A New Method for (4+1) Dimensional Fokas Equation

Seyma Tuluce Demiray[1,*] and *Hasan Bulut*[2]

[1] Department of Mathematics, Osmaniye Korkut Ata University, Osmaniye, Turkey

[2] Department of Mathematics, Fırat University, Elazığ, Turkey

Abstract. In this paper, modified $\exp\left(-\Omega\left(\xi\right)\right)$-expansion function method (MEFM) has been tackled for procuring exact solutions of (4+1) dimensional Fokas equation. Hyperbolic function solutions and dark soliton solutions of (4+1) dimensional Fokas equation have been found by means of this method. Moreover, by the help of Mathematica 9, some graphical simulations were given to clarify the behavior of these solutions.

1 Introduction

Nonlinear evolution equations (NLEEs) are considerably used to identify a variety of physical circumstances in the areas such as quantum field theory, hydrodynamics, chemical kinematics, geochemistry, electricity, elastic media and plasma physics.

Recently, most of researchers have submitted to acquire exact solutions of NLEEs many methods such as G'/G-expansion method [1], modified extended tanh-function method [2], sine-cosine method [3], exp-function method [4], modified simple equation method [4], extended trial equation method [5], generalized Kudryashov method [6] . In this study, MEFM [7] will be implemented to seek exact solutions of (4+1) dimensional Fokas equation.

We consider (4+1) dimensional Fokas equation [8-14],

$$4u_{tx}-u_{xxxy}+u_{xyyy}+12u_{x}u_{y}+12uu_{xy}-6u_{zw}=0. \tag{1}$$

This equation has been obtained by Fokas by expanding the Lax pairs of the integrable Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equations to some higher-dimensional nonlinear wave equations [8]. The importance of Eq. (1) suggests that the idea of complexifying time can be considered in the context of modern field theories via the existence of integrable nonlinear equations in four spatial dimensions involving complex time [8].

* Corresponding author: seymatuluce@gmail.com

Our target in this work is to gain exact solutions of (4+1) dimensional Fokas equation. In Sec. 2, we clarify general structure of MEFM. In Sec. 3, we get exact solutions of this equation by means of MEFM.

2 General Structure of Method

Step 1. We handle PDE as follows:

$$P\left(u,u_x,u_y,u_z,u_w,u_t,u_{xx},u_{yy},u_{zz},u_{ww},u_{tt},\cdots\right)=0, \tag{2}$$

where $u=u\left(x,y,z,w,t\right)$ is an unknown function, P is a polynomial in $u\left(x,y,z,w,t\right)$ and its derivatives, in which the highest order derivatives and nonlinear terms are included and the subscripts demonstrate the partial derivatives. Then, we get traveling transformation

$$u\left(x,y,z,w,t\right)=U\left(\xi\right),\quad \xi=\alpha x+\beta y+\gamma z+\delta w+\varepsilon t. \tag{3}$$

Using Eq. (3), we can turn Eq. (2) into a nonlinear ordinary differential equation (NODE) described by:

$$NODE\left(U,U',U'',U''',\cdots\right)=0, \tag{4}$$

where $NODE$ is a polynomial of U and its derivatives and the superscripts demonstrate the ordinary derivatives according to ξ.

Step 2. Assume the traveling wave solution of Eq. (4) can be shown as follows:

$$U\left(\xi\right)=\frac{\sum_{i=0}^{N}A_i\left[\exp\left(-\Omega\left(\xi\right)\right)\right]^i}{\sum_{j=0}^{M}B_j\left[\exp\left(-\Omega\left(\xi\right)\right)\right]^j}=\frac{A_0+A_1\exp\left(-\Omega\right)+\cdots+A_N\exp\left(\mathrm{N}\left(-\Omega\right)\right)}{B_0+B_1\exp\left(-\Omega\right)+\cdots+B_M\exp\left(\mathrm{M}\left(-\Omega\right)\right)}, \tag{5}$$

where $A_i,B_j,\left(0\le i\le N,0\le j\le M\right)$ are constants to be described later, such that $A_N\neq 0,B_M\neq 0,$ and $\Omega=\Omega\left(\xi\right)$ is solution of the following ordinary differential equation:

$$\Omega'\left(\xi\right)=\exp\left(-\Omega\left(\xi\right)\right)+\mu\exp\left(\Omega\left(\xi\right)\right)+\lambda. \tag{6}$$

There are the following solution families of Eq. (6):

Family1: When $\mu\neq 0,\ \lambda^2-4\mu>0,$

$$\Omega\left(\xi\right)=\ln\left(\frac{-\sqrt{\lambda^2-4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\xi+E\right)\right)-\frac{\lambda}{2\mu}\right). \tag{7}$$

