Approximate solutions for the Bagley-Torvik fractional equation with boundary conditions using the Polynomial Least Squares Method

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Abstract. In this paper we apply the recently introduced Polynomial Least Squares Method (PLSM) to compute approximate analytical polynomial solutions for the Bagley-Torvik fractional equation with boundary conditions.
The Bagley–Torvik equation may be used to model the motion of real physical systems such as the motion of a thin rigid plate immersed in a Newtonian fluid. In order to emphasize the accuracy of PLSM, we included a comparison with previous approximate solutions obtained for the Bagley-Torvik fractional equation by means of other approximation method.

1 Introduction

Solving the equations modeling various problems which arise in the fields of science and engineering have always been a priority for the researchers. In recent years there has been a growing interest in the computation of numerical solutions of the equations (with the help of various software programs) by various methods. One of the goals of such a numerical computation is the minimization of the approximation errors leading to numerical solutions as close as possible to the exact ones. Many of the physical phenomena from various area such as: fluid flow, viscoelasticity, electromagnetic waves, acoustics, mathematical biology etc, are modeled by fractional differential equations. In general it is difficult to solve fractional differential equations and usually the computation of an exact solution impossible([12],[13]). This is why different methods are being studied to determine numerical solutions for the fractional differential equation ([5],[7], [9], [10], [11], [14]). One of the recently method used to compute approximate analytical polynomial solutions for fractional differential equation is the Polynomial Least Squares Method (PLSM). The PLSM has been used by C Bota and B Caruntu in 2015 to compute an approximate analytical solution of the fractional order brusselator system ([6]). Subsequently the same researchers used the PLSM to find approximate solutions for the quadratic Riccati differential equation of fractional order ([8]),and other types of equations ([1]).
In this paper the PLSM is employed to find approximate analytical polynomial solutions for

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the Bagley-Torvik fractional equation with boundary conditions.
We consider the following Bagley-Torvik equation with boundary conditions:

$$y''(x) + p(x)y'(x) + q(x)D_x^{\alpha}y(x) + r(x)y(x) = f(x), \tag{1}$$

with $n-1 < \alpha < n$, $n \in \mathbb{N}$, and

$$y(0) = \mu_1, y(1) = \mu_2, \tag{2}$$

where μ_1 and μ_2 are constants , the functions $p, q, r \in C_{[a,b]}$ and f are given such that the problem ((1)-(2)) has a single continuous solution.

$D_x^{\alpha}y(x)$ denotes Caputo's fractional derivative:

$$D_x^{\alpha}y(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \int_0^x (x-\zeta)^{-(\alpha-n+1)} \cdot y^{(n)}(\zeta)d\zeta, \quad n-1 < \alpha \leq n. \tag{3}$$

with Γ the Euler Gamma function, is the unique function that simultaneously satisfies

$$\Gamma(z) = \int_o^{\infty} e^{-x}x^{z-1}dx,$$

$\Gamma(1) = 1$ and $\Gamma(z+1) = z\Gamma(z)$ for all complex numbers z except the non-positive integers, and $\Gamma(n=z) = (n-1)!$ for all z positive integers numbers $z = n, n \in \mathbb{N}$.
In the next section we will introduce PLSM for the Bagley Torvik fractional equation with boundary conditions and in the third section we will compare the approximate solutions obtained by using PLSM with previous approximate solutions obtained by means of other approximation methods.
The computations show that the approximate analytical polynomial solutions computed by using PLSM present an error smaller than the error of the corresponding solutions from other methods.

2 The Polynomial Least Squares Method

We denote by $\tilde{y}$ an approximate solution of equation (1). In this case the error obtained by replacing the exact solution y with the approximation $\tilde{y}$ is given by the remainder:

$$\mathcal{R}(\tilde{y}) = \tilde{y}''(x) + p(x)\tilde{y}'(x) + q(x)D^{\alpha}\tilde{y}(x) + r(x)\tilde{y}(x) - f(x) \tag{4}$$

$$x \in [a,b], \tilde{y}(a) = \nu_0, \tilde{y}(b) = \nu_1$$

where $p(x), q(x)$ and $r(x)$ are continious functions, and ν_0 and ν_1 are constantes and $q(x) \neq 0$. D^{α}, with $\alpha \notin \mathbb{Z}$ denotes the fractional differential operator of order α in Caputo's sense and given by:

$$D^{\alpha}\tilde{y}(x) = \frac{1}{\Gamma(k-\alpha)} \cdot \int_0^x (x-\zeta)^{(k-\alpha-1)} \cdot \tilde{y}^{(k)}(\zeta)d\zeta$$

$k-1 < \alpha \leq k$ where $k \in \mathbb{N}$.

