0$, then $w_\gamma^\alpha\left(\Delta_i^m,(p),f\right)-\lim y_l=R$ unique.

Proof. Omitted.

Theorem 3.5. Let $\gamma=(l_k)$ be a lacunary sequence, m and i be two non-negative integers and $0<\alpha\leq 1$. The sequence spaces $w_\gamma^\alpha\left(\Delta_i^m,p\right), S_\gamma^\alpha\left(\Delta_i^m\right)$ and $w_\gamma^\alpha\left(\Delta_i^m,(p),f\right)$ are neither solid nor symmetric, nor sequence algebras for $m\geq 1$.

Proof. Let $\gamma=\left(2^k\right)$ and $i=1$, then $\left(y_l\right)=(l^{m-1})\in w_\gamma^\alpha\left(\Delta_i^m,p\right)$ but $\left(\alpha_l y_l\right)\notin w_\gamma^\alpha\left(\Delta_i^m,p\right)$ when $\alpha_l=\left(-1\right)^l$ for all $l\in\mathbb{N}$. Hence $w_\gamma^\alpha\left(\Delta_i^m,p\right)$ is not solid. The other cases can be proved on considering similar examples.

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