# Numerical solution of the first order nonlinear differential equations with the mixed nonlinear conditions by using PLSM (comparison with Bernstein polynomials method)

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**Abstract.** We use the Polynomial Least Squares Method (PLSM), which allows us to compute analytical approximate polynomial solutions for nonlinear ordinary differential equations with the mixed nonlinear conditions. The accuracy of the method is illustrated by a comparison with approximate solutions previously computed using Bernstein polynomials method.

## 1 Introduction

In recent years, in many practical applications in various fields for example physics, mechanics, chemistry, and biology, the problems under studied are modeled using the first order nonlinear differential equations.

In order to find approximate solutions of these equations, many methods were proposed, such as:

- Bernstein polynomials method [2], [3];
- Fermat collocation method [5];
- Taylor collocation method [6];
- Legendre collocation method [4].

In the present paper we use the Polynomial Least Squares Method (PLSM) to compute analytical approximate polynomial solutions for the first order nonlinear differential equation:

$$P(t)x(t) + Q(t)x'(t) + R(t)x^2(t) + S(t)x(t)x'(t) + T(t)(x'(t))^2 = g(t) \qquad (1)$$

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having the mixed nonlinear condition:

$$\alpha x(a) + \beta x(b) + \gamma(x(a))^2 + \tau(x(b))^2 + \xi x(a)x(b) = \lambda \tag{2}$$

where the following are functions: $P(t), Q(t), R(t), S(t), T(t), g(t)$ and the following are constants: $\alpha, \beta, \gamma, \tau, \xi, \lambda$.

In the next section we will introduce the Polynomial Least Squares Method (PLSM, [1]), which allows us to find analytical approximate polynomial solutions for the problem (1)-(2), and in the third section we compare our approximate solutions with approximate solutions presented using Bernstein polynomials method ([2]).

## 2 The Polynomial Least Squares Method

We use the Polynomial Least Squares Method (PLSM).
This method was introduced by BOTA and CĂRUNTU in [1].
For the problem (1)-(2) we consider the operator:

$$D(x) = P(t)x(t) + Q(t)x'(t) + R(t)x^2(t) + S(t)x(t)x'(t) + T(t)(x'(t))^2 - g(t). \tag{3}$$

If $\tilde{x}$ is an approximate solution of equation (1), we evaluate the error obtained by replacing the exact solution $x$ with the approximate one $\tilde{x}$ as the remainder:

$$R(t, \tilde{x}) = D(\tilde{x}(t)), \quad t \in \mathbb{R}. \tag{4}$$

For $\epsilon \in \mathbb{R}_+$, we will compute approximate polynomial solutions $\tilde{x}$ of the problem (1)-(2) on the interval [a,b]. We impose for $\tilde{x}$ the following conditions:

$$|R(t, \tilde{x})| < \epsilon \tag{5}$$

$$\alpha\tilde{x}(a) + \beta\tilde{x}(b) + \gamma(\tilde{x}(a))^2 + \tau(\tilde{x}(b))^2 + \xi\tilde{x}(a)\tilde{x}(b) = \lambda. \tag{6}$$

**Definition 2.1.** *We call an* $\epsilon$-approximate polynomial solution *of the problem* (1)-(2) *an approximate polynomial solution* $\tilde{x}$ *satisfying the relations* (5),(6).

**Definition 2.2.** *We call a* weak $\delta$-approximate polynomial solution *of the problem* (1)-(2) *an approximate polynomial solution* $\tilde{x}$ *satisfying the relation:*

$$\int_a^b R^2(t, \tilde{x})dt \leq \delta, \quad \delta \in \mathbb{R}_+ \tag{7}$$

*together with the initial condition* (6).

