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Stability Analysis, Numerical and Exact Solutions of the (1+1)-Dimensional NDMBBM Equation

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Abstract. A newly propose mathematical approach is presented in this study. We utilize the new approach in investigating the solutions of the (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation. The new analytical technique is based on the popularly known sinh-Gordon equation and a wave transformation. In developing this new technique at each every steps involving integration, the integration constants are considered to not be zero which gives rise to new form of travelling wave solutions. The (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony is used in modelling an approximation for surface long waves in nonlinear dispersive media. We construct some new trigonometric function solution to this equation. Moreover, the finite forward difference method is utilized in investigating the numerical behavior of this equation by taking one of the obtained analytical solutions into consideration. We finally, give a comprehensive conclusions.

1 Introduction

Various complex aspects such as plasma physics, fluid dynamics, biological and chemical phenomena can be expressed in form of nonlinear partial differential equations (NPDEs). Because of the nonlinearity nature of these equations its sometimes difficult to find their exact and numerical solutions, but their solutions play a vital roles in our real life. Therefore, its important to seek their solutions. Various analytical and numerical technique have been formulated and applied to acquire the solutions of such type of models [1–16].

This study investigates the analytical solutions of the (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony (NDMBBM) equation arising in nonlinear dispersive media [18] by using a newly proposed technique called the New sinh-Gordon function method (ShGFM). The (1+1)-dimensional NDMBBM equation is used in modelling an approximation for surface long waves in nonlinear dispersive

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media [18]. The new ShGFM is based on the sinh-Gordon equation and a wave transformation. In this new technique, when simplifying the sinh-Gordon equation, we consider the integration constant to not be zero. On the other hand, we utilize the finite difference method (FDM) in examining the numerical behavior of (1+1)-dimensional NDMBBM.

2 The ShGFM

In this section, we present the analysis of the proposed approach on how it can be applied in exploring search for the new solutions of various nonlinear models. To obtain the new travelling wave solutions of any given nonlinear model by using the proposed approach, we do the following;

Consider the following equation;

$$F(u,\ uu_x,\ u^2u_t,\ uu_{tt},\ \ldots) = 0, \tag{1}$$

where u is a function of the independent variable x, t and the subscripts indicate the derivative of u with respect to the independent variables and F is the polynomial of the function u.

Consider the following sinh-Gordon equation [17]:

$$u_{xt} = \lambda sinh(u), \tag{2}$$

where u is the unknown function of x, t and $\lambda \in \mathbb{R} \setminus \{0\}$.

Using the following wave transformation

$$u = U(\zeta), \quad \zeta = \mu(x + ct) \tag{3}$$

on Eq. (2), we get the following nonlinear ordinary differential equation (NODE);

$$U^{''} = \frac{\lambda}{c\mu^2} sinh(U), \tag{4}$$

where $U = U(\zeta)$, ζ is the width, μ is the height and c the velocity of the travelling wave. Integrating Eq. (4), we get the following;

$$\Big[\Big(\frac{U}{2}\Big)^{'}\Big]^2 = \frac{2\lambda}{c\mu^2} sinh^2\Big(\frac{U}{2}\Big) + \frac{\lambda q}{c\mu^2}, \tag{5}$$

where q is the integration constant, resulting from integrating Eq. (4).

Setting $\varphi = \frac{U}{2}$ and $p = \frac{2\lambda}{c\mu^2} = \frac{\lambda q}{c\mu^2}$, we get the following;

$$\varphi^{'} = \sqrt{p}\, cosh(\varphi). \tag{6}$$

Eq. (6) is a variables separable equation, simplifying it, produces the following two significant equations;

$$cosh(\varphi) = tan\Big(\sqrt{p}(\zeta + d)\Big), \tag{7}$$

$$sinh(\varphi) = sec\Big(\sqrt{p}(\zeta + d)\Big), \tag{8}$$

where d is the integration constant, resulting from integrating Eq. (6).

