

# An approach with Lagrange identity of the mixed problem in theory of strain gradient thermoelasticity

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**Abstract.** In our paper we first define the mixed initial-boundary values problem in the theory of strain gradient thermoelasticity. With the help of an identity of Lagrange's type, we then prove some theorems regarding the uniqueness of the solution of this mixed problem and also two results regarding the continuous dependence of solutions on initial data and on the charges. We must outline that we obtain these qualitative results without recourse to any laws of conservation of energy and without recourse to any boundedness assumptions on the coefficients. It is equally important to note that we do not impose restrictions on the elastic coefficients regarding their positive definition.

**Keywords:** Lagrange's identity; strain gradient thermoelasticity; uniqueness; continuous dependence results

## 1. Introduction

We want to underline that in the constitutive equations in the strain gradient thermoelasticity theory it is contain the second order gradient, of course, along with the first gradient, because both have contributions to dissipation. The strain gradient theory of thermoelasticity is suitable to approach main problems regarding the size effects and to characterize the evolution of chiral elastic materials which include auxenic media, carbon nanotube, bones, porous composite, honeycomb structures Some papers in this domain are: [5-14]. There are many papers in the theory of elasticity of in the theory of thermoelasticity dedicated to the uniqueness of solutions or/and to continuous dependence results, but we need to say that these results are based almost exclusively on the hypothesis that the tensors of the thermoelastic coefficients are positive definite. In other studies, the uniqueness or continuous dependence of solutions are obtained by using a specific law for the conservation of the energy. Green and Lindsay supplemented in [15] the conditions arising from thermoelasticity with some assumptions on positive or negative definiteness, in order to prove an uniqueness theorem. We want to consider that our study is the continuation of many studies which are based on the different improvements of the Lagrange identity, of which it is worth mentioning the papers [16-18]. From the studies dedicated to Cesaro means, to uniqueness and to continuous dependence results, we remember [19-21]. Other results for different kind of micro-structures can be found in the papers [22-29].

In our study we address the mixed problem in the context of strain gradient thermoelasticity in a new manner, namely our approach is based on the identity of Lagrange. So, we can

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prove that the solution of the mixed problem in this context is unique and obtain the continuous dependence of solutions with respect to loads, that is, body forces and heat supply. Other continuous dependence of solutions result it is obtained, with respect to initial data. All the results are deduced in the case of bounded domains from the Euclidean three dimensional space, but they can be extended without much difficulty in the case of boundless domains, with some restrictions on behavior to infinity. Again, we outline that the results are obtained without recourse to any hypotheses regarding the boundedness of the coefficients or to a law for the conservation of the energy, as well as avoid use of definiteness assumptions on the thermoelastic coefficients.

## 2. Basic equations

Let us consider a bounded domain  $D$  of three-dimensional Euclidean space  $R^3$  which is occupied by the reference configuration of an anisotropic homogeneous linear elastic body, from strain gradient thermoelasticity theory. The closure of the regular domain  $D$  is denoted by  $\bar{D}$  and the boundary of  $D$  is  $\partial D$ , that is, we have  $\bar{D} = D \cup \partial D$ .

To characterize the evolution of such kind of media, we will use the vector of the displacement with the components  $v_i$  and the variation of the temperature, denoted by  $\vartheta$ , which is measured from absolute temperature in the reference state,  $T_0$ , which is assumed be a constant. We will use two strain tensors, of components  $e_{mn}$  and  $\mu_{mnr}$ , respectively, which are also called the kinematic characteristics of the body. These are introduced with the help of the kinematic equations:

$$e_{mn} = \frac{1}{2}(v_{m,n} + v_{n,m}), \mu_{mnr} = v_{r,mn} \quad (1)$$

We also will use two tensors of stress, namely the classic stress tensor of components  $t_{mn}$  and the hyperstress tensor of components  $\sigma_{mnr}$ , both tensors defined over  $D$ . Having the stress tensors and the strain tensors, we can highlight the connections between them, through the constitutive equations, which for an anisotropic and homogeneous strain gradient thermoelastic body, have the following form:

$$\begin{aligned} t_{mn} &= a_{mnlk}e_{kl} + b_{mnrkl}\mu_{rkl} - \alpha_{mn}\vartheta, \\ \sigma_{mnr} &= b_{klmnr}e_{kl} + c_{klsmnr}\mu_{kls} - \beta_{mnr}\vartheta, \\ \eta &= \alpha_{mn}e_{mn} + \beta_{mnr}\mu_{mnr} + \frac{a}{T_0}\vartheta - d_m\vartheta_{,m}, \\ q_m &= T_0(d_m\dot{\vartheta} - \kappa_{mn}\vartheta_{,n}). \end{aligned} \quad (2)$$

In our following considerations we will use some basic notations and theoretical results in a manner similar to that used by Iesan in his known book [30].

