Lacunary Statistical Convergence of Order alpha and Lacunary Strongly Summable Sequences of Order alpha of Generalized Difference Sequences

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Abstract. The notion of the \( \alpha \) th order \( \Delta^m_i - \) lacunary statistical convergence and \( \alpha \) th order lacunary strongly \( (\Delta^m_i, p) \) – summable sequences was introduced by Altınok et al. [1]. They also gave significant inclusion correlation associated with the aforementioned sequence spaces. In this paper, our aim is to exploit some other important relations between the given notions.

1 Introduction

The idea of statistical convergence was firstly determined by Fast [12] and redefined independently by Schoenberg [28]. The concept of the density of the set of positive integers \( \mathbb{N} \) was used as the main motivation to create this idea. For any subset \( F \) of the \( \mathbb{N} \), the density of \( F \) is defined as

\[
\beta(F) = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \omega_F(l)
\]

where \( \omega_F \) is the characteristic function of \( F \). It is needless to say that this definition is valid only for the existence of the limit. Let \( y = (y_l) \) be a given any sequence then it is statistically converged to \( R \), if for every \( \varepsilon > 0 \), \( \beta\{l \in \mathbb{N} : |y_l - R| \geq \varepsilon\} = 0 \). Furthermore, the relation and some application of this definition with the summability theory was presented by using the perspective of the sequence space by Çınar et al. ([3], [6], [30]), Connor [5], Fridy [14], Et et al. ([2], [11], [30]), Işik et al. ([17], [18], [19], [20], [21]). Gadjiev and Orhan [16] stated firstly the order of the sequence’s statistical

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convergence. Based on this research, Çolak [4] studied the \( \alpha \) th order of the statistical convergence and \( \alpha \) th order of the strong \( p-Cesàro \) summability.

The lacunary sequence is defined by \( h_k = (l_k - l_{k-1}) \to \infty \) as \( k \to \infty \), where \( \gamma = (l_k) \) is an increasing sequence of integer. The interval denoted by \( \gamma \) is determined by \( I_k = (l_{k-1}, l_k] \) and \( q_k \) is the abbreviation of the ratio of \( \frac{l_k}{l_{k-1}} \) ([13], [15], [20], [29]). The introduction of the difference sequence space and its modification and extension were recently conducted by Kizmaz [23] and Et and Çolak [7]. Tripathy et al. [31] obtained the generalized shape of the difference sequences by considering the former difference sequences in the following form

\[
Y(\Delta^m_i) = \{ y = (y_i) : (\Delta^m_i y_i) \in Y \}
\]

where \( Y \) is regarded as a sequence, \( i \) and \( m \) are any two arbitrary positive integers, and

\[
\Delta^m_i y_i = \sum_{w=0}^{m} (-1)^w \binom{m}{w} y_{i+tw}.
\]

We rather choose to take \( Y(\Delta^m) \) in lieu of \( Y(\Delta^m_i) \) and \( \Delta^m_i y \) in lieu of \( \Delta^m_i y_i \) if we assume that \( i = 1 \). Other well known studies can be found in the following papers ([8], [9], [10], [17], [22], [25]).

### 2 Main Results

This section is the core of the study since it contains main results ans theorems.

**Definition 2.1.** [1] Let \( \gamma = (l_k) \) be a lacunary sequence such that \( m \) and \( i \) are choosen as arbitrarily non-negative integers and \( 0 < \alpha \leq 1 \) be given. If a real number \( R \) is defined as follows

\[
\lim_{k \to \infty} \frac{1}{h_k^\alpha} \left| \left\{ l \in I_k : |\Delta^m_i y_i - R| \geq \varepsilon \right\} \right| = 0,
\]

then the sequence \( y = (y_i) \) is said to be \( \Delta^m_i \) - lacunary statistically convergent of order \( \alpha \) (or \( S^\alpha_\gamma (\Delta^m_i) \) - convergent).

Thus it is obvious that \( S^\alpha_\gamma (\Delta^m_i) - \lim y_i = R \). The set of all \( \Delta^m_i \) - lacunary statistically convergent sequences will be denoted by \( S^\alpha_\gamma (\Delta^m_i) \).

\( \Delta^m_i \) - lacunary statistical convergence is well defined for \( 0 < \alpha \leq 1 \), but it is not well defined for \( \alpha > 1 \) in general. It is evidently true that every \( \Delta^m_i \) - convergent sequence is lacunary \( \Delta^m_i \) - statistically convergent of order \( \alpha \) \( (0 < \alpha \leq 1) \), but the converse does not hold.