To find the new solutions to Eq. (1), we consider the following two equations;

$$U(\varphi) = \sum_{i=1}^{m} cosh^{i-1}(\varphi)[B_i sinh(\varphi) + A_i cosh(\varphi)] + A_0, \tag{9}$$

$$U(\zeta) = \sum_{i=1}^{m} tan^{i-1}\Big(\sqrt{p}(\zeta + d)\Big)\Big[B_i sec\Big(\sqrt{p}(\zeta + d)\Big) + A_i tan\Big(\sqrt{p}(\zeta + d)\Big)\Big] + A_0. \tag{10}$$

The value of m in Eq. (9) and (10) is determined by using the balancing technique thereby considering the highest power nonlinear term and the highest derivative in the reduced NODE. Setting each summation of the coefficients of $sinh^i(\varphi)cosh^j(\varphi)$, $(0 \leq i \leq n,\ 0 \leq j \leq n)$ with the same power to zero, yields a group of algebraic equations. Simplifying this group of algebraic equations with the aid of the Wolfram Mathematical package, gives the values of the coefficients $A_i,\ B_i,\ \mu,\ c$ and p. Substituting the values of $A_i,\ B_i,\ \mu,\ c$ and p into Eq. (10) along with the value of m, yields the new travelling wave solutions to Eq. (1).

3 The FDM

To use the finite forward difference method their is needs to give the following notations;

• Δx, which is the spatial step.
• Δt, which is the time step.
• $x_i = a + i\Delta x,\ i = 0, 1, 2, ..., N$ points are the coordinates of mesh and $N = \frac{b-a}{\Delta x}$, $t_j = j\Delta x,\ j = 0, 1, 2, ..., M$ and $M = \frac{T}{\Delta t}$.
• The functions $u\,(x, t)$ stands for the value of this solution at the grid point which is $u\,(x_i, t_j) \approx u_{i,j}$.
• $u_{i,j}$ stands for the numerical solution of the exact value of $u\,(x, t)$ at the point (x_i, t_j).

We have the following finite difference operators that are associated to the considered equation:

$$H_t u_{i,j} = u_{i,j+1} - u_{i,j}, \tag{11}$$

$$H_x u_{i,j} = u_{i+1,j} - u_{i,j}, \tag{12}$$

$$H_{xxx} u_{i,j} = u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}. \tag{13}$$

Thus, the approximate form of the partial derivatives through the finite difference operators is given by:

$$\left.\frac{\partial u}{\partial t}\right|_{i,j} = \frac{H_t u_{i,j}}{\Delta t} + O((\Delta t)), \tag{14}$$

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{H_x u_{i,j}}{\Delta x} + O((\Delta x)), \tag{15}$$

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_{i,j} = \frac{H_{xxx} u_{i,j}}{(\Delta x)^3} + O((\Delta x)^2), \tag{16}$$

Therefore, we can write Eq. (24) in the finite difference operator as:

$$\frac{H_t u_{i,j}}{\Delta t} + \frac{H_x u_{i,j}}{\Delta x} - \alpha u_{i,j}^2 \frac{H_x u_{i,j}}{\Delta x} + \frac{H_{xxx} u_{i,j}}{(\Delta x)^3} = 0. \tag{17}$$

Utilizing Eqs. (11), (12) and (13), transforms Eq. (17) to the following indexed form:

$$u_{i+1,j} = \frac{1}{2(\Delta t)\Big(1 - (\Delta t)^2 + \alpha(\Delta x)^2 u_{i,j}^2\Big)}\Big(- (\Delta t)u_{i-2,j} + 2(\Delta t)u_{i-1,j}$$
$$-2(\Delta x)^3 u_{i,j} - 2(\Delta x)^2(\Delta t)u_{i,j} + 2\alpha(\Delta x)^2(\Delta t)u_{i,j}^3 + 2(\Delta x)^3 u_{i,j+1} + (\Delta t)u_{i+2,j}\Big), \tag{18}$$

where the initial value $u_{i,0}$ is $u_0(x_i)$.

4 Von-Neumann Stability Analysis

Here, we utilize the Von-Neumann stability analysis technique in investigating the stability of Eq. (24). When the Fourier method of analyzing stability is used, and ξ^n is considered as the amplication factor, the growth factor of a typical Fourier mode is given as:

$$u_m^n = \xi^n e^{im\Phi}, \tag{19}$$

where $i = \sqrt{-1}$, ξ^n is taken as the amplication factor.