**Remark 2.1.** *From the Weierstrass approximation theorem it follows that there exists the sequence of polynomials,*

$$P_m(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m, \quad c_i \in \mathbb{R}, \quad i = 0, 1, \cdots, m$$

*satisfying the condition:*

$$\alpha P_m(a) + \beta P_m(b) + \gamma(P_m(a))^2 + \tau(P_m(b))^2 + \xi P_m(a)P_m(b) = \lambda \tag{8}$$

*such that*

$$\lim_{m\to\infty} D(P_m(t)) = 0.$$

**Theorem 2.1.** *The problem* (1)-(2) *admits a sequence of weak approximate polynomial solutions.*

We will compute a weak $\epsilon$-approximate polynomial solution, in the sense of the definition 2, of the type:

$$\tilde{x}(t) = \sum_{k=0}^{m} d_k t^k \tag{9}$$

where $d_0, d_1, \cdots, d_m$ are constants which are calculated using the following steps:

(1) In the first step we substitute the approximate solution (9) in the equation (1) and obtain the remainder:

$$\begin{aligned}\mathcal{R}(t, d_0, d_1, \cdots, d_m) = R(t, \tilde{x}) = P(t)\tilde{x}(t) + Q(t)\tilde{x}'(t) + R(t)\tilde{x}^2(t) + \\ + S(t)\tilde{x}(t)\tilde{x}'(t) + T(t)(\tilde{x}'(t))^2 - g(t).\end{aligned} \tag{10}$$

(2) We attach to problem (1)-(2) the real functional:

$$\mathcal{J}(d_1, d_2, d_3 \cdots, d_m) = \int_a^b \mathcal{R}^2(t, d_0, d_1, \cdots, d_m)dt. \tag{11}$$

where $d_0$ is computed as functions of $d_1, d_2, d_3, \cdots, d_m$ using the initial condition (6).

(3) We compute $d_1^0, d_2^0, d_3^0, \cdots d_m^0$ as values which give the minimum of the functional (11) and the value of $d_0^0$ as functions of $d_1^0, d_2^0, d_3^0, \cdots, d_m^0$ using the initial condition.

(4) Using the constants $d_0^0, d_1^0, \cdots, d_m^0$ thus determined, we consider the polynomial:

$$T_m(t) = \sum_{k=0}^{m} d_k^0 t^k. \tag{12}$$

**Theorem 2.2.** *The sequence of polynomials* $T_m(t)$ *from* (12) *satisfies the property:*

$$\lim_{m\to\infty} \int_a^b R^2(t, T_m(t))dt = 0. \tag{13}$$

*Moreover,* $\forall \epsilon > 0, \exists m_o \in \mathbb{N}$ *such that* $\forall m \in \mathbb{N}, m > m_0$ *it follows that* $T_m(t)$ *is a* weak $\epsilon$-approximate polynomial solution *of the problem* (1)-(2).

*Proof.* Based on the way the coefficients of polynomial $T_m(t)$ are computed and taking into account the relations (10)-(12), the following inequalities are satisfied:

$$0 \le \int_a^b R^2(t, T_m(t))dt \le \int_a^b R^2(t, P_m(t))dt, \quad \forall m \in \mathbb{N}. \tag{14}$$

It follows that:

$$0 \le \lim_{m\to\infty} \int_a^b R^2(t, T_m(t))dt \le \lim_{m\to\infty} \int_a^b R^2(t, P_m(t))dt = 0, \quad (\forall) m \in \mathbb{N}. \tag{15}$$

We obtain:

$$\lim_{m\to\infty}\int_a^b R^2(t, T_m(t))dt = 0. \tag{16}$$

From this limit we obtain that $(\forall)\epsilon > 0, \exists m_o \in \mathbb{N}$ such that $(\forall)m \in \mathbb{N}, m > m_0$. It follows that $T_m(t)$ is a *weak $\epsilon$-approximate polynomial solution* of the problem (1)-(2). □

**Remark 2.2.** *Taking into account the above remark, in order to find $\epsilon$-approximate polynomial solutions of the problem* (1)-(2) *by using the Polynomial Least Squares Method (PLSM), we will first determine weak approximate polynomial solutions, $\tilde{x}$ foollowing the steps* 1 *to* 4 *previously described. If $|R(t, \tilde{x})| < \epsilon$, then $\tilde{x}$ is also an $\epsilon$-approximate polynomial solution of the problem.*

## 3 Applications

### 3.1 Application 1

We consider the nonlinear differential equation:

$$3x^2(t) - tx'(t) - 3x(t) = 3t^6 \tag{17}$$

with the nonlinear condition $x'(0) + x(0) = 2$, $x \in [0, 1]$.