First, the equations of motion in strain gradient thermoelasticity have the general form (see also, [30]):

$$t_{mn,n} + \sigma_{mnr,nr} + F_m = \rho\ddot{v}_m \quad (3)$$

The equation of energy is given by

$$\vartheta_0\dot{\eta} = \frac{1}{\rho}q_{m,m} + r. \quad (4)$$

The functions used in the previous equations have the following meanings:

- $F_m$  are the components of the body force per unit volume;
- $\rho$  is the density of mass;

- $\eta$  is the notation for the entropy;
- $r$  is the supply of heat;
- $q_m$  are the components of the heat flux.

The thermoelastic coefficients  $a_{mnkl}$ ,  $b_{mnrkl}$ ,  $c_{mnrkls}$ ,  $\alpha_{mn}$ ,  $\beta_{mnk}$ ,  $d_m$ ,  $a$  and  $\kappa_{mn}$  are constants for the characterization of materials from a constitutive point of view and these obey the following relations of symmetry:

$$\begin{aligned} a_{mnkl} &= a_{klmn} = a_{nmkl}, & b_{mnrkl} &= b_{nmrkl} = b_{mnrk}, \\ c_{mnrkls} &= c_{klsnmr} = c_{nmrkl}, & \alpha_{mn} &= \alpha_{nm}, & \beta_{mnr} &= \beta_{nmr}, & \kappa_{mn} &= \kappa_{nm}. \end{aligned} \quad (5)$$

From Clausius-Duhem inequality, called also the inequality of production of entropy, we can write:

$$\kappa_{mn}\dot{\xi}_m\dot{\xi}_n \geq 0, \quad \forall \dot{\xi}_m.$$

In all well-defined points of the set  $\partial D$  we consider a surface traction of components  $t_k$  and a scalar heat flux denoted by  $q$ , with the help of notations:

$$t_k = t_{kl}n_l, \quad q = q_l n_l,$$

where,  $n_l$  are the components of the normal vector of  $\partial D$ .

Together with the differential relations (1-4) we introduce a system of initial data of the form:

$$v_m(0, x) = a_m(x), \quad \dot{v}_m(0, x) = b_m(x), \quad \vartheta(0, x) = \sigma(x), \quad x \in \bar{D}. \quad (6)$$

Also, there are prescribed the following boundary conditions:

$$\begin{aligned} v_m &= \bar{v}_m \text{ on } [0, t_0) \times \partial D_1, & t_k &= t_{kl}n_l = \bar{t}_k \text{ on } [0, t_0) \times \partial D_1^c, \\ \vartheta &= \bar{\vartheta} \text{ on } [0, t_0) \times \partial D_2, & q &= q_l n_l = \bar{q} \text{ on } [0, t_0) \times \partial D_2^c, \end{aligned} \quad (7)$$

where the instant of time  $t_0$  can take the infinite value.

Also, the subsets  $\partial D_1$  and  $\partial D_2$ , respectively  $\partial D_1^c$  and  $\partial D_2^c$ , are subsurfaces of the set  $\partial D$  satisfying the following conditions:

$$\begin{aligned} \partial D_1 \cap \partial D_1^c &= \partial D_2 \cap \partial D_2^c = \emptyset, \\ \partial D_1 \cup \partial D_1^c &= \partial D_2 \cup \partial D_2^c = \partial D. \end{aligned}$$

Assume that  $a_m$ ,  $b_m$ ,  $\sigma$ ,  $\bar{v}_m$ ,  $\bar{t}_k$ ,  $\bar{\vartheta}$  and  $\bar{q}$  are given, regular functions, on the domain on which they are defined.

In the following considerations we will use some hypotheses of regularity, as follows:

- (i) the functions which define the thermoelastic coefficients of class  $C^1$  on  $\bar{D}$ ;
- (ii)  $\varrho$  is a function of class  $C^0$  on  $\bar{D}$ ;
- (iii)  $F_m$  and  $r$  are functions of class  $C^0$  on  $[0, t_0) \times \bar{D}$ ;
- (iv)  $a_m$ ,  $b_m$  and  $\sigma$  are functions of class  $C^0$  on  $\bar{D}$ ;
- (v)  $\bar{v}_m$  and  $\bar{\vartheta}$  are functions of class  $C^0$  on  $[0, t_0) \times \partial D_1$  and  $[0, t_0) \times \partial D_2$ , respectively;
- (vi)  $\bar{t}_k$  and  $\bar{q}$  are piecewise regular functions on  $[0, t_0) \times \partial D_1^c$  and  $[0, t_0) \times \partial D_2^c$ , respectively and continuous in time.

