Digital signal processing in telecommunications based on parametric discrete Fourier transform

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Abstract. A generalization of the discrete Fourier transform in the form of a parametric discrete Fourier transform is proposed. The analytical and stochastic properties of the introduced discrete transformation are investigated. An example of the application of the parametric discrete Fourier transform in telecommunications is given - a generalization of the well-known Herzel algorithm

1 Introduction

The methods of vector¹ and spectral² digital signal processing (DSP) play a crucial role in information technologies, which are currently widely used in telecommunications [1-6]. The relevance of the digitalization of telecommunications is mainly due to two reasons. Firstly, DSP methods and algorithms can effectively solve the problems of processing and transmitting incoming information to a distance. Secondly, DSP methods and algorithms make it possible to obtain reliable information (information) about the state of telecommunication complexes and communication channels, as well as analyze and process information about the processes and phenomena occurring in them.

As practice has shown, the classical Fourier analysis based on the discrete Fourier transform (DFT), along with a number of advantages, also has fundamental disadvantages, which are manifested in the form of specific effects (overlap effects, paling, leakage, scallop effect).

The disadvantages of the DFT are determined by the nature of the DFT, as well as the properties of its basis - the system of discrete exponential functions (DEF). In [7-9], it was shown that the use of well-known methods to reduce the influence of negative effects inherent in DFTs, unfortunately, does not solve the problem of digital Fourier - signal processing.

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¹ Digital vector signal processing – a set of discrete Fourier methods – processing of discrete signals that allow you to get complete information about the harmonic components of the studied signals (the amplitude, frequency, phase of each of the components and / or their changes over time).

² Digital spectral signal processing is a multitude of discrete Fourier methods – processing of discrete signals that allow one to obtain only energy spectra and / or their change in time.

The aim of this work is to solve the problem of digital Fourier - signal processing in telecommunications by generalizing the DFT in the form of a parametric discrete Fourier transform (DFT-P).

2. Parametric Discrete Fourier Transform

Parametric Discrete Fourier Transform ((DFT-P) – is the decomposition of a discrete signal in a system of parametric DEF (DEF-P):

$$def_p(k, n, \theta) = W_N^{(k+\theta)n} = \exp\left[-j \frac{2\pi}{N} (k+\theta)n\right], \quad k, n = \overline{0, (N-1)}, \quad 0 \leq \theta < 1. \quad (1)$$

DFT-P in matrix form is given by the following equation:

$$S_{N,\theta} = \frac{1}{N} F_{N,\theta} X_N; \quad 0 \leq \theta < 1, \quad (2)$$

where $X_N = [x(0), x(1), \dots, x(N-1)]^T$ – discrete signal $x(n)$, $n = \overline{0, N-1}$, represented as a vector of a linear N -dimensional space; T – vector transpose sign;
 $S_{N,\theta} = [s(0, \theta), s(1, \theta), \dots, s((N-1), \theta)]^T$ – decomposition coefficients of a discrete signal $x(n)$, $n = \overline{0, N-1}$, according to the DEF-P system defined by a matrix $F_{N,\theta}$, presented as a vector of a linear N -dimensional space; $F_{N,\theta}$ – matrix defining the DEF-P system:

$$F_{N,\theta} = \begin{matrix} & \begin{matrix} 0 & 1 & \dots & (N-1) \end{matrix} & n \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ (N-1) \end{matrix} & \begin{bmatrix} 1 & W_N^\theta & \dots & W_N^{\theta(N-1)} \\ 1 & W_N^{(1+\theta)} & \dots & W_N^{(1+\theta)(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1+\theta)} & \dots & W_N^{(N-1+\theta)(N-1)} \end{bmatrix} \end{matrix} \quad (3)$$

DFT-P in algebraic form is given by the following relation:

$$S_N(k, \theta) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) W_N^{(k+\theta)n}, \quad k = \overline{0, N-1}. \quad (4)$$

There is an inverse DFT-P (ODPF-P), which in matrix and algebraic form is determined respectively by the following relation:

$$X_N = F_{N,\theta}^* S_{N,\theta}; \quad 0 \leq \theta < 1, \quad (5)$$

$$\text{where } F_{N,\theta}^* = \begin{matrix} & \begin{matrix} 0 & 1 & \dots & (N-1) \end{matrix} & k \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ (N-1) \end{matrix} & \begin{bmatrix} 1 & 1 & \dots & 1 \\ W_N^{-\theta} & W_N^{-(1+\theta)} & \dots & W_N^{-(N-1+\theta)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)\theta} & W_N^{-(N-1)(1+\theta)} & \dots & W_N^{-(N-1)(N-1+\theta)} \end{bmatrix} \end{matrix}, \quad (6)$$

$$x(n) = \sum_{k=0}^{N-1} S_N(k, \theta) W_N^{-(k+\theta)n}, \quad n = \overline{0, N-1}, \quad 0 \leq \theta < 1. \quad (7)$$

DFT-P allows us to generalize the concept of signal periodicity in the form of a parametric signal periodicity:

$$x_\theta(n) = x(n \bmod N) W_N^{\theta N \operatorname{int}[n/N]}, \quad (8)$$

where $\operatorname{int}[\cdot]$ is the function of removing the fractional part of number and returning integer value.

