Investigation of Dark and Bright Soliton Solutions of Some Nonlinear Evolution Equations

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Abstract. In this paper, generalized Kudryashov method (GKM) is used to find the exact solutions of (1+1) dimensional nonlinear Ostrovsky equation and (4+1) dimensional Fokas equation. Firstly, we get dark and bright soliton solutions of these equations using GKM. Then, we remark the results we found using this method.

1 Introduction

During the past years, soliton solutions are considerably important issue in biophysics, geophysical sciences, chemical kinematics, optical fibers, technology of space, electricity, elastic media and several topics in nonlinear sciences.

In recent years, most authors have presented several methods to find the soliton solutions of NLEEs such as (G'/G)-expansion method [1], exp-function method [2], the tanh method [3] and many more. In this work, GKM [4] will be used to find exact solutions of (1+1) dimensional nonlinear Ostrovsky equation.

Firstly, we consider the following (1+1) dimensional nonlinear Ostrovsky equation,

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0. \tag{1}$$

Eq. (1) is a model for weakly nonlinear surface and internal waves in a rotating ocean. This equation has been introduced by Vakhnenko and Parkers [5]. They have found completely integrable of Eq. (1) by inverse scattering method [6]. Then, some authors have used various methods to find travelling wave solutions of this equation [7-11].

Secondly, we handle (4+1) dimensional Fokas equation [12-18],

$$4u_{tx} - u_{xxxy} + u_{xyyy} + 12u_x u_y + 12uu_{xy} - 6u_{zw} = 0. \tag{2}$$

This equation has been obtained by Fokas by expanding the Lax pairs of the integrable Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equations to some higher-dimensional nonlinear wave equations [12]. The importance of Eq. (1) suggests that the idea of complexifying time can be considered in the context of modern field theories via the existence of integrable nonlinear equations in four spatial dimensions involving complex time [12].

The layout of this paper is organized as follows. In Sec. 2, we give basic facts of GKM. In Sec. 3, we implement GKM to (1+1) dimensional nonlinear Ostrovsky equation

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and (4+1) dimensional Fokas equation. In Sec. 4, we report the results we obtained using this method.

2 Basic Facts of GKM

Step 1. We consider PDE as follows:

$$P\left(u,u_t,u_x,u_{xx},u_{xxx},\cdots\right)=0. \tag{3}$$

Then, we get traveling transformation

$$u\left(x,t\right)=u\left(\xi\right),\xi=x-ct, \tag{4}$$

where c is arbitrary constant. Eq.(3) was reduced to a nonlinear ordinary differential equation:

$$N(u,u',u'',u''',\cdots)=0, \tag{5}$$

where the prime denotes differentiation with respect to ξ.

Step 2. Suggest that the exact solutions of Eq.(5) can be found as follows;

$$u\left(\xi\right)=\frac{\sum\limits_{i=0}^{N}a_iQ^i\left(\xi\right)}{\sum\limits_{j=0}^{M}b_jQ^j\left(\xi\right)}=\frac{A\left[Q(\xi)\right]}{B\left[Q(\xi)\right]}, \tag{6}$$

where Q is $\frac{1}{1\pm e^{\xi}}$. We highlight that the function Q is solution of Eq. (7)

$$Q_{\xi}=Q^2-Q. \tag{7}$$

Taking Eq.(6) into account, we gain

$$u'\left(\xi\right)=\frac{A'Q'B-AB'Q'}{B^2}=Q'\left[\frac{A'B-AB'}{B^2}\right]=\left(Q^2-Q\right)\left[\frac{A'B-AB'}{B^2}\right], \tag{8}$$

$$u''\left(\xi\right)=\frac{Q^2-Q}{B^2}\left[(2Q-1)\left(A'B-AB'\right)+\frac{Q^2-Q}{B}\left[B\left(A''B-AB''\right)-2B'A'B+2A\left(B'\right)^2\right]\right], \tag{9}$$