Family2: When $\mu\neq 0,\ \lambda^2-4\mu<0,$

$$\Omega\left(\xi\right)=\ln\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}\left(\xi+E\right)\right)-\frac{\lambda}{2\mu}\right). \tag{8}$$

Family3: When $\mu=0,\ \lambda\neq 0,$ and $\lambda^2-4\mu>0,$

$$\Omega\left(\xi\right)=-\ln\left(\frac{\lambda}{\exp\left(\lambda\left(\xi+E\right)\right)-1}\right). \tag{9}$$

Family4: When $\mu\neq 0,\ \lambda\neq 0,$ and $\lambda^2-4\mu=0,$

$$\Omega\left(\xi\right)=\ln\left(-\frac{2\lambda\left(\xi+E\right)+4}{\lambda^2\left(\xi+E\right)}\right). \tag{10}$$

Family5: When $\mu=0,\ \lambda=0,$ and $\lambda^2-4\mu=0,$

$$\Omega(\xi) = \ln(\xi + E). \tag{11}$$

such that $A_0, A_1, A_2, \cdots A_N, B_0, B_1, B_2, \cdots B_M, E, \lambda, \mu$ are constants to be described later. The positive integers N and M can be identified by taking into consideration the homogeneous balance between the highest order derivatives and the nonlinear terms arising in Eq. (5).

Step 3: Embedding Eqs. (6) and (7–11) into Eq. (5), we attain a polynomial of $\exp(-\Omega(\xi))$. We compensate all the coefficients of same power of $\exp(-\Omega(\xi))$ to zero. This process provides a system of equations which can be solved to obtain $A_0, A_1, A_2, \cdots A_N, B_0, B_1, B_2, \cdots B_M, E, \lambda, \mu$ by using Wolfram Mathematica 9. Putting the values of $A_0, A_1, A_2, \cdots A_N, B_0, B_1, B_2, \cdots B_M, E, \lambda, \mu$ into Eq. (5), the general solutions of Eq. (5) fulfil the determination of the solution of Eq. (1).

3 MEFM for (4+1) Dimensional Fokas Equation

In order to solve Eq. (1), we use the travelling wave transform

$$u(x, y, z, w, t) = U(\xi), \quad \xi = \alpha x + \beta y + \gamma z + \delta w + \varepsilon t. \tag{12}$$

where $\alpha, \beta, \gamma, \delta$ and ε are arbitrary constants. Then, we reduce Eq. (1) to following equation

$$\left(\alpha\beta^3 - \alpha^3\beta\right)u'' + \left(4\alpha\varepsilon - 6\gamma\delta\right)u + 6\alpha\beta u^2 = 0. \tag{13}$$

Using balance principle in Eq. (13), we obtain

$$N = M + 2 \,. \tag{14}$$

If we take $M = 1$ so $N = 3$, we can acquire

$$U = \frac{A_0 + A_1\exp(-\Omega) + A_2\exp(2(-\Omega)) + A_3\exp(3(-\Omega))}{B_0 + B_1\exp(-\Omega)}, \tag{15}$$

$$\begin{aligned} U' = {} & \frac{\left[A_1\exp(-\Omega)(-\Omega') + A_2\exp(2(-\Omega))(-2\Omega') + A_3\exp(3(-\Omega))(-3\Omega')\right]\left[B_0 + B_1\exp(-\Omega)\right]}{\left[B_0 + B_1\exp(-\Omega)\right]^2} \\ & - \frac{\left[A_0 + A_1\exp(-\Omega) + A_2\exp(2(-\Omega)) + A_3\exp(3(-\Omega))\right]\left[B_1\exp(-\Omega)(-\Omega')\right]}{\left[B_0 + B_1\exp(-\Omega)\right]^2} = \frac{\Upsilon}{\Psi}, \\ U'' = {} & \frac{\Upsilon'\Psi - \Upsilon\Psi'}{\Psi^2}, \\ & \vdots \end{aligned} \tag{16}$$

where $A_3 \neq 0$ and $B_1 \neq 0$. Substituting Eqs. (15) and (16) in Eq. (13), we attain a system of algebraic equations from the coefficients of polynomial of $\exp(-\Omega(\xi))$. By solving this system of algebraic equations by using Wolfram Mathematica 9, it yields us the following coefficients:

Case 1.

$$\begin{aligned} & A_0 = \frac{\mu A_3 B_0}{B_1}, A_1 = A_3\left(\mu + \frac{\lambda B_0}{B_1}\right), A_2 = A_3\left(\lambda + \frac{B_0}{B_1}\right), A_3 = A_3, B_0 = B_0, B_1 = B_1, \\ & \alpha = -\frac{\sqrt{A_3 + \beta^2 B_1}}{\sqrt{B_1}}, \varepsilon = \frac{\beta\left(\lambda^2 - 4\mu\right)A_3 - \dfrac{6\gamma\delta B_1^{3/2}}{\sqrt{A_3 + \beta^2 B_1}}}{4B_1}, \lambda = \lambda, \mu = \mu. \end{aligned} \tag{17}$$