**Definition 2.2.** [1] Let \( \gamma = (l_k) \) be a lacunary sequence such that \( m \) and \( i \) are choosen as arbitrarily non-negative integers and \( \alpha, p > 0 \) be given. If a real number \( R \) is defined as follows
The lacunary sequence is defined by Definition 2.2. We rather choose to take an increasing sequence of integer. The interval denoted by \([10], \[17], \[22], \[25]\). Thus it is obvious that sequences in the following form is regarded as a sequence, \(i\) and \(m\) are any two arbitrary positive integers, and \(\alpha \leq i < m\). Let \(\gamma = (l_k)\) be a lacunary sequence such that \(\limsup_{k \to \infty} \frac{l_k}{l_{k-1}} < \infty\), then \(w^\alpha_\gamma(\Delta_i^m, p) \subseteq w^\alpha_\gamma(\Delta_i^m, p)\).

**Theorem 2.3.** Let \(0 < \alpha \leq 1\) and \(\gamma = (l_k)\) be a lacunary sequence. If \(\limsup_{k \to \infty} \frac{l_k}{l_{k-1}} < \infty\), then \(w^\alpha_\gamma(\Delta_i^m, p) \subseteq w^\alpha_\gamma(\Delta_i^m, p)\).

**Theorem 2.4.** Let \(\gamma = (l_k)\) and \(\gamma' = (s_k)\) be two lacunary sequences such that \(I_k \subseteq J_k\) for all \(k \in \mathbb{N}\) and let \(\alpha\) and \(c\) be fixed real numbers such that \(0 < \alpha \leq c \leq 1\),

(i) If \(\liminf_{k \to \infty} \frac{S_k^\alpha}{l_k^c} > 0\) \hspace{1cm} (1)

then \(S^\alpha_\gamma(\Delta_i^m) \subseteq S^\alpha_\gamma(\Delta_i^m)\),

(ii) If \(\lim_{k \to \infty} \frac{l_k}{g_k} = 1\) \hspace{1cm} (2)

then \(S^\alpha_\gamma(\Delta_i^m) \subseteq S^\alpha_\gamma(\Delta_i^m)\), where \(I_k = (l_{k-1}, l_k]\), \(J_k = (s_{k-1}, s_k]\), \(g_k = l_k - l_{k-1}\) and \(l_k = s_k - s_{k-1}\).

**Theorem 2.5.** Let \(\gamma = (l_k)\) and \(\gamma' = (s_k)\) be two lacunary sequences such that \(I_k \subseteq B_k\) for all \(k \in \mathbb{N}\), \(\alpha\) and \(c\) be fixed real numbers such that \(0 < \alpha \leq c \leq 1\) and \(0 < p < \infty\). Then we get

(i) If \(1\) holds then \(w^\alpha_\gamma(\Delta_i^m, p) \subseteq w^\alpha_\gamma(\Delta_i^m, p)\),

(ii) If \(2\) holds and \(y \in \ell_\infty(\Delta_i^m)\) then \(w^\alpha_\gamma(\Delta_i^m, p) \subseteq w^\alpha_\gamma(\Delta_i^m, p)\).

**Theorem 2.6.** Let \(\gamma = (l_k)\) and \(\gamma' = (s_k)\) be two lacunary sequences such that \(I_k \subseteq B_k\) for all \(k \in \mathbb{N}\), \(\alpha\) and \(c\) be fixed real numbers such that \(0 < \alpha \leq c \leq 1\) and \(0 < p < \infty\). Then
Let \((1)\) holds, if a sequence is strongly \(w_{\gamma}^{x}(\Delta_{i}^{m}, p)\)–summable to \(R\), then it is \(S_{\gamma}^{x}(\Delta_{i}^{m})\)– convergent to \(R\),

(ii) Let \((2)\) holds, if a \(\Delta_{i}^{m}\)–bounded sequence is \(S_{\gamma}^{x}(\Delta_{i}^{m})\)– convergent to \(R\) then it is strongly \(w_{\gamma}^{x}(\Delta_{i}^{m}, p)\)–summable to \(R\).

**Theorem 2.7.** If \(y \in w^{\alpha}(\Delta_{i}^{m}, p) \cap w_{\gamma}^{x}(\Delta_{i}^{m}, p)\) and \(\limsup_{k \to \infty} \frac{k}{l_{k}} < \infty\), then \(w_{\gamma}^{x}(\Delta_{i}^{m}, p) - \lim y_{i} = w^{\alpha}(\Delta_{i}^{m}, p) - \lim y_{i} = R\).