By using the constitutive equations (2), the motion equation (3) and the equation of energy (4) become

$$\begin{aligned} a_{mnkl}v_{k,ln} + b_{mnrkl}v_{k,lnr} + \alpha_{mn}\dot{\vartheta}_{,n} + b_{mnrkl}v_{k,lnr} + \\ + c_{mnrkls}v_{k,lnrs} + \beta_{mnr}\dot{\vartheta}_{,nr} + F_m = \varrho\ddot{v}_m, \\ (\kappa_{mn}\dot{\vartheta}_{,n})_{,m} + \varrho r = c\dot{\vartheta} + T_0(\alpha_{mn}\dot{v}_{m,n} + \beta_{mnr}\dot{v}_{r,mn}). \end{aligned} \quad (8)$$

Let us denote by  $\mathcal{P}$  the mixed problem with initial and boundary values, from the strain gradient theory of thermoelasticity, in the domain  $D_0 = [0, t_0) \times D$ , which consists of the system of partial differential relations (8) for all  $(t, x) \in D_0$ , the boundary data (7) and the initial relations (6). Any solution of this problem is an ordered array  $(v_m, \vartheta)$ .

### 3. Basic results

Let  $v(t, x)$  and  $w(t, x)$  be two functions of class  $C^1$  regarding the variable  $t$ . By a simple check, we can see that the next equality takes place:

$$\frac{d}{dt} [v(t)\dot{w}(t) - \dot{v}(t)w(t)] = v(t)\ddot{w}(t) - \ddot{v}(t)w(t),$$

where, for simplicity, we have omitted to write the time variable and the spatial variables of the functions  $v(t, x)$  and  $w(t, x)$ .

In the previous identity, we will replace the functions  $v(t, x)$  and  $w(t, x)$  by the functions  $V_m(t, x)$  and  $W_m(t, x)$ , respectively, considering that the new two functions are also of class  $C^1$ , regarding the variable  $t$ . If we integrate the resulting equality, we are led to the following equality, known as Lagrange's identity:

$$\begin{aligned} \int_B \varrho(x) [V_m(t, x)\dot{W}_m(t, x) - \dot{V}_m(t, x)W_m(t, x)] dV = \\ = \int_D \varrho(x) [V_m(0, x)\dot{W}_m(0, x) - \dot{V}_m(0, x)W_m(0, x)] dV + \\ + \int_0^t \int_B \varrho(x) [V_m(\tau, x)\ddot{W}_m(\tau, x) - \ddot{V}_m(\tau, x)W_m(\tau, x)] dV d\tau. \end{aligned} \tag{9}$$

We introduce the following notations:

$$\begin{aligned} w_m = v_m^{(2)} - v_m^{(1)}, \quad \mu = \vartheta^{(2)} - \vartheta^{(1)} \\ t_{mn} = t_{mn}^{(2)} - t_{mn}^{(1)}, \quad \sigma_{mnk} = \sigma_{mnk}^{(2)} - \sigma_{mnk}^{(1)}, \quad S = \eta^{(2)} - \eta^{(1)}, \\ p_m = q_m^{(2)} - q_m^{(1)}, \quad f_m = F_m^{(2)} - F_m^{(1)}, \quad R = r^{(2)} - r^{(1)}, \end{aligned} \tag{10}$$

in which we denoted by  $(v_m^{(\nu)}, \vartheta^{(\nu)})$ ,  $\nu = 1, 2$  two solutions that verify of the mixed problem  $\mathcal{P}$ , corresponding to the same boundary relations and same initial relations, but to heat supplies and to different body forces, namely:  $(F_m^{(\nu)}, r^{(\nu)})$ ,  $\nu = 1, 2$ , respectively.