Embedding Eq. (17) together with Eqs. (3) and (7) in Eq. (15), we find new hyperbolic function solutions for Eq. (1) as follows:

$$u_1\left(x,y,z,w,t\right)=\frac{\mu\left(-\lambda^2+4\mu\right)A_3}{B_1\left(\lambda\cosh\left[f\left(x,y,z,w,t\right)\right]+\sqrt{\lambda^2-4\mu}\,\sinh\left[f\left(x,y,z,w,t\right)\right]\right)^2},\quad(18)$$

where

$$f\left(x,y,z,w,t\right)=\frac{1}{2}\sqrt{\lambda^2-4\mu}\left[E+y\beta+z\gamma+w\delta+\frac{t\beta\left(\lambda^2-4\mu\right)A_3}{4B_1}-\frac{2xA_3+B_1\left(2x\beta^2+3t\gamma\delta\right)}{2\sqrt{B_1}\left(\sqrt{A_3+\beta^2B_1}\right)}\right],$$

and $\mu\neq 0,\lambda^2-4\mu>0.$

Case 2.

Substituting Eq. (19) together with Eqs. (3) and (7) in Eq. (15), we obtain new dark soliton solutions for Eq. (1) as follows:

$$u_2\left(x,y,z,w,t\right)=\frac{\left(\lambda^2-4\mu\right)A_3\left(\lambda^2-6\mu+2\lambda\sqrt{\lambda^2-4\mu}\tanh\left[g\left(x,y,z,w,t\right)\right]+\left(\lambda^2+2\mu\right)\tanh^2\left[g\left(x,y,z,w,t\right)\right]\right)}{6B_1\left(\lambda+\sqrt{\lambda^2-4\mu}\,\tanh\left[g\left(x,y,z,w,t\right)\right]\right)^2},\quad(20)$$

where

$$g\left(x,y,z,w,t\right)=\frac{1}{2}\sqrt{\lambda^2-4\mu}\left[E+y\beta+z\gamma+w\delta-\frac{t\beta\left(\lambda^2-4\mu\right)A_3}{4B_1}-\frac{2xA_3+B_1\left(2x\beta^2+3t\gamma\delta\right)}{2\sqrt{B_1}\left(\sqrt{A_3+\beta^2B_1}\right)}\right],$$

and $\mu\neq 0,\lambda^2-4\mu>0.$

Case 3.

$$A_0=\left(\alpha^2-\beta^2\right)\mu B_0,\ A_1=\frac{A_2B_0-\left(\alpha^2-\beta^2\right)\left(B_0^2-\mu B_1^2\right)}{B_1},\ A_3=\left(\alpha^2-\beta^2\right)B_1,$$

$$\lambda=\frac{A_2-\left(\alpha^2-\beta^2\right)B_0}{\left(\alpha^2-\beta^2\right)B_1},\ \varepsilon=\frac{1}{4}\left(\frac{6\gamma\delta}{\alpha}-4\alpha^2\beta\mu+4\beta^3\mu+\frac{\beta\left(A_2-B_0\left(\alpha^2-\beta^2\right)\right)^2}{\left(\alpha^2-\beta^2\right)B_1^2}\right),\quad(21)$$

$$A_2=A_2,\ B_0=B_0,\ B_1=B_1,\ \lambda=\lambda,\ \mu=\mu.$$

Putting Eq. (21) together with Eqs. (3) and (7) in Eq. (15), we find new hyperbolic function solutions for Eq. (1) as follows:

$$u_3\left(x,y,z,w,t\right)=-\frac{\left(\alpha^2-\beta^2\right)\mu\left(\left(A_2-\left(\alpha^2-\beta^2\right)B_0\right)^2-4\left(\alpha^2-\beta^2\right)^2\mu B_1^2\right)\operatorname{sec}h^2\left[h\left(x,y,z,w,t\right)\right]}{\left[A_2-\left(\alpha^2-\beta^2\right)\left(B_0-B_1\sqrt{-4\mu+\frac{\left(A_2-\left(\alpha^2-\beta^2\right)B_0\right)^2}{\left(\alpha^2-\beta^2\right)^2B_1^2}}\right)\tanh\left[h\left(x,y,z,w,t\right)\right]\right]^2},\quad(22)$$

