(λ,μ) Statistical Convergence of Order γ for Double Sequences of Functions

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Abstract. In this present work, we investigate the concepts of Δ^r-statistical convergence of order $\tilde{\gamma}$ for double sequences of real valued functions. Also some definitions we given for Δ^r-statistical Cauchy sequence for double sequences of real valued functions and some relations between $S_{(\lambda,\mu)}^{\tilde{\gamma}}\left(\Delta^r, f\right)$ statistical convergence and strong $\left[V^{\tilde{\gamma}}, \lambda, \mu\right]_q\left(\Delta^r, f\right)-$ summability are given.

1 Introduction

Zygmund [20] defined statistical convergence in 1935. After that defined statistical convergence for sequences of real numbers was given by Steinhaus [19] and Fast [6] later reintroduced by Schoenberg [14] independently. Then, in terms of the sequence spaces is examined and correlated with the summability theory by Connor [9], Çinar et. al. [1], Çolak [2], Et et. al. ([3],[4],[15],[16]), Fridy [7], Mursaleen [12], and many others [8],[10],[13],[17],[18]). Statistical convergence is related to density of subsets of the set ℕ of natural numbers. The density of a subset D of ℕ is defined by

$$d(D)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_D(K)$$

provided that the limit texists. x = (x_n) sequences of real numbers is statistically convergent to ξ if for every ε > 0, $d(\{n\in \mathrm{N}:|x_n-\xi|\geq\varepsilon\})=0$

Pringsheim[21] defined the convergence of double sequences in 1900 as follows :

$x=(x_{nm})$ be a double sequences , for every $\varepsilon>0$ there exists $N\in$ ℕ such that $|x_{nm}-\xi|<\varepsilon$ whenever $n,m>N$. In this situation we write $P-\lim x=\xi.$

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A double sequence $x=(x_{nm})_{n,m=0}^{\infty}$ can be defined as bounded if there exists a positive real number T such that $|x_{nm}|<T$ for all n and m, i.e. $\|x\|=\sup_{n,m\geq 0}|x_{nm}|<\infty.$

Let $K\subseteq \mathbb{N}\text{x }\mathbb{N}$ and $K(r,s)=\{(n,m)\ :\ n\leq r, m\leq s\}$. The double natural density of K is defined by

$$d^2(K)=P-\lim_{r,s}\frac{1}{rs}|K(r,s)|,\ \text{if the limit exists.}\ [12]$$

The idea of difference sequences was given Kızmaz [11] and were generalized by Et and Çolak [5].

Throught this work unless said otherwise by for all $n,m\in \mathrm{N}_{n_0}$ we mean for all $n,m\in \mathrm{N}_{n_0}$ except finite numbers of positive integers where $\mathrm{N}_{n_0}=\{n_0,n_0+1,n_0+2,...\}$ for some $n_0\in\mathbb{N}=\{1,2,3,...\}$

2 Main Result

Definition 1 Let $\widetilde{\gamma}\in(0,1]$ be given. A double sequence of functions $\{f_{kl}\}$ is said to be $\Delta^r-(\lambda,\mu)$ statistically convergent of order $\widetilde{\gamma}$ to the function f on a set D if for every $\varepsilon>0$ and for every $x\in D$

$$\lim_{uv}\frac{1}{\lambda_u^s\mu_v^t}\left|\{(k,l)\in I_u xJ_v\ :\ |\Delta^r f_{kl}(x)-f(x)|\geq\varepsilon\}\right|=0.$$

where $I_u=[u-\lambda_u+1,u], J_v=[v-\mu_v+1,v]$ Then we can write $S_{(\lambda,\mu)}^{\widetilde{\gamma}}-\lim\Delta^r f_{kl}(x)=f(x)$ on D. The function f is $\Delta^r-(\lambda,\mu)$ statistical limit of order $\widetilde{\gamma}$ of the sequence $\{f_{kl}\}$. The set of all $\Delta^r-(\lambda,\mu)$ statistically convergent sequences of functions order $\widetilde{\gamma}$ will be denoted by $S_{(\lambda,\mu)}^{\widetilde{\gamma}}(\Delta^r,f)$. Then we write shortly $\Delta^r f_{kl}(x)\overset{S_{(\lambda,\mu)}^{\widetilde{\gamma}}}{\rightarrow}f(x)$.

$\Delta^r-\ (\lambda,\mu)$ statistical convergence of order $\widetilde{\gamma}$ is well defined for $\widetilde{\gamma}\in(0,1]$, but is not well defined for $\widetilde{\gamma}\succ 1$. $\{f_{kl}\}$ sequence of functions defined as follows

$$f_{kl}(x)=\begin{cases}1 & k+l=2n\\ x^{k+l} & k+l\neq 2n\end{cases}\quad n=1,2,3...,x\in[0,\tfrac{1}{2}]$$

For $\widetilde{\gamma}\succ 1,$ $S_{(\lambda,\mu)}^{\widetilde{\gamma}}-\lim\Delta f_{kl}(x)=2$ and $S_{(\lambda,\mu)}^{\widetilde{\gamma}}-\lim\Delta f_{kl}(x)=-2$ which is impossible.