For $\epsilon \in R_+$, we will compute approximate polynomial solutions $\tilde{y}$ of the boundary-value problem ((1)-(2)) on the interval [a, b].

Definition 1. *We call an* ϵ-approximate polynomial solution *of the problem ((1)-(2)) an approximate polynomial solution* $\tilde{y}$ *satisfying the relations:*

$$|\mathcal{R}(\tilde{y})| < \epsilon \tag{5}$$

$$\tilde{y}(a) = \mu_1, \tilde{y}(b) = \mu_2. \tag{6}$$

We call a *weak* ϵ*-approximate polynomial solution* of the boundary-value problem ((1)-(2)) an approximate polynomial solution $\tilde{y}$ satisfying the relation:

$$\int_a^b |\mathcal{R}(\tilde{y})| dx \leq \epsilon \tag{7}$$

together with the boundary conditions (6).

Definition 2. *Let* $P_m(x) = c_0 + c_1(x) + c_2(x) + \cdots + c_m x^m, c_i \in \mathbb{R}, i = 0, 1, \cdots, m$ *be a sequence of polynomials satisfying the conditions:* $P_m(a) = \mu_1, \quad P_m(b) = \mu_2.$
We call the sequence of polynomials $P_m(x)$ *convergent to the solution of the problem ((1)-(2)) if* $\lim_{m\to\infty} D(P_m(x)) = 0.$

We observe that from the hypothesis of the boundary-value problem ((1)-(2)) it follows that there exists a sequence of polynomials $P_m(x)$ which converges to the solution of the problem, according to the Weierstrass Theorem on Polynomial Approximation ([17], [18]).

We will compute a weak ϵ - approximate polynomial solution, in the sense of the Definition 1, of the type:

$$\tilde{y}(x) = \sum_{k=0}^{m} d_k x^k \tag{8}$$

where $d_0, d_1, \cdots, d_m$ are constants which are calculated using the following steps:
- By substituting the approximate solution (8) in the equation (1) we obtain the expression:

$$\mathcal{R}(\tilde{y}) = \tilde{y}''(x) + p(x)\tilde{y}'(x) + q(x)D^{\alpha}\tilde{y}(x) + r(x)\tilde{y}(x) - f(x). \tag{9}$$

If we could find $d_0, d_1, \cdots, d_m$ such $\mathcal{R}(\tilde{y}) = 0, \tilde{y}(a) = \mu_1, \tilde{y}(b) = \mu_2$, then by substituting $d_0, d_1, \cdots, d_m$ in (8) we obtain the solutions of the boundary-value problem ((1)-(2)).
- Then we attach to the problem ((1)-(2)) the following functional:

$$J(d_2, d_3 \cdots, d_m) = \int_a^b \mathcal{R}^2(\tilde{y}) dx \tag{10}$$

where d_0, d_1 are computed as functions of $d_2, d_3 \cdots d_m$ using the boundary conditions (6).
- We compute the values $d_2^0, d_3^0, \cdots d_m^0$ as the values which give the minimum of the functional (10), and the values of d_0^0 and d_1^0 as functions of $d_2^0, d_3^0, \cdots d_m^0$ using the boundary conditions.
- With constants $d_2^0, d_3^0, \cdots d_m^0$ previously determined we consider the polynomial:

$$M_m(x) = \sum_{k=0}^{m} d_k^0 x^k \tag{11}$$

Theorem 1. *The sequence of polynomials* $M_m(x)$ *from (11) satisfies the property:*

$$\lim_{x\to\infty} \int_a^b \mathcal{R}^2(M_m(x)) dx = 0 \tag{12}$$

moreover, $\forall \epsilon > 0, \exists m_o \in \mathbf{N}, m > m_0$ *it follows that* $M_m(x)$ *is a* weak ϵ-approximate polynomial solution *of the boundary-value problem ((1)-(2)).*