**Proof.** Let \(w_{\gamma}^{x}(\Delta_{i}^{m}, p) - \lim y_{i} = R\) and \(w^{\alpha}(\Delta_{i}^{m}, p) - \lim y_{i} = R'\), and suppose that \(R \neq R'\). Since \(\limsup_{k \to \infty} \frac{k}{l_{k}} < \infty\), from Theorem 2.3 we already know that \(w_{\gamma}^{x}(\Delta_{i}^{m}, p) \subseteq w^{\alpha}(\Delta_{i}^{m}, p)\). Since \((\Delta_{i}^{m} y - R') \in w_{\gamma, 0}^{x}(\Delta_{i}^{m}, p)\), it follows that \((\Delta_{i}^{m} y - R') \in w_{0}^{x}(\Delta_{i}^{m}, p)\) and therefore \(\frac{1}{t^{\alpha}} \sum_{b=1}^{t} |\Delta_{i}^{m} y_{b} - R' | \to 0\). Then we have

\[
\frac{1}{t^{\alpha}} \sum_{b=1}^{t} |\Delta_{i}^{m} y_{b} - R' | + \frac{1}{t^{\alpha}} \sum_{b=1}^{t} |\Delta_{i}^{m} y_{b} - R | \geq \frac{1}{t^{\alpha}} |R - R' | > 0,
\]

and hence \(R = R'\).

3 Results Related to Modulus Function

Nakano developed the mathematical background of the notion of the modulus \([26]\). Then this notion is used for the construction of some sequence spaces by Maddox \([24]\) and Ruckle \([27]\). Now, we present the inclusion relations among the sets of lacunary strongly \((\Delta_{i}^{m}, p)\)–summable sequences of order \(\alpha\) and \(\Delta_{i}^{m}\)–lacunary statistically convergent sequences of order \(\alpha\) with respect to the modulus function \(f\).

**Definition 3.1.** Let \(f\) be a modulus function, \(p = (p_{i})\) be a sequence of strictly positive real numbers and \(\alpha > 0\). Now we define

\[w_{\gamma}^{x}(\Delta_{i}^{m}, (p), f) = \left\{ y = (y_{i}) : \lim_{k \to \infty} \frac{1}{g_{k}^{\alpha} \sum_{l \in l_{k}}} [f(\Delta_{i}^{m} y_{l} - R)]^{p_{i}} = 0, \text{ for some } R \right\} \]

If \(y \in w_{\gamma}^{x}(\Delta_{i}^{m}, (p), f)\), then we say that \(y\) is strongly \(w_{\gamma}^{x}(\Delta_{i}^{m}, (p), f)\)–summable with respect to the modulus function \(f\). If \(p_{i} = p\) for all \(l \in \mathbb{N}\) and \(f(y) = y\) we shall write \(w_{\gamma}^{x}(\Delta_{i}^{m}, p)\) instead of \(w_{\gamma}^{x}(\Delta_{i}^{m}, (p), f)\) and in the special case \(f(y) = y\) we shall write \(w_{\gamma}^{x}(\Delta_{i}^{m}, p)\) instead of \(w_{\gamma}^{x}(\Delta_{i}^{m}, (p), f)\).
In the following theorems we shall assume that the sequence \( (p_i) = p \) is bounded and \( 0 < g = \inf_i p_i \leq p_i \leq \sup_i p_i = G < \infty \).

**Theorem 3.2.** Let \( m \) and \( i \) be two non-negative integers, \( \gamma = (l_k) \) be a lacunary sequence, \( \alpha \) and \( c \) be fixed real numbers such that \( 0 < \alpha \leq c \leq 1 \) and \( f \) be a modulus function, then \( w_\gamma^\alpha \left( \Delta_i^m, (p), f \right) \subset S_\gamma^c \left( \Delta_i^m \right) \).