The exact solution of this problem is the following: $x_e(t) = 1 - t^3$.

Using the steps described in the previous section we performed the following computations:

$(i)$ We computed a polynomial solution of the form

$$x_{PLSM} = d_0 + d_1t + d_2t^2 + d_3t^3. \tag{18}$$

$(ii)$ From the initial condition $(x(0))^2 + x(0) = 2$, $x \in [0, 1]$ we obtained $d_0 = 1$ and the approximation becomes:

$$x_{PLSM} = 1 + d_1t + d_2t^2 + d_3t^3. \tag{19}$$

$(iii)$ The corresponding remainder is:

$$\mathcal{R}(t) = -3t^6 - t(d_1 + 2d_2t + 3d_3t^2) - 3(1 + d_1t + d_2t^2 + d_3t^3)+ \\ + 3(1 + d_1t + d_2t^2 + d_3t^3)^2. \tag{20}$$

$(iv)$ The corresponding function to problem (17) is:

$$\mathcal{J}(d_1, d_2, d_3) = \frac{9}{13} - \frac{3d_1}{2} - \frac{2d_1^2}{3} + 3d_1^3 + \frac{9d_1^4}{5} - \frac{2d_2}{3} - \frac{13d_1d_2}{5} + 6d_1^2d_2+ \\ + 6d_1^3d_2 - \frac{79d_2^2}{55} + 4d_1d_2^2 + \frac{54d_1^2d_2^2}{7} + \frac{6d_2^3}{7} + \frac{9d_1d_2^3}{2} + d_2^4- \\ - \frac{36d_1d_3}{11} + 4d_1^2d_3 + \frac{36d_1^3d_3}{7} - 3d_2d_3 + \frac{36d_1d_2d_3}{7} + \frac{27d_1^2d_2d_3}{2}+ \\ + \frac{3d_2^2d_3}{2} + 12d_1d_2^2d_3 + \frac{18d_2^3d_3}{5} - \frac{18d_3^2}{13} + \frac{3d_1d_3^2}{2} + 6d_1^2d_3^2+ \\ + \frac{2d_2d_3^2}{3} + \frac{54d_1d_2d_3^2}{5} + \frac{54d_2^2d_3^2}{11} + \frac{36d_1d_3^3}{11} + 3d_2d_3^3 + \frac{9d_3^4}{13}. \tag{21}$$

$(v)$ To find the minimum of this function we compute the stationary points as the solutions of the system:

$$\begin{cases} \frac{\partial \mathcal{J}}{\partial d_1} = 0 \\ \frac{\partial \mathcal{J}}{\partial d_2} = 0 \\ \frac{\partial \mathcal{J}}{\partial d_3} = 0 \end{cases} \tag{22}$$

Since the only stationary point is $d_1 = 0$, $d_2 = 0$ and $d_3 = -1$ it is easy to show that this point is indeed a minimum. We have the Polynomial Least Squares Method (PLSM) solution:

$$x_{PLSM} = 1 - t^3 \tag{23}$$

which is actually the exact solution of the problem.

## 3.2 Application 2

Consider the following nonlinear differential equation

$$(x'(t))^2 + 3tx'(t) - x(t) = -\frac{3}{2}t^2 \tag{24}$$

with the nonlinear initial condition $2x(0) - (x(0))^2 = 1$, $x \in [0, 1]$.