Proof. Based on the way the polynomials $M_m(x)$ are computed and taking into account the relations (9-12), the following inequalities are satisfied:

$$0 \leq \int_a^b \mathcal{R}^2(M_m(x))dx \leq \int_a^b \mathcal{R}^2(P_m(x))dx, \forall m \in \mathbf{N},$$

where $P_m(x)$ is the sequence of polynomials introduced in Definition 2.
It follows that :

$$0 \leq \lim_{x\to\infty} \int_a^b \mathcal{R}^2(M_m(x))dx \leq \lim_{x\to\infty} \int_a^b \mathcal{R}^2(P_m(x))dx = 0$$

We obtain:

$$\lim_{x\to\infty} \int_a^b \mathcal{R}^2(M_m(x))dx = 0$$

From this limit we obtain that $\forall\epsilon > 0, \exists m_o \in \mathbf{N}, m > m_0$ it follows that $M_m(x)$ is a *weak ϵ-approximate polynomial solution* of the boundary-value problem ((1)-(2)). q.e.d. □

Remark 1. *Taking into account the above remark, in order to find ϵ-approximate polynomial solutions of the boundary-value problem ((1)-(2)) by using the Polynomial Least Squares Method we will first determine weak approximate polynomial solutions, $\tilde{y}$. If $|\mathcal{R}(\tilde{y})| < \epsilon$ then $\tilde{y}$ is also an ϵ approximate polynomial solution of the problem.*

3 Applications

In this section we present the Polynomial Least Squares Method to compute approximate solutions for the Bagley-Torvik fractional differential equation with boundary conditions. The comparison with approximate solutions obtained for the Bagley-Torvik fractional differential equations by means of other approximation methods illustrates the accuracy of the Polynomial Least Squares Method.

3.1 Application 1

We consider the problem:

$$\begin{cases} y''(x) + 0.5 \cdot D^{0.5}y(x) + y(x) = f(x) \\ y(0) = 0, \; y(1) = 2 \end{cases} \qquad for\; x \in \; [0, 1] \tag{13}$$

where $f(x) = 3 + x^2 \cdot \left(\frac{x^{-0.5}}{\Gamma(2.5)} + 1\right)$.
The exact solution of this problem is $y(x) = x^2 + 1$.
An approximate solution of this problem with absolute errors larger than 10^{-7} was proposed by Alkan, Yildirim and Secer in [2] and with absolute errors larger than 10^{-5} by Zahra and Elkholy in [3].
Using PLSM we computed a solution of the type (8): $\tilde{y}(x) = d_0 + d_1 \cdot x + d_2 \cdot x^2$.
Taking into account the boundary conditions $\tilde{y}(0) = 1$ and $\tilde{y}(1) = 2$, we obtain $d_0 = 1$, $d_1 = 1 - d_2$ and the approximation becomes: $\tilde{y}(x) = 1 + (1 - d_2) \cdot x + d_2 \cdot x^2$.
The corresponding remainder (9) is : $\mathcal{R}(\tilde{y}) = \dfrac{(d_2 - 1) \cdot (\sqrt{x}(4 \cdot x - 3) + 3\sqrt{\pi}(2 + x \cdot (x - 1)))}{3\sqrt{\pi}}$.
Obviously the value $d_2 = 1$ leads to a zero remainder and by replacing this value in $\tilde{y}(x)$ we obtain the exact solution of the problem: $\tilde{y}(x) = x^2 + 1$.

3.2 Application 2

We consider the following problem:

$$\begin{cases} y''(x) + D^{\frac{3}{2}} y(x) + y(x) = f(x) \\ y(0) = 0, \quad y(1) = 1 \end{cases} \qquad for\ x \in [0,1]. \tag{14}$$

where $f(x) = 2 + 4 \cdot \sqrt{\frac{x}{\pi}} + x^2$

The exact solutions of this problem is $y(x) = x^2$.
An approximate solution of this problem with absolute errors larger than 10^{-15} was proposed by Karaaslan, Celiker and Kurulay in [4].
Using PLSM we compute a solution of type $\tilde{y}(x) = d_0 + d_1 \cdot x + d_2 \cdot x^2 + d_3 \cdot x^3$.
This solutions should satisfy the boundary conditions of (14). Imposing the conditions $\tilde{y}(0) = 0$ and $\tilde{y}(1) = 1$ we obtain $d_0 = 0$ and $d_1 = 1 - d_2 - d_3$ and the approximation becomes: $\tilde{y}(x) = (1 - d_2 - d_3) \cdot x + d_2 \cdot x^2 + d_3 \cdot x^3$.
The remainder operator (9) in this case is:

$$\mathcal{R}(\tilde{y}) = d_2 \cdot (2 + x \cdot (x-1)) + x \cdot (1 - x + d_3 \cdot (5 + x^2)) + \frac{4\sqrt{x} \cdot (d_2 + 2 \cdot d_3 \cdot x - 1)}{\sqrt{\pi}}$$

We observe that the values $d_2 = 1$ and $d_3 = 0$ lead to a zero remainder and thus, by replacing this values in $\tilde{y}(x)$ the exact solution of the problem is obtained : $\tilde{y}(x) = x^2$.
However, if the exact solution is not known, as it is the case for practical problems, such an observation is probably impossible to make. Using PLSM we proceed to compute the functional (10) (whose expression is too large to reproduce here):

$$J(d_2, d_3) = \int_0^1 \mathcal{R}^2(\tilde{y}) dx$$

We will find the values of d_2 and d_3 by minimizing this functional. In order to do that we will first find the critical points associated to (10). The critical points are the solutions of the system:

$$\begin{cases} \frac{\partial J}{\partial d_2} = 0 \\ \frac{\partial J}{\partial d_3} = 0 \end{cases},$$

where $\frac{\partial J}{\partial d_2} = \frac{707\cdot\pi\cdot(2\cdot d_2+3\cdot d_3-2)+1120\cdot(3\cdot d_2+4\cdot d_3-3)+64\cdot\sqrt{\pi}\cdot(64\cdot d_2+97\cdot d_3-64)}{210\cdot\pi}$ and $\frac{\partial J}{\partial d_3} = \frac{24640\cdot(2\cdot d_2+3\cdot d_3-2)+64\cdot\sqrt{\pi}\cdot(1067\cdot(d_2-1)+1860\cdot d_3)+11\cdot\pi\cdot(2121\cdot(d_2-1)+4400\cdot d_3)}{2310\cdot\pi}$.

We solve this system and obtain again the values $d_2 = 1$ and $d_3 = 0$ as the single stationary point of the system. It is trivial to show that this stationary point is actually the minimum of the functional and thus we are able to obtain the exact solution of the problem: $\tilde{y}(x) = x^2$.

3.3 Application 3

We consider the boundary-value problem:

$$\begin{cases} y''(x) + D^{0.3} y(x) = f(x), for x \in [0,1] \\ y(0) = 0, y(1) = 1 \end{cases}. \tag{15}$$

where $f(x) = -6x + 2 + x^{3.7}\left(\frac{6}{\Gamma(3.7)} - x^{2.7}\frac{2}{\Gamma(2.7)}\right)$
The exact solutions of this problem is $y(x) = x^2(1-x)$.
Using the PLSM we gate the following result for the remainder operator (9)

$$\mathcal{R}(\tilde{y}) = \frac{10x^{1.7}(153(d_2+d_3) - 180x(d_2-1) - 200x^2(d_3+1) + 2142x(d_2 - 1 + 3(1+d_3) + \Gamma(0.7))}{1071 \cdot \Gamma(0.7)}$$

for a solution of the type $\tilde{y}(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3$.
Respecting the boundary conditions of the problem $\tilde{y}(0) = 0$ and $\tilde{y}(1) = 1$ we obtain $d_0 = 0$, $d_1 = -d_2 - d_3$ and

$$\tilde{y}(x) = (-d_2 - d_3)x + d_2 x^2 + d_3 x^3.$$

We compute the functional (10) and minimize it resulting the aproximate solutions of the problem (15): $\tilde{y}(x) = 9.51518 \cdot 10^{-10} x + 1 \cdot x^2 - 1 \cdot x^3$.
An approximate solution of this problem was proposed by Alkan, Yildirim and Secer in [2] witch using the sinc-collocation method. In Table 1 we presents the values of the absolute errors corresponding to the solution by Alkan et al - Error (SCM), and our solution-Error(PLSM).