To examine the stability of the numerical scheme, the nonlinear term $u^2 u_x$ in the (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation has been linearized by making the quantity $\widehat{u} = u^2$ a local constant. Thus the nonlinear term in the equation changes into $\widehat{u}u_x$ in this case Eq. (24) becomes

$$u_t + u_x - \alpha\widehat{u}u_x + u_{xxx} = 0. \tag{20}$$

Inserting Eq. (14), (15), (16) and the Fourier mode (19) into the recurrence relation (20), yields

$$\xi = X + iY, \tag{21}$$

where

$$X = \frac{1}{(\Delta x)}\Big((\Delta t) - \alpha(\Delta t)\widehat{u} + (\Delta x) - (\Delta t)cos(\Phi)(1 - \alpha\widehat{u})\Big) \tag{22}$$

and

$$Y = \frac{(\Delta t)}{(\Delta x)^3}\Big(2 - (\Delta x)^2 + \alpha(\Delta x)^2\widehat{u} - sin(2\Phi)\Big). \tag{23}$$

According to the Fourier stability, for the given scheme to be stable, the condition $|\xi| \leq 1$ must be satisfied and $|\xi|^2 = X^2 + Y^2$.

5 L_2 and L_∞ Error Norms

For the test problem used in this study, numerical solutions of the (1+1)-dimensional NDMBBM equtaion have been investigated and all the computations are carried out in the Wolfram Mathematica 11. To show that how the analytical results and numerical results are close to each other we use L_2 and L_∞ error norms. The error L_2 norms defined as [20]

$$L_2 = \left\|u^{exact} - u^{numeric}\right\|_2 = \sqrt{h\sum_{j=0}^{N}\left|u_j^{exact} - u_j^{numeric}\right|^2},$$

and L_∞ error norm defined as

$$L_\infty = ||u^{exact} - u^{numeric}||_\infty = \mathop{Max}_{j}|u_j^{exact} - u_j^{numeric}|.$$

6 Theoretical Calculations

In this section, we present the applications of the ShGFM to the (1+1)-dimensional NDMBBM equation [18] given by:

$$u_t + u_x - \alpha u^2 u_x + u_{xxx} = 0, \tag{24}$$

carrying the following wave transformation on Eq. (24)

$$u(x,t) = U(\zeta), \quad \zeta = \mu(x - ct), \tag{25}$$

yields the following NODE:

$$3(1-c)U - \alpha U^3 + 3\mu^2 U^{''} + K = 0, \tag{26}$$

where K is an integration constant.

Balancing U^3 and $U^{''}$, yields $m = 1$. With Eq. (35) and $m = 1$, we obtain the following:

$$U(\varphi) = B_1 sinh(\varphi) + A_1 cosh(\varphi) + A_0, \tag{27}$$

differentiating Eq. (27) twice, we obtain the following:

$$U^{''}(\varphi) = 2pB_1 cosh^2(\varphi)sinh(\varphi) + pA_1 sinh^2(\varphi)cosh(\varphi) + pA_1 cosh^3(\varphi). \tag{28}$$

Inserting Eq. (27) and (28) into Eq. (26), gives an equation in hyperbolic functions form. From this equation, we collect a set of algebraic equations by equating each sum of the coefficients of the hyperbolic functions with the same power to zero. We solve the set of algebraic equations and obtain the values of the coefficients involved. We obtain the following two cases of the values for the coefficients;

Case 1.

$$A_0 = 0, A_1 = -\frac{\sqrt{\frac{3}{2}p}}{\sqrt{\alpha}}\mu, B_1 = \frac{\sqrt{\frac{3}{2}p}}{\sqrt{\alpha}}\mu, K = 0, c = 1 + \frac{p\mu^2}{2},$$

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Case 2.

$$A_0 = 0, A_1 = -\frac{\sqrt{3(c-1)}}{\sqrt{\alpha}}, B_1 = -\frac{\sqrt{3(c-1)}}{\sqrt{\alpha}}, K = 0, \mu = \frac{\sqrt{2(c-1)}}{\sqrt{p}}.$$

With the values of the coefficients in case 1, we obtain the following solution

$$u_1(x,t) = \frac{\sqrt{\frac{3}{2}p}}{\sqrt{\alpha}}\mu\Bigg(sec\Big[\sqrt{p}\Big(d+\mu\Big(x-\Big(1+\frac{p\mu^2}{2}\Big)t\Big)\Big)\Big] \\ -tan\Big[\sqrt{p}\Big(d+\mu\Big(x-\Big(1+\frac{p\mu^2}{2}\Big)t\Big)\Big)\Big]\Bigg). \tag{29}$$

With the values of the coefficients in case 2, we obtain the following solution

$$u_2(x,t) = -\frac{\sqrt{3(c-1)}}{\sqrt{\alpha}}\Bigg(sec\Big[\sqrt{p}\Big(d+\frac{\sqrt{2(c-1)}}{\sqrt{p}}(x-ct)\Big)\Big] \\ +tan\Big[\sqrt{p}\Big(d+\frac{\sqrt{2(c-1)}}{\sqrt{p}}(x-ct)\Big)\Big]\Bigg). \tag{30}$$

Remark 1. We observed that the values of the integration constant "K" in both the two cases is zero, but we still attain to the solutions of the (1+1)-dimensional NDMBBM which we claim to be new solutions.