The exact solution of this problem is $x_e(t) = 1 - t - \frac{1}{2}t^2$.

Using our method we performed the following computations:

$(i)$ We computed a polynomial solution of the form

$$x_{PLSM} = d_0 + d_1 t + d_2 t^2. \tag{25}$$

$(ii)$ From the initial condition $2x(0) - (x(0))^2 = 1$, $x \in [0, 1]$, we obtained $d_0 = 1$ and the approximation becomes:

$$x_{PLSM} = 1 + d_1 t + d_2 t^2. \tag{26}$$

$(iii)$ The corresponding remainder is:

$$\mathcal{R}(t) = -1 - d_1 t + \frac{3t^2}{2} - d_2 t^2 + 3t(d_1 + 2d_2 t) + (d_1 + 2d_2 t)^2. \tag{27}$$

$(iv)$ The corresponding function to the problem (24) is:

$$\mathcal{J}(d_1, d_2) = \frac{9}{20} - \frac{d_1}{2} + \frac{d_1^2}{3} + 2d_1^3 + d_1^4 - \frac{d_2}{3} + 4d_1 d_2 + \frac{26d_1^2 d_2}{3} + 4d_1^3 d_2 + \frac{71d_2^2}{15} + \\ + 14d_1 d_2^2 + 8d_1^2 d_2^2 + 8d_2^3 + 8d_1 d_2^3 + \frac{16d_2^4}{5}. \tag{28}$$

$(v)$ The values $d_1 = -1$ and $d_2 = -\frac{1}{2}$ lead to a zero remainder and by replacing this value in $x_{PLSM}$ we obtain the exact solution of the problem:

$$x_{PLSM} = 1 - t - \frac{1}{2}t^2. \tag{29}$$

### 3.3 Application 3

Lets consider following nonlinear differential equation

$$t^2(x'(t))^2 + x'(t) - t^2(x(t))^2 = e^t \tag{30}$$

with the nonlinear initial condition $2x(0) + 3(x(0))^2 = 4$, $x \in [0, 1]$.

The exact solution of this problem is $x_e(t) = e^t$.

Using our method we obtained the following polynomial approximate solution of (30):

$$x_{PLSM} = 1 + 0.721523t + 0.939892t^2. \tag{31}$$

Table 1 present the values of the absolute errors corresponding to the solution obtaind by H. Gurler and S. Yalcinbas using the Bernstein method - (Error Bernstein [2]) and our solution using the Polynomial Least Squares Method - (Error PLSM).

**Table 1.** Numerical results

| x | $Error\ Bernstein$[2] | $Error\ PLSM$ |
|---|---|---|
| 0.1 | $1.709 \times 10^{-4}$ | $2.172 \times 10^{-6}$ |
| 0.2 | $1.402 \times 10^{-3}$ | $2.204 \times 10^{-6}$ |
| 0.3 | $4.858 \times 10^{-3}$ | $4.138 \times 10^{-6}$ |
| 0.4 | $1.182 \times 10^{-2}$ | $2.676 \times 10^{-6}$ |
| 0.5 | $2.372 \times 10^{-2}$ | $2.048 \times 10^{-7}$ |
| 0.6 | $4.211 \times 10^{-2}$ | $8.852 \times 10^{-7}$ |
| 0.7 | $6.875 \times 10^{-2}$ | $4.600 \times 10^{-8}$ |
| 0.8 | $1.055 \times 10^{-1}$ | $1.236 \times 10^{-6}$ |
| 0.9 | $1.546 \times 10^{-1}$ | $1.075 \times 10^{-6}$ |
| 1 | $2.182 \times 10^{-1}$ | $8.384 \times 10^{-7}$ |

## 4 Conclusions

The computations performed show that Polynomial Least Squares Method (PLSM) is a rapid and efficient method of computing approximate polynomial solutions for nonlinear differential equations. The applications presented have demonstrated this and the accuracy of the method.

## References

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