Table 1. Numerical results for Application 3

x	$Error(SCM)$	$Error(PLSM)$
0.1	3.06×10^{-3}	2.46×10^{-10}
0.2	3.08×10^{-4}	7.24×10^{-10}
0.3	3.49×10^{-3}	1.32×10^{-9}
0.4	3.24×10^{-3}	1.94×10^{-9}
0.5	1.49×10^{-4}	2.47×10^{-9}
0.6	2.99×10^{-3}	2.80×10^{-9}
0.7	3.78×10^{-3}	2.82×10^{-9}
0.8	1.66×10^{-3}	2.43×10^{-9}
0.9	2.61×10^{-4}	1.53×10^{-9}
1	0	0

3.4 Application 4

We consider the problem:

$$\begin{cases} y''(x) + x^2 y' - D^{0.7}y(x) + y(x) = f(x), for x \in \Omega \\ y(0) = 0, y(1) = 1 \end{cases} . \tag{16}$$

where $f(x) = 5x^6 - 3x^5 - x^4 + 20x^3 - 12x^2 - \frac{120}{\Gamma(5.3)}x^{4.3} + \frac{24}{\Gamma(4.3)}x^{3.3}$
The exact solutions of this problem is $y(x) = x^4(x-1)$.
With PLSM the remainder operator is :

$$\begin{aligned}\mathcal{R}(\tilde{y}) = &\frac{1}{1141427 \cdot \Gamma(1.3)} \cdot (x^{0.3}(141427(d_3 + d_4 + d_5) - 10879d_2(20x - 13) + \\ &+ 200x^2(-1419d_3 - 40x(43 + 43d_4 + 50(d_5 - 1)x))) + 141427(2d_2 - \\ &- (d_2 - 5d_3 + d_4 + d_5)x - (d_3 - 11d_4 + d_5 - 12)x^2 + (2d^2 + d^3 + \\ &+ 20(d_5 - 1))x^3 + (1 + 3d_3 + d_4)x^4 + (3 + 4d_4 + d_5)x^5 + 5(d_5 - 1)X^6) \cdot \Gamma(1.3))\end{aligned}$$

for a solution of the type $\tilde{y}(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + d_5 x^5$.
Respecting the boundary conditions of the problem $\tilde{y}(0) = 0$ and $\tilde{y}(1) = 1$
we obtain $d_0 = 0$, $d_1 = -d_2 - d_3 - d_4 - d_5$ and

$$\tilde{y}(x) = (-d_2 - d_3 - d_4 - d_5)x + d_2 x^2 + d_3 x^3 + d_4 x^4 + d_5 x^5.$$

We compute the functional (10) and minimize it obtaining the values
$d_2 = 1.3919 \cdot 10^{-13}$, $d_3 = 5.10892 \cdot 10^{-13}$, $d_4 = -1$ and $d_5 = 1$.
The approximate solutions of the problem (16) is:
$\tilde{y}(x) = -1.22172 \cdot 10^{-15} x + 1.3919 \cdot 10^{-13} \cdot x^2 + 5.10892 \cdot 10^{-13} \cdot x^3 - x^4 + x^5$.
In Table 2 we presents the values of the absolute errors corresponding to the solution by Alkan et al [2] - Error (SCM), and our solution- Error(PLSM).

Table 2. Numerical results for Application. 4

x	$Error(SCM)$	$Error(PLSM)$
0.1	2.82×10^{-3}	8.17×10^{-16}
0.2	1.73×10^{-3}	2.13×10^{-15}
0.3	3.31×10^{-4}	2.72×10^{-15}
0.4	1.15×10^{-3}	2.25×10^{-15}
0.5	1.75×10^{-3}	1.00×10^{-15}
0.6	2.36×10^{-3}	4.30×10^{-16}
0.7	1.49×10^{-3}	1.36×10^{-15}
0.8	2.66×10^{-3}	1.36×10^{-15}
0.9	4.88×10^{-3}	6.10×10^{-16}
1	0	0

Conclusion

In this paper we show the utility of the Polynomial Least Squares Method for solving the Bagley-Torvik fractional equation with boundary conditions. In the numerical examples the comparison with approximate solutions obtained for the Bagley-Torvik fractional differential equations by means of other approximation methods illustrates the accuracy of the Polynomial Least Squares Method.

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