where

$$h\left(x,y,z,w,t\right)=\frac{1}{2}\sqrt{-4\mu+\frac{\left(A_2-\left(\alpha^2-\beta^2\right)B_0\right)^2}{\left(\alpha^2-\beta^2\right)^2B_1^2}}\left[E+x\alpha+y\beta+z\gamma+w\delta+\frac{1}{4}t\left(\frac{6\gamma\delta}{\alpha}-4\alpha^2\beta\mu+4\beta^3\mu+\frac{\beta\left(A_2-B_0\left(\alpha^2-\beta^2\right)\right)^2}{\left(\alpha^2-\beta^2\right)B_1^2}\right)\right],$$

and $\mu\neq 0,\lambda^2-4\mu>0.$

Remark. The exact solutions of Eq. (1) were obtained via MEFM and were controlled by use of Mathematica Release 9. As far as we know, the solutions of Eq. (1) that we found in this study, are new and are not indicated before.

Fig.1. The 3D and 2D surfaces of Eq. (18) for
$\lambda = 0.3,\ \mu = -0.5, \beta = 1, \gamma = 2, \delta = 3, A_3 = 2, B_1 = 1$
$E = 0.5, y = 0.3, z = 0.2, w = 0.1,\ -15 < x < 15,\ -15 < t < 15$ and $-25 < x < 25, t = 0.01$ for 2D surface.

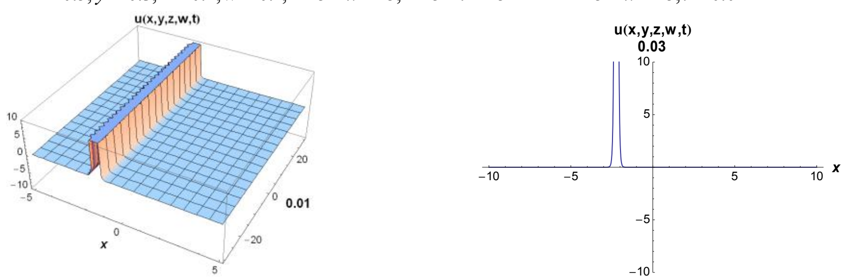

Fig.2. The 3D and 2D surfaces of Eq. (22) for
$\mu = -0.4, \alpha = 2, \beta = 1, \gamma = 3, \delta = 2, A_2 = 0.4, B_0 = 0.03, B_1 = 0.01$
$E = 0.25, y = 0.3, z = 0.2, w = 0.1,\ -5 < x < 5,\ -30 < t < 30$ and $-10 < x < 10, t = 0.03$ for 2D surface.

4 Conclusion

In this paper, we apply MEFM to obtain exact solutions of (4+1) dimensional Fokas equation. Then, for proper parameters, we draw 2D and 3D surfaces of some exact solutions of this equation by using Mathematica Release 9. This method yields us to make complicated and tedious algebraic calculations. According to these informations, this method has been influential for the analytical solutions of (4+1) dimensional Fokas equation. Also, MEFM is a strong mathematical device in the way of finding new exact solutions. The graphical demonstrations clearly show validness of proposed method.

References

1. M. Younis, S. T. R. Rizvi, Optik, **126**, 5812-5814 (2015).
2. E. H. M. Zahran, M. M. A. Khater, Appl. Math. Model., **40**, 1769-1775 (2016).
3. S. Bibi, S. T. Mohyud-Din, J. Assoc. Arab Univ. Basic Appl., **15**, 90-93 (2014).
4. K. Khan, M. A. Akbar, Ain Shams Engineering Journal, **5**, 247-256 (2014).
5. S. Tuluce Demiray, H. Bulut, Wave Random Complex, **25(1)**, 75-90 (2015)
6. S. Tuluce Demiray, H. Bulut,. Opt. Quant. Electron., **49(117)**, 1-8 (2017).
7. H. M. Baskonus, H Bulut, A. Atangana, Smart Mater. Struct., **25(3)**, 1-8 (2016).
8. A. S. Fokas, Phys. Rev. Lett., **96(19),** Article ID:190201 (2006).
9. Z. Z. Yang, Z. Y. Yan, Commun. Theor. Phys. (Beijing,) **51(5),** 876-880 (2009).
10. J. Lee, R. Sakthivel, L. Wazzan, Modern Phys. Lett. B, **24(10),** 1011-1021 (2010).
11. H. Kim, R. Sakthivel, Rep. Math. Phys., **70(1),** 39-50 (2012).
12. Y. He, Math. Probl. Eng., **2014,** Article ID: 972519, 11 pages (2014).
13. S. Zhang, C. Tian, W. Y. Qian, Pramana J. Phys., **86(6),** 1259-1267 (2016).
14. M. O. Al-Amr, S. El-Ganaini, Comput. Math. Appl., **74**, 1274-1287 (2017).