$$\begin{aligned}u'''\left(\xi\right)=&\left(Q^2-Q\right)^3\left[\frac{\left(A'''B-AB'''-3A''B'-3B''A'\right)B+6B\left(AB''+B'A'\right)}{B^3}-\frac{6A\left(B'\right)^3}{B^4}\right]\\&+3\left(Q^2-Q\right)^2\left(2Q-1\right)\left[\frac{B\left(A''B-AB''\right)-2B'A'B+2A\left(B'\right)^2}{B^3}\right]\\&+\left(Q^2-Q\right)\left(6Q^2-6Q+1\right)\left[\frac{A'B-AB'}{B^2}\right].\end{aligned} \tag{10}$$

Step 3. The solution of Eq.(5) can be expressed as follows:

$$u\left(\xi\right)=\frac{a_0+a_1Q+a_2Q^2+\cdots+a_NQ^N+\cdots}{b_0+b_1Q+b_2Q^2+\cdots+b_MQ^M+\cdots}. \tag{11}$$

To compute the values M and N in Eq.(11) that is the pole order for the general solution of Eq.(5), we develop comparably as in the classical Kudryashov method on balancing the highest order nonlinear terms in Eq.(5) and we can establish a relation between M and N. We can find values of M and N.

Step 4. Substituting Eq.(6) into Eq.(5) ensures a polynomial $R(Q)$ of Q. Equating the coefficients of $R(Q)$ to zero, we get a system of algebraic equations. Solving this system, we can identify c and the coefficients $a_0, a_1, a_2, \cdots, a_N, b_0, b_1, b_2, \cdots, b_M$. Thus, we gain the exact solutions to Eq.(5).

3 Application of GKM to (1+1) Dimensional Nonlinear Ostrovsky Equation and (4+1) Dimensional Fokas Equation

In order to solve Eq. (1), we use the travelling wave transform

$$u(x,t) = u(\xi), \xi = x - ct, \tag{12}$$

where c is arbitrary constant.
Substituting Eq. (13) into Eq. (1),

$$u_t = -cu', \ u_x = u', u_{xt} = -cu'', u_{xxt} = -cu''', \tag{13}$$

we get the following equation

$$3uu'' - 3(u')^2 + u^3 = 0. \tag{14}$$

After Eq. (1) has been turned into Eq. (14), Substituting Eqs. (6) and (9) into Eq. (14) and balancing the highest order nonlinear terms of uu'' and u^3 in Eq. (14), then the following relation is acquired

$$2N - 2M + 2 = 3N - 3M \Rightarrow N = M + 2. \tag{15}$$

The exact solution of Eq.(1) is obtained as follows;

$$a_0 = 0, \ a_1 = -a_2 + 6b_1, \ a_3 = -6b_1, \ b_0 = -\frac{a_2}{6} + b_1, \tag{16}$$

Substituting Eq. (16) into Eq. (11), we get dark soliton solutions of Eq.(1)

$$u_1(x,t) = \frac{3}{2}\left(1 - \left[\tanh\left(\frac{x-ct}{2}\right)\right]^2\right), \tag{17}$$

$$u_2(x,t) = \frac{3}{2}\left(1 - \left[\coth\left(\frac{x-ct}{2}\right)\right]^2\right), \tag{18}$$

In an effort to find travelling wave solutions of the Eq. (2), we get the transformation via the wave variables

$$u(x,y,z,w,t) = u(\xi), \xi = \alpha x + \beta y + \gamma z + \delta w + \varepsilon t, \tag{19}$$

where $\alpha, \beta, \gamma, \delta$ and ε are arbitrary constants.