Let us prove that (7) is really the inverse transformation with respect to (4). To do this, multiply the left and right sides of the equality by $W_N^{-(k+\theta)m}$ and add up the result by k :

$$\sum_{k=0}^{N-1} S_N(k, \theta) W_N^{-(k+\theta)m} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x(n) W_N^{(k+\theta)n} W_N^{-(k+\theta)m},$$

or

$$\sum_{k=0}^{N-1} S_N(k, \theta) W_N^{-(k+\theta)m} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) W_N^{-\theta(m-n)m} \sum_{k=0}^{N-1} W_N^{-(m-n)k},$$

Given the orthogonality of the DEF-P functions, we finally obtain:

$$x(m) = \sum_{k=0}^{N-1} S_N(k, \theta) W_N^{-(k+\theta)m}, \quad m = \overline{0, N-1},$$

which coincides with (7).

The main properties of DEF-P.

1. DEF-P, unlike DEF, are not functions of two equal variables p and l . Therefore, the DEF-P matrix is asymmetric.
2. DEF-P are periodic in variable p and parametrically periodic in variable l with period N :

$$\begin{aligned} \operatorname{def}_p(p \pm kN, l, \theta) &= \operatorname{def}_p(p, l, \theta), \\ \operatorname{def}_p(p, l \pm kN, \theta) &= \operatorname{def}_p(p, l, \theta) W_N^{\pm \theta Nk}. \end{aligned}$$

3. The DEF-P system is not multiplicative in the variable p :

$$\operatorname{def}_p(p, l, \theta) \operatorname{def}_p(m, l, \theta) \neq \operatorname{def}_p(p+m, l, \theta);$$

and is multiplicative in variable l :

$$\operatorname{def}_p(p, l, \theta) \operatorname{def}_p(p, m, \theta) = \operatorname{def}_p(p, l+m, \theta), \quad m = \overline{0, N-1}$$

4. The average value of DEF-P in the variable p is equal to zero at $l \neq 0$,

$$\sum_{p=0}^{N-1} \operatorname{def}_p(p, l, \theta) = \exp\left(-j \frac{2\pi}{N} \theta l\right) \frac{1 - \exp(-j2\pi l)}{1 - \exp(-j \frac{2\pi}{N} l)}$$

and the variable l is not equal to zero:

$$\sum_{l=0}^{N-1} \operatorname{def}_p(p, l, \theta) = \frac{1 - \exp[-j2\pi l(p+\theta)]}{1 - \exp[-j \frac{2\pi}{N} (p+\theta)]}.$$

5. The DEF-P system is orthogonal in both variables:

$$\sum_{l=0}^{N-1} W_N^{(p+\theta)l} [W_N^{(m+\theta)l}]^* = \frac{1 - W_N^{-(m-p)N}}{1 - W_N^{-(m-p)}} = \begin{cases} N & \text{при } p=m; \\ 0 & \text{при } p \neq m; \end{cases}$$

$$\sum_{p=0}^{N-1} W_N^{(p+\theta)l} [W_N^{(p+\theta)k}]^* = W_N^{-\theta(k-l)} \frac{1 - W_N^{-(k-l)N}}{1 - W_N^{-(k-l)}} \begin{cases} N & \text{при } k=l; \\ 0 & \text{при } k \neq l. \end{cases}$$

6. The DEF-P system is a complete system, since the number of linearly independent functions is equal to the dimension of the set of discrete signals.

Let us prove that the linearity, shift, correlation, and Parseval theorems are valid for DFT-P. We introduce the symbolic notation for DFT-P and DDPF-P, defined respectively by relations (4) and (7), as well as for $x_\theta(n)$ relation (8):

$$x_\theta(n) \Leftrightarrow S_N(k, \theta).$$

Linearity theorem.

DFT-P is linear by definition. Indeed,

$$\text{if } x_\theta(n) \Leftrightarrow S_N(k, \theta) \text{ и } y_\theta(n) \Leftrightarrow Q_N(k, \theta),$$

$$\text{then } \lambda_1 \cdot x_\theta(n) + \lambda_2 \cdot y_\theta(n) = \lambda_1 \cdot S_N(k, \theta) + \lambda_2 \cdot Q_N(k, \theta).$$

Shift theorem.

$$\text{if } x_\theta(n) \Leftrightarrow S_N(k, \theta),$$

$$\text{then } x_\theta(n+m) \Leftrightarrow W_N^{-(k+\theta)m} S_N(k, \theta).$$

Proof.