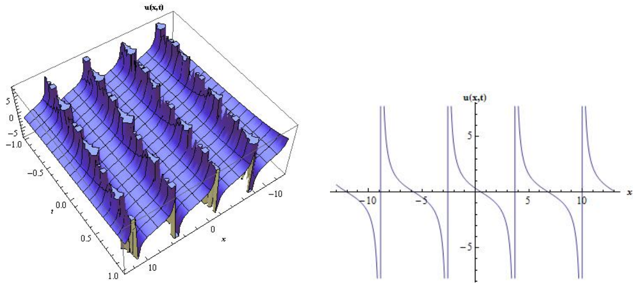


Figure 1. The singular periodic waves surface of Eq. (29) under the values $p = d = \alpha = \mu = 1$, $-13 < x < 13$, $-1 < t < 1$ and $t = 0$ for the 2D graphic.

7 Numerical Example

In this section, we investigate the numerical solutions of Eq. (24) by considering Eq. (29). Substituting these datum $p = 1, \mu = 1, d = 1, \alpha = 1$ $0 < x < 1$ and $0 < t < 1$ into Eq. (29), we obtain the following special exact solution:

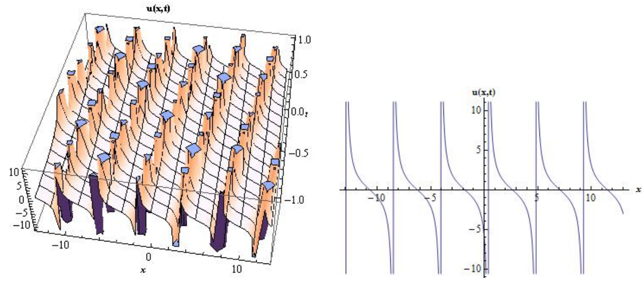


Figure 2. The singular periodic waves surface of Eq. (30) under the values $p = d = \alpha = 1$, $c = 2$, $-13 < x < 13$, $-1 < t < 1$ and $t = 0$ for the 2D graphic.

$$u(x,t) = \sqrt{\frac{3}{2}}\Big(sec\Big[1 - \frac{3}{2}t + x\Big] - tan\Big[1 - \frac{3}{2}t + x\Big]\Big). \tag{31}$$

At $t = 0$, Eq. (31) becomes:

$$u_0(x) = \sqrt{\frac{3}{2}}\Big(sec\Big[1 + x\Big] - tan\Big[1 + x\Big]\Big). \tag{32}$$

Inserting $(\Delta x) = (\Delta t) = 0.02$ into Eq. (18), yields the following indexed form:

$$u_{i+1,j} = \frac{1}{-124950. - 50.u_{i,j}^2}\Big(62500. - u_{i-2,j} - 125000.u_{i-1,j} + 50.u_{i,j} \\ -50.u_{i,j}^3 - 50.(-u_{i,j} + u_{i,j+1}) - 62500.u_{i+2,j}\Big). \tag{33}$$

Table 1.

(Δx)	(Δt)	Numerical solution	Exact Solution	Eror
0	0.01	0.379271	0.379392	1.20954×10^{-4}
0.01	0.01	0.365894	0.366011	1.16185×10^{-4}
0.02	0.01	0.352597	0.352709	1.11472×10^{-4}
0.03	0.01	0.339376	0.339483	1.06814×10^{-4}
0.04	0.01	0.326229	0.326331	1.02208×10^{-4}
0.05	0.01	0.313151	0.313248	9.76527×10^{-5}
0.06	0.01	0.300139	0.300232	9.31453×10^{-5}

Table 1: Numerical, exact solutions and absolute errors of Eq. (24) by considering Eq. (29), under the value $\Delta(x) = 0.02$ and $0 \leq x \leq 1$.