The constitutive equations become:

$$\begin{aligned} t_{mn} &= a_{mnkl}w_{k,l} + b_{mnrkl}w_{r,kl} + \alpha_{mn}\mu, \\ \sigma_{mnr} &= b_{klmnr}w_{k,l} + c_{mnrkls}w_{k,ls} + \beta_{mnr}\mu, \\ \eta &= -\alpha_{mn}v_{m,n} - \beta_{mnr}v_{r,mn} + \frac{a}{T_0}\mu - b_m\mu_{,m}, \\ p_m &= T_0(b_m\dot{\mu} - \kappa_{mn}\mu_{,n}). \end{aligned} \tag{11}$$

In this way, we see that the differences  $(w_m, \mu)$  verify the equations and conditions that follow:

- the equation of motion:

$$\begin{aligned} \varrho\ddot{w}_m = a_{mnkl}w_{k,ln} + b_{mnrkl}w_{r,kln} + \alpha_{mn}\mu_{,n} + \\ + b_{klmnr}w_{r,klm} + c_{klsmnr}w_{r,nskl} + \beta_{mnr}\mu_{,nr} + f_m; \end{aligned} \tag{12}$$

- the equation of energy:

$$a\dot{\chi} + \vartheta_0(\alpha_{mn}\dot{w}_{m,n} + \beta_{mnr}\dot{w}_{r,mn}) = (\kappa_{mn}\mu_{,n})_{,m} + \varrho R; \tag{13}$$

- the initial conditions:

$$w_m(0, x) = 0, \quad \dot{w}_m(0, x) = 0, \quad \mu(0, x) = 0, \quad x \in \bar{D}; \tag{14}$$

- the boundary conditions:

$$\begin{aligned} w_m(t, x) &= 0 \text{ on } [0, t_0) \times \partial D_1, \quad t_{mk}(t, x)n_k = 0 \text{ on } [0, t_0) \times \partial D_1^c, \\ \mu(t, x) &= 0 \text{ on } [0, t_0) \times \partial D_2, \quad p_k(t, x)n_k = 0 \text{ on } [0, t_0) \times \partial D_2^c. \end{aligned} \tag{15}$$

Now, we can find a Lagrange identity corresponding to the difference of two solutions of the mixed initial boundary value problem  $\mathcal{P}$ .

**Theorem 1.** *Let us consider the difference  $(w_m, \chi)$  of two solutions of the mixed problem  $\mathcal{P}$ . Corresponding to this difference the Lagrange identity receives the form that follows:*

$$\begin{aligned} &2 \int_D \varrho w_m(t) \dot{w}_m(t) dV + \int_D \frac{1}{T_0} \kappa_{mn} \left( \int_0^t \mu_{,m}(s) ds \right) \left( \int_0^t \mu_{,n}(s) ds \right) dV = \\ &= \int_0^t d\tau \int_D \varrho [w_m(2t - \tau) f_m(\tau) - w_m(\tau) f_m(2t - \tau)] dV + \\ &\quad + \int_0^t \int_D \frac{\varrho}{T_0} \left[ \mu(\tau) \int_0^{2t-\tau} R(s) ds - \mu(2t - \tau) \int_0^\tau R(s) ds \right] dV d\tau, \quad t \in [0, \frac{t_0}{2}). \end{aligned} \tag{16}$$

First application of the identity (16) is the result of uniqueness, from next theorem, with regards to the solution of the mixed initial boundary value problem  $\mathcal{P}$ . For this we need to suppose that the tensor of conductivity  $\kappa_{mn}$  is positive definite, that is, we have

$$\kappa_{mn} x_m x_n \geq k_0 x_m x_m, \quad \forall x_m,$$

where  $k_0 > 0$  is a constant.

**Theorem 2** *Let us suppose that the relations of symmetry (5) are fulfilled and the set  $\partial D_2$  is not empty or the specific heat  $a(x)$  is nonzero on  $B$ . Thus, the mixed problem  $\mathcal{P}$  in strain gradient thermoelasticity has at most one solution.*

In our next theorem, we will prove the first result regarding the continuous dependence of the solutions of the problem  $\mathcal{P}$ , in relation to heat supply and body force, on the compact sub-intervals of the interval  $[0, t_0)$ .