Theorem 2 Let $\{f_{kl}\}$ and $\{g_{kl}\}$ are two double sequences of real valued functions defined on a set D and $\widetilde{\gamma}\in(0,1]$.

(i) If $\Delta^r f_{kl}(x) \overset{S^{\tilde{\gamma}}_{(\lambda,\mu)}}{\rightarrow} f(x)$ and $c \in \mathrm{R}$, then $c\Delta^r f_{kl}(x) \overset{S^{\tilde{\gamma}}_{(\lambda,\mu)}}{\rightarrow} cf(x)$,

(ii) If $\Delta^r f_{kl}(x) \overset{S^{\tilde{\gamma}}_{(\lambda,\mu)}}{\rightarrow} f(x)$ and $\Delta^r g_{kl}(x) \overset{S^{\tilde{\gamma}}_{(\lambda,\mu)}}{\rightarrow} g(x)$, then $\Delta^r f_{kl}(x) + \Delta^r g_{kl}(x) \overset{S^{\tilde{\gamma}}_{(\lambda,\mu)}}{\rightarrow} f(x) + g(x)$.

Theorem 3 $\tilde{\gamma}, \tilde{\beta} \in (0,1]$ are given such that $\tilde{\gamma} \le \tilde{\beta}$ then $S^{\tilde{\gamma}}_{(\lambda,\mu)}(\Delta^r, f) \subseteq S^{\tilde{\beta}}_{(\lambda,\mu)}(\Delta^r, f)$ which is proper.

Proof. It's easy to show that inclusion is provided. To show that the inclusion is strict define a double sequence $\{f_{kl}\}$ by

$$f_{kl}(x) = \begin{cases} 1 & k,l = n^2 \\ \frac{k^2 l^2 x}{1+k^3 l^3 x^2} & k,l \neq n^2 \end{cases}.$$

Then $S^{\tilde{\beta}}_{(\lambda,\mu)} - \lim \Delta f_{jk}(x) = 0$ for $\tilde{\beta} \in \left(\frac{1}{2}, 1\right]$, but $f(x) \notin S^{\tilde{\gamma}}_{(\lambda,\mu)}(\Delta, f)$ for $\tilde{\gamma} \in \left(0, \frac{1}{2}\right]$.

Corollary 4 If $\{f_{kl}\} \in S^{\tilde{\gamma}}_{(\lambda,\mu)}(\Delta^r, f)$, then $\{f_{kl}\} \in S_{(\lambda,\mu)}(\Delta^r, f)$

Definition 5 Let $\tilde{\gamma} \in (0,1]$. The sequence $\{\Delta^r f_{kl}\}$ is a (λ, μ) statistically Cauchy sequence of order $\tilde{\gamma}$ provided that for every $\varepsilon > 0$ there are two numbers $N(= N(\varepsilon)), M(= M(\varepsilon))$ such that

$$\left|\Delta^r f_{kl}(x) - \Delta^r f_{NM}(x)\right| < \varepsilon \quad a.a.(k,l)\ (\tilde{\gamma}) \text{ and for each } x \in D$$

i.e.

$$\lim_{uv} \frac{1}{\lambda_u^s \mu_v^t} \left|\left\{(k,l) \in I_u x J_v \ : \ \left|\Delta^r f_{kl}(x) - \Delta^r f_{NM}(x)\right| \ge \varepsilon \right\}\right| = 0.$$

for each $x \in D$.

Definition 6 Suppose that $\tilde{\gamma} \in (0,1]$ and u is a positive real number. A double sequence of functions $\{\Delta^r f_{kl}\}$ is strongly $(V^{\tilde{\gamma}}, \lambda, \mu)_q$ summable of order $\tilde{\gamma}$ (or $[V^{\tilde{\alpha}}, \lambda, \mu]_q (\Delta^r, f)-$ summable) if there is a function f such that

$$\lim_{uv\to\infty} \frac{1}{\lambda_u^s \mu_v^t} \sum_{k\in I_u} \sum_{l\in J_v} \left|\Delta^r f_{kl}(x) - f(x)\right|^q = 0.$$

Thus it can be written $[V^{\tilde{\gamma}}, \lambda, \mu]_q - \lim \Delta^r f_{kl}(x) = f(x)$ on D. The set of all strongly $(V^{\tilde{\gamma}}, \lambda, \mu)_q (\Delta^r, f)-$ summable double sequences of functions of order $\tilde{\gamma}$ is shown by $[V^{\tilde{\gamma}}, \lambda, \mu]_q (\Delta^r, f)$. We can write shortly $\Delta^r f_{kl}(x) \overset{[V^{\tilde{\gamma}}, \lambda, \mu]_q}{\rightarrow} f(x)$.