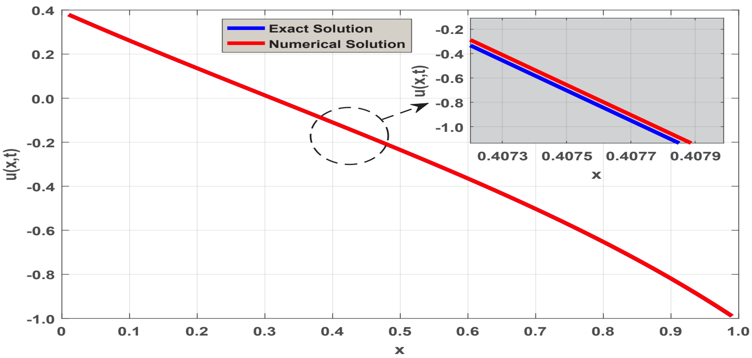
L_2 and L_∞ error norm table

Table 2.

$x_i = t_j$	L_2	L_∞
0.2	6.22622×10^{-3}	1.14559×10^{-2}
0.02	2.49150×10^{-4}	3.95029×10^{-4}
0.05	3.74149×10^{-4}	7.48387×10^{-4}
0.01	1.52415×10^{-5}	3.03436×10^{-5}
0.001	1.63392×10^{-7}	1.18747×10^{-6}

Table 2: L_2 and L_∞ error norm when $0 \leq h \leq 1$ and $0 \leq x \leq 1$

Figure 3. Numerical and exact solution of Eq. (24) by considering Eq. (29).



8 Results and Discussions

Various analytical techniques have been invested to solve the (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation and solutions of different structures were obtained. Khan *et al.* [18] used the modified simple equation method in solving this equation and some hyperbolic and trigonometric solutions were obtained. Baskonus *et al.* [19] used the generalized Kudryashov method in investigating some new solutions to this equation and some hyperbolic, trigonometric and exponential function solutions were obtained. In this study, we used the ShGFM and construct some new analytical solutions with trigonometric structure. We present below the trigonometric function solutions obtained in the three above mentioned references.

Results obtained by Khan *et al.* [18]:

$$u_1(x,t) = \pm\sqrt{\frac{3(c-1)}{\alpha}}tan[\varpi_1(x,t)], \tag{34}$$

$$u_2(x,t) = \pm\sqrt{\frac{3(c-1)}{\alpha}}cot[\varpi_1(x,t)], \tag{35}$$

where $\varpi_1(x,t) = \sqrt{\frac{(c-1)}{2}}(x-ct)$.

Results obtained by Baskonus *et al.* [19]:

$$u_1(x,t) = \pm i\sqrt{\frac{6}{\alpha}}cosec[i(-x+2t)], \tag{36}$$

$$u_2(x,t) = \pm 3\sqrt{\frac{3}{2\alpha}}\Big(-i\ tan\Big[\frac{3}{4}i(2x+7t)\Big]\Big), \tag{37}$$

$$u_3(x,t) = \pm\sqrt{\frac{6}{\alpha}}(i\ cot[i(x+t)]). \tag{38}$$

We observed that the results obtained by using ShGFM are newly constructed solutions when compared the available results in the literature. Although, in Eq. (30) when $sec\Big[\sqrt{p}\Big(d+\frac{\sqrt{2(c-1)}}{\sqrt{p}}(x-ct)\Big)\Big] \to 0$ and we choose $d=0$, we get the following:

$$u(x,t) = -\sqrt{\frac{3(c-1)}{\alpha}}tan\Big[\sqrt{2(c-1)}(x-ct)\Big] \tag{39}$$

which is closely the same with the result obtained in Eq. (34) by Khan *et al.* [18].

In Table 1, we present the values of the numerical and exact solutions obtained by using finite forward difference method.

In table 2, we present the L_2 and L_∞ error norms for the numerical solutions of Eq. (24).

9 Conclusions

In this study, we proposed a new analytical technique called a new sinh-Gordon function method. With the aid of the Wolfram Mathematica package, we utilized this new technique in investigating some new solutions to the (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation. We obtained some new analytical solutions with trigonometric function structure. We observed that our results are newly constructed when compared with the existing results in the literature. We performed the numerical simulations of the obtained solutions by choosing suitable parameters. We performed the numerical simulation of all the obtained analytical solutions in this study. On the other hand we compute the exact and numerical approximations of the studied equation using the finite forward difference method. We checked the stability of the numerical scheme by using the Fourier-Von Neumann analysis. In table 1, we present the results for both the numerical and exact solutions obtained by considering Eq. (29). We give the L_2 and L_∞ error norms analysis of this equation in table 2. The comparison between the numerical and analytical solutions can be observed from table 1, which is also supported by fig. 3. To the best of our knowledge, the application of this new method which is giving new results has not been applied and submitted to the literature.

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