Substituting Eq. (20) into Eq. (2),

$$\begin{aligned} &u_x = \alpha u', u_y = \beta u', u_{tx} = \alpha\varepsilon u'', \ u_{xy} = \alpha\beta u'', \\ &u_{zw} = \gamma\delta u'', u_{xxxy} = \alpha^3 \beta u^{(4)}, u_{xyyy} = \alpha\beta^3 u^{(4)}, \end{aligned} \tag{20}$$

we get the following equation

$$\left(\alpha\beta^3 - \alpha^3\beta\right)u'' + \left(4\alpha\varepsilon - 6\gamma\delta\right)u + 6\alpha\beta u^2 = 0. \tag{21}$$

After Eq. (2) has been turned into Eq. (21), Substituting Eqs. (6) and (9) into Eq. (21) and balancing the highest order nonlinear terms between u'' and u^2 in Eq. (21), then the following relation is acquired

$$N - M + 2 = 2N - 2M \Rightarrow N = M + 2. \tag{22}$$

The exact solution of Eq.(2) is obtained as follows;

Case 1.

$$\begin{gathered} a_0 = 0,\ a_1 = \left(-\alpha^2 + \beta^2\right) b_0, a_2 = \left(\alpha^2 - \beta^2\right)\left(b_0 - b_1\right), \\ a_3 = \left(\alpha^2 - \beta^2\right) b_1,\ \varepsilon = \frac{\alpha\beta^3 - \alpha^3\beta + 6\gamma\delta}{4\alpha}, \end{gathered} \tag{23}$$

Inserting Eq. (23) into Eq. (11), we get bright soliton solutions of Eq.(2)

$$u_1\left(x, y, z, w, t\right) = \frac{\left(-\alpha^2 + \beta^2\right)}{4}\left[\sec h^2\left(\frac{\alpha x + \beta y + \gamma z + \delta w + \varepsilon t}{2}\right)\right], \tag{24}$$

$$u_2\left(x, y, z, w, t\right) = \frac{\left(\alpha^2 - \beta^2\right)}{4}\left[\csc h^2\left(\frac{\alpha x + \beta y + \gamma z + \delta w + \varepsilon t}{2}\right)\right]. \tag{25}$$

Case 2.

$$\begin{gathered} a_0 = \frac{1}{6}\left(\alpha^2 - \beta^2\right) b_0,\ a_1 = \frac{1}{6}\left(\alpha^2 - \beta^2\right)\left(-6b_0 + b_1\right), a_2 = \left(\alpha^2 - \beta^2\right)\left(b_0 - b_1\right), \\ a_3 = \left(\alpha^2 - \beta^2\right) b_1,\ \varepsilon = \frac{-\alpha^3\beta + \alpha\beta^3 + 6\gamma\delta}{4\alpha}, \end{gathered} \tag{26}$$

Embedding Eq. (26) into Eq. (11) , we get dark soliton solution of Eq.(2)

$$u_3\left(x, y, z, w, t\right) = \frac{\left(\alpha^2 - \beta^2\right)}{6}\left(-\frac{1}{2} + \frac{3}{2}\left[\tanh\left(\frac{\alpha x + \beta y + \gamma z + \delta w + \varepsilon t}{2}\right)\right]^2\right), \tag{27}$$

$$u_4\left(x, y, z, w, t\right) = \frac{\left(\alpha^2 - \beta^2\right)}{6}\left(-\frac{1}{2} + \frac{3}{2}\left[\coth\left(\frac{\alpha x + \beta y + \gamma z + \delta w + \varepsilon t}{2}\right)\right]^2\right). \tag{28}$$

Remark. If we compare the exact solutions of Eq. (1) and Eq. (2) reported by the other authors, our solution (17) is a similar solution with the solution (4.15) in [7]. Also, our solution (24) is a similar solution with the solution (10) in [15]. To our knowledge, other solutions of Eq. (1) we reported here are new and are not trackable in the former literature.

4 Conclusion

In this paper, we gain dark soliton solutions of (1+1) dimensional nonlinear Ostrovsky equation and (4+1) dimensional Fokas equation using GKM. Applications show that this method can successfully solve this equation. Thus, we can conclude that not only this method has a significiant role in observing NLEEs but also it is highly strong with regard to yielding analytical solutions of NLEEs.

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