Teorem 7 Assume that q is a positive real number and $\widetilde{\gamma},\widetilde{\beta}\in(0,1]$ such that $\widetilde{\gamma}\leq\widetilde{\beta}$. Then $\left[V^{\widetilde{\gamma}},\lambda,\mu\right]_q\left(\Delta^r,f\right)\subseteq\left[V^{\widetilde{\beta}},\lambda,\mu\right]_q\left(\Delta^r,f\right)$ is proper for some $\widetilde{\gamma}=(s,t)$ and $\widetilde{\gamma}=(u,v)$ such that $\widetilde{\gamma}\prec\widetilde{\beta},\;\; s,t,u,v\in R$.

Proof. It's easy to show that inclusion is provided.. To show that the inclusion is strict define a double sequence $\{f_{kl}\}$ by

$$f_{kl}(x)=\begin{cases}\frac{klx}{1+klx} & k,l=n^2\\ 0 & k,l\neq n^2\end{cases}\;\; x\in[1,2].$$

the sequence $\{\Delta f_{kl}\}$ is strongly $\left(V^{\widetilde{\beta}},\lambda,\mu\right)_q$ summable of order $\widetilde{\beta}$, for $\widetilde{\beta}\in\left(\frac{1}{2},1\right]$, the sequence $\{\Delta f_{kl}\}$ is not strongly $\left(V^{\widetilde{\gamma}},\lambda,\mu\right)_q$ summable of order $\widetilde{\gamma}$, for $\widetilde{\gamma}\in\left(0,\frac{1}{2}\right]$ where $\lambda_u=u,\;\;\mu_v=v.$

Corollary 8 Let $\widetilde{\gamma},\widetilde{\beta}\in(0,1]$ and q be a positive real number. Then

(i) if $\widetilde{\gamma}\cong\widetilde{\beta}$, then $\left[V^{\widetilde{\gamma}},\lambda,\mu\right]_q\left(\Delta^r,f\right)\subseteq\left[V^{\widetilde{\beta}},\lambda,\mu\right]_q\left(\Delta^r,f\right)$

(ii) $\left[V^{\widetilde{\gamma}},\lambda,\mu\right]_q\left(\Delta^r,f\right)\subseteq\left[V,\lambda,\mu\right]_q\left(\Delta^r,f\right)$ for each $\widetilde{\gamma}\in(0,1]$ and $0<u<\infty$.

Theorem 9 Let $\widetilde{\gamma},\widetilde{\beta}\in(0,1]$ such that $\widetilde{\gamma}\leq\widetilde{\beta}$ and u be a positive real number. If a double sequence of functions $\{f_{kl}\}$ is in the $\left[V^{\widetilde{\gamma}},\lambda,\mu\right]_q\left(\Delta^r,f\right)$ then $\{f_{kl}\}$ is also in the $S^{\widetilde{\beta}}_{(\lambda,\mu)}\left(\Delta^r,f\right)$.

Corollary 10 Let $\widetilde{\gamma}\in(0,1]$ be given and and u be a positive real number. If $\{f_{kl}\}\in\left[V^{\widetilde{\gamma}},\lambda,\mu\right]_q\left(\Delta^r,f\right)$, then $\{f_{kl}\}\in S^{\widetilde{\gamma}}_{(\lambda,\mu)}\left(\Delta^r,f\right)$.

Theorem 11 Let $\lambda=(\lambda_u)$, $\mu=(\mu_v)$ $\sigma=(\sigma_u),\rho=(\rho_v)$ be four sequences in Λ such that $\lambda_u\leq\sigma_u$, $\mu_v\leq\rho_v$ for all $u,v\in\mathbb{N}_{n_o}$ $0<\widetilde{\gamma}\leq\widetilde{\beta}\leq 1$ and $\{f_{kl}\}$ be a double sequence of real valued functions defined on a set D.

(i) If

$$\liminf_{uv\to\infty}\frac{\lambda_u^{\widetilde{\alpha}}\mu_v^{\widetilde{\alpha}}}{\sigma_u^{\widetilde{\beta}}\rho_v^{\widetilde{\beta}}}>0$$

then $S^{\widetilde{\beta}}_{(\lambda\mu)}\left(\Delta^r,f\right)\subseteq S^{\widetilde{\gamma}}_{(\sigma,\rho)}\left(\Delta^r,f\right)$,

(ii) If

$$\lim_{u\to\infty}\frac{\lambda_u^{\widetilde{\alpha}}}{\sigma_u^{\widetilde{\beta}}}=1,\lim_{v\to\infty}\frac{\mu_v^{\widetilde{\alpha}}}{\rho_v^{\widetilde{\beta}}}=1 \quad and \quad \lim_{u\to\infty}\frac{\sigma_u}{\sigma_u^{\widetilde{\beta}}}=1,\lim_{v\to\infty}\frac{\rho_v}{\rho_v^{\widetilde{\beta}}}=1$$

then $S_{(\lambda\mu)}^{\widetilde{\gamma}}(\Delta^r,f)\subseteq S_{(\sigma,\rho)}^{\widetilde{\beta}}(\Delta^r,f)$.

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