Denote $W_N^{-(k+\theta)m} S_N(k, \theta)$ by $S_{m,N}(k, \theta)$. DFT parametric N – periodic sequence is equal to:

$$S_{m,N}(k, \theta) = \frac{1}{N} \sum_{n=0}^{N-1} x_\theta(n+m) \cdot W_N^{(k+\theta)n}.$$

Let $m+n=l$ then:

$$S_{m,N}(k, \theta) = \frac{1}{N} \sum_{l=m}^{N+m-1} x_\theta(n+m) \cdot W_N^{(k+\theta)(l-m)} = W_N^{-(k+\theta)m} \cdot \frac{1}{N} \sum_{l=m}^{N+m-1} x_\theta(l) \cdot W_N^{(k+\theta)l}.$$

Using the properties of DEF-P, it is easy to establish the validity of the following relations:

$$\frac{1}{N} \sum_{n=m}^p x_\theta(n) \cdot W_N^{(k+\theta)n} = \frac{1}{N} \sum_{n=0}^{N-1} x_\theta(n) \cdot W_N^{(k+\theta)n}; |p-m| = N-1$$

$$\sum_{k=m}^p S_N(k, \theta) \cdot W_N^{-(k+\theta)n} = \sum_{k=0}^{N-1} S_N(k, \theta) \cdot W_N^{-(k+\theta)n}; |p-m| = N-1.$$

Given these ratios, we obtain:

$$x_{\theta}(n+m) \Leftrightarrow W_N^{-(k+\theta)m} S_N(k, \theta);$$

similarly:

$$x_{\theta}(n-m) \Leftrightarrow W_N^{(k+\theta)m} S_N(k, \theta).$$

Correlation theorem.

$$\text{if } a : x_{\theta}(n) \Leftrightarrow S_N(k, \theta) \text{ и } y_{\theta}(n) \Leftrightarrow Q_N(k, \theta),$$

then the DFT-P of the circular (cyclic) correlation defined by the relation:

$$z_{\theta}(n) = \frac{1}{N} \sum_{m=0}^{N-1} x_{\theta}(m) \cdot y_{\theta}(n+m);$$

$$\text{equally: } U_N(k, \theta) = S_N^*(k, \theta) Q_N(k, \theta); \text{ где } z_{\theta}(n) \Leftrightarrow U_N(k, \theta).$$

Proof.

DFT-P sequence $z_{\theta}(n)$ is equal to:

$$U_N(k, \theta) = \frac{1}{N^2} \sum_{m=0}^{N-1} x_{\theta}(m) \cdot \sum_{n=0}^{N-1} y_{\theta}(n+m) \cdot W_N^{(k+\theta)n}.$$

It follows from the shift theorem that:

$$U_N(k, \theta) = Q_N(k, N) \cdot \frac{1}{N} \sum_{m=0}^{N-1} x_{\theta}(m) W_N^{-(k+\theta)m};$$

or

$$U_N(k, \theta) = S_N^*(k, \theta) Q_N(k, \theta).$$

Parseval's theorem.

$$\text{if } a : x_{\theta}(n) \Leftrightarrow S_N(k, \theta),$$

$$\text{then } \frac{1}{N} \sum_{n=0}^{N-1} x_{\theta}^2(n) = \sum_{k=0}^{N-1} |S_N(k, \theta)|^2.$$

Proof.

The validity of the Parseval theorem for DFT-P follows directly from the correlation theorem. Indeed, if put $x_{\theta}(n) = y_{\theta}(n)$, then:

$$\frac{1}{N} \sum_{m=0}^{N-1} x_{\theta}(m) \cdot x_{\theta}(n+m) = \sum_{k=0}^{N-1} |S_N(k, \theta)|^2 \cdot W_N^{-(k+\theta)m},$$

whence at $m=0$ follows:

$$\frac{1}{N} \sum_{n=0}^{N-1} x_{\theta}^2(n) = \sum_{k=0}^{N-1} |S_N(k, \theta)|^2.$$

3. Generalization of the Goertzel algorithm based on parametric Discrete Fourier Transform

Goertzel's algorithm implements the DFT as an IIR filter and is widely used in solving problems of detecting and decoding DTMF (Dual-Tone Multi-Frequency) used in telecommunications. Herzel's algorithm (filter), in addition to being fixed in the set of frequencies being analyzed: $\{2\pi k/N\}$, $k=0, N-1$, where $k=0, N-1$, N – the number of signal samples, has a more significant drawback. The fact is that the Herzel filter is on the verge of stability.

Two versions of the generalized Goertzel algorithm based on the DFT-P (Fig. 1, a, b) are proposed, which are free from these drawbacks.

Fig. 1. IIR filter structures that implement the generalized Goertzel algorithm at the $(k + \theta)$ frequency at $0 \leq \theta < 1$

4. Conclusion

A new, efficient and effective mathematical apparatus for digital spectral and vector signal processing in telecommunication information technologies is proposed – parametric Discrete Fourier Transform.

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