To this aim, we consider two solutions  $(v_m^{(\nu)}, \vartheta^{(\nu)})$ ,  $\nu = 1, 2$ , of the mixed problem  $\mathcal{P}$ , corresponding to the same boundary data and the same initial conditions, but to different different body force and heat supply, namely:  $(F_m^{(\nu)}, r^{(\nu)})$ ,  $\nu = 1, 2$ . We will use the notations:

$$f_m = F_m^{(2)} - F_m^{(1)}, \quad R = r^{(2)} - r^{(1)}.$$

**Theorem 3** *Suppose that the relations of symmetry (5) take place. Assuming there exist the constants  $Q_1, Q_2, M_1$  and  $M_2$  and there exists  $t^* \in (0, t_0)$  so that*

$$\begin{aligned} \int_0^{t^*} \int_D \varrho w_m(t) w_m(t) dV dt &\leq Q_1^2, \quad \int_0^{t^*} \int_D \frac{\varrho}{T_0} \mu^2(t) dV dt \leq Q_2^2, \\ \int_0^{t^*} \int_D \varrho f_m(t) f_m(t) dV dt &\leq M_1^2, \quad \int_0^{t^*} \int_D \frac{\varrho}{T_0} \left( \int_0^t R(\xi) d\xi \right)^2 dV dt \leq M_2^2, \end{aligned} \tag{17}$$

we obtain the following estimation:

$$\begin{aligned} &\int_D \varrho w_m(\tau) w_m(\tau) dV + \int_0^\tau \int_D \frac{1}{\vartheta_0} \kappa_{mn} \left( \int_0^s \mu_{,m}(\xi) d\xi \right) \left( \int_0^s \mu_{,n}(\xi) d\xi \right) dV d\tau \leq \\ &\leq t^* Q_1 \left[ \int_0^{t^*} \int_D \varrho f_m(s) f_m(s) dV ds \right]^{1/2} + t^* Q_2 \left[ \int_0^{t^*} \int_D \frac{\varrho}{T_0} \left( \int_0^s R(\xi) d\xi \right)^2 dV ds \right]^{1/2}, \end{aligned} \tag{18}$$

where  $\tau \in [0, t^*/2)$  and  $w_m(\tau)$  and  $\mu(\tau)$  are the differences defined in (27).

Now, we intend to prove the continuous dependence of the solutions with respect to the initial data.

To this aim, we consider two solutions of the mixed problem  $\mathcal{P}$

$$(v_m^{(1)}, \vartheta^{(1)}), (v_m^{(1)} + w_m, \vartheta^{(1)} + \mu)$$

corresponding to the same boundary conditions and to same heat supply and body force, but to different initial conditions, namely

$$(v_m^{(1)}(0), \dot{v}_m^{(1)}(0), \vartheta^{(1)}(0)), (v_m^{(1)}(0) + \alpha_m^0, \dot{v}_m^{(1)}(0) + \beta_m^0, \vartheta^{(1)}(0) + \delta^0),$$

where the perturbations  $(\alpha_m^0, \beta_m^0, \delta^0)$  satisfy the following conditions: there exist the constants  $M_3$  and  $M_4$  such that:

$$\int_D \varrho (\alpha_m^0 \alpha_m^0 + \beta_m^0 \beta_m^0) dV \leq M_3^2, \int_B \frac{T_0}{\varrho} \eta_0^2 dV \leq M_4^2$$

where we denoted by  $\eta_0$  the following expression:

$$\eta_0 = \frac{a}{T_0} \delta^0 - \alpha_{mn} \alpha_{n,m}^0 - \beta_{mnk} \beta_{k,mn}^0.$$

With the help of the perturbation  $w_m$  and  $\mu$ , we introduce the functions  $V_m(t)$  and  $\Theta(t)$  by means of the notations:

$$V_m(t) = \int_0^t \int_0^s w_m(\tau) d\tau ds, \Theta(t) = \int_0^t \int_0^s \mu(\tau) d\tau ds. \tag{19}$$

**Theorem 4** *Suppose that symmetry relations take place and the functions  $(V_m, \Theta)$  satisfy the restrictions (30). Then the following estimate takes place:*

$$\begin{aligned} & \int_D \varrho V_m(t) V_m(t) dV + \int_0^t \int_D \frac{1}{T_0} \kappa_{mn} \left( \int_0^s \Theta_{,m}(\xi) d\xi \right) \left( \int_0^s \Theta_{,n}(\xi) d\xi \right) dV ds \leq \\ & \leq t^* Q_1 \left[ \left( t^* + \frac{t^{*2}}{2} \right) \int_D \varrho \alpha_m^0 \alpha_m^0 dV + \left( \frac{t^{*2}}{2} + \frac{t^{*3}}{3} \right) \int_D \varrho \beta_m^1 \beta_m^1 dV \right]^{1/2} + \\ & + t^{*7/2} Q_2 \frac{1}{\sqrt{20}} \left( \int_D \frac{T_0}{\varrho} \eta_0^2 dV \right)^{1/2}, \quad t \in \left[ 0, \frac{t^*}{2} \right]. \end{aligned} \tag{20}$$

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