**Proof.** Let \( y \in w_\gamma^\alpha \left( \Delta_i^m, (p), f \right) \) and let \( \varepsilon > 0 \) be given and \( \Sigma_1 \) and \( \Sigma_2 \) denote the sums over \( l \in I_k, \quad \left| \Delta_i^m y_l - R \right| \geq \varepsilon \) and \( l \in I_k, \quad \left| \Delta_i^m y_l - R \right| < \varepsilon \), respectively. Since \( g_k^\alpha \leq g_k^c \) for each \( k \) we may write

\[
\frac{1}{g_k^\alpha} \sum_{l \in I_k} \left[ f \left( \left| \Delta_i^m y_l - R \right| \right) \right]^{p_l} = \frac{1}{g_k^\alpha} \left[ \sum_1 f \left( \left| \Delta_i^m y_l - R \right| \right) \right]^{p_l} + \sum_2 f \left( \left| \Delta_i^m y_l - R \right| \right)^{p_l} \\
\geq \frac{1}{g_k^c} \left[ \sum_1 f \left( \left| \Delta_i^m y_l - R \right| \right) \right]^{p_l} + \sum_2 f \left( \left| \Delta_i^m y_l - R \right| \right)^{p_l} \\
\geq \frac{1}{g_k^c} \left| \left\{ l \in I_k : \left| \Delta_i^m y_l - R \right| \geq \varepsilon \right\} \right| \min \left( f (\varepsilon) \right)^G, f (\varepsilon)^G \right).
\]

Hence \( y \in S_\gamma^c \left( \Delta_i^m \right) \).

**Theorem 3.3.** Let \( \gamma = (l_k) \) be a lacunary sequence, \( m \) and \( i \) be two non-negative integers and \( 0 < \alpha \leq 1 \). If \( \lim_{k \to \infty} \frac{g_k}{g_k^\alpha} = 1 \) and the modulus \( f \) is bounded, then \( S_\gamma^\alpha \left( \Delta_i^m \right) \subset w_\gamma^\alpha \left( \Delta_i^m, (p), f \right) \).

**Proof.** Let \( y \in S_\gamma^\alpha \left( \Delta_i^m \right) \) and suppose that \( f \) is bounded and \( \varepsilon > 0 \) be given. Since \( f \) is bounded there exists an integer \( K \) such that \( f (y) \leq K \), for all \( y \geq 0 \). Then we may write

\[
\frac{1}{g_k^\alpha} \sum_{l \in I_k} \left[ f \left( \left| \Delta_i^m y_l - R \right| \right) \right]^{p_l} = \frac{1}{g_k^\alpha} \left[ \sum_1 f \left( \left| \Delta_i^m y_l - R \right| \right) \right]^{p_l} + \sum_2 f \left( \left| \Delta_i^m y_l - R \right| \right)^{p_l} \\
\leq \max \left( K^G, K^G \right) \frac{1}{g_k^\alpha} \left| \left\{ l \in I_n : \left| \Delta_i^m y_l - R \right| \geq \varepsilon \right\} \right| \\
+ \frac{g_k}{g_k^\alpha} \max \left( f (\varepsilon)^G, f (\varepsilon)^G \right).
\]

and so \( w_\gamma^\alpha \left( \Delta_i^m, (p), f \right) \).
Theorem 3.4. Let $\gamma = (l_k)$ be a lacunary sequence, $m$ and $i$ be two non-negative integers and $0 < \alpha \leq 1$. If $\gamma = (y_j)$ is strongly $w_\gamma^\alpha (\Delta_i^m, (p), f)$– summable to $R$ with respect to the modulus function $f$ and $\lim p_j > 0$, then $w_\gamma^\alpha (\Delta_i^m, (p), f) \lim y_j = R$ unique.

Proof. Omitted.

Theorem 3.5. Let $\gamma = (l_k)$ be a lacunary sequence, $m$ and $i$ be two non-negative integers and $0 < \alpha \leq 1$. The sequence spaces $w_\gamma^\alpha (\Delta_i^m, p), S_\gamma^\alpha (\Delta_i^m)$ and $w_\gamma^\alpha (\Delta_i^m, (p), f)$ are neither solid nor symmetric, nor sequence algebras for $m \geq 1$.

Proof. Let $\gamma = (2^k)$ and $i = 1$, then $(y_j) = (l^{m-1}) \in w_\gamma^\alpha (\Delta_i^m, p)$ but $(\alpha_i y_j) \notin w_\gamma^\alpha (\Delta_i^m, p)$ when $\alpha_i = (-1)^l$ for all $l \in \mathbb{N}$. Hence $w_\gamma^\alpha (\Delta_i^m, p)$ is not solid. The other cases can be proved on considering similar examples.

References

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The other cases can be proved on considering similar examples.

Proof. Let

\[ f(\gamma, \Delta) = \alpha \quad \text{and} \quad \gamma \Delta = (\gamma, \Delta) \]

\[ \alpha \leq 0 \]

References