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Tykhonov triples and convergence results for history-dependent variational inequalities

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Abstract. We deal with the Tykhonov well-posedness of a time-dependent variational inequality defined on the unbounded interval of time $\mathbb{R}_+ = [0, +\infty)$, governed by a history-dependent operator. To this end we introduce the concept of Tykhonov triple, provide three relevant examples, then we state and prove the corresponding well-posedness results. This allows us to deduce various corollaries which illustrate the continuous dependence of the solution with respect to the data. Our results provide mathematical tools in the analysis of a large number of history-dependent problems which arise in Mechanics, Physics and Engineering Sciences. To give an example, we consider a mathematical model which describes the equilibrium of a viscoelastic body in frictionless contact with a rigid foundation.

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1 Introduction

The concept of well-posedness in the sense of Tykhonov was introduced in [23] for a minimization problem and then it has been generalized for different optimization problems. References in the field include [10, 26], [11] and [2], where the concepts of extended well-posendess, Levitin-Polyak well-posendess and generic well-posedness for minimization problems have been introduced, respectively. For more details on well-posedness of optimization problems we refer the readers to the monographs [3, 15]. The well-posedness in the sense of Tykhonov (well-posedness, for short) has been extended to variational inequalities in [13, 14] and to hemivariational inequalities in [6]. General results concerning the well-posedness of various classes of inequalities can be found in [5, 9, 20, 24, 25], for instance. A careful analysis of the above mentionned papers reveals the fact that the concept of well-posedness for a given problem is based on two main ingredients: the existence and uniqueness of solution of the problem and the convergence to it of any approximating sequence. Based on this remark, a general framework which unifies the view on well-posedness for abstract problems in metric spaces was recently considered in [21, 22].

The approch used in [22] was based on the concept of Tykhonov triple $\mathcal{T} = (I, \Omega, C)$ where I is a set of parameters, Ω represents a family approximating sets and C is a set of

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sequences which defines a criterion of convergence. Following the arguments presented there, the well-posedness of a problem $\mathcal{P}$ with respect to the Tykhonov triple $\mathcal{T}$ implicitely provides a convergence result, namely the convergence of the approximating sequences. This remark suggests that the Tykhonov triples could be used to prove some convergence results, in the following way: given a problem $\mathcal{P}$ which has a unique solution u and given a sequence $\{u_n\}$, we are looking for an appropriate Tykhonov triple $\mathcal{T}$ with whom problem $\mathcal{P}$ is well posed and, moreover, $\{u_n\}$ is a $\mathcal{T}$-approximating sequence. Then, using the well-posedness of $\mathcal{P}$ with respect to $\mathcal{T}$, we deduce that $\{u_n\}$ converges to u. Note that this strategy can be applied for a general problem $\mathcal{P}$ which could be either a nonlinear equation, a variational inequality, a hemivariational inequality, an inclusion, a sweeping process, a saddle point, a fixed point or an optimization problem, defined on a metric space. Details, examples and appropriate reference can be found in our recent papers [8, 22]. Nevertheless, this represents only a theoretical principle since the choice of such an appropriate Tykhonov triple remains an open question which depends on the problem we consider and the convergence result we are interested in, as well.

In this current paper we intend to illustrate the principle above and to underlie the importance of the choice of the Tykhonov triple in the study of various convergence results. Even if the method we present can be used in the study of various problems, as mentioned, we restrict here to the study of a particular class of time-dependent variational inequalities in Hilbert spaces, for simplicity. Everywhere in this paper we use $\mathbb{N}$ to represent the set of positive integers, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{R}_+$ to denote the set of real positive numbers, i.e., $\mathbb{R}_+ = [0, +\infty)$. Unless stated otherwise, X will be a real Hilbert space. We use $(\cdot,\cdot)_X$, $\|\cdot\|_X$ and 0_X for the inner product, the associated norm and the zero element of X, respectively. We also denote by $C(\mathbb{R}_+; X)$ the space of continuous functions on $\mathbb{R}_+$ with values in X and, for $U \subset X$, we use $C(\mathbb{R}_+; U)$ for the set of continuous functions on $\mathbb{R}_+$ with values in U. Recall that $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e., as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. The convergence of a sequence $\{v_n\}$ to an element v, in the space $C(\mathbb{R}_+; X)$, can be described as follows:

$$\begin{cases} v_n \to v \ \text{ in } \ C(\mathbb{R}_+; X) \ \text{ as } \ n \to \infty \ \text{ if and only if} \\ \max\limits_{t\in[0,m]} \|v_n(t) - v(t)\|_X \to 0 \ \text{ as } \ n \to \infty, \ \text{ for all } \ m \in \mathbb{N}. \end{cases} \tag{1.1}$$

We also mention that, unless stated otherwise, all the limits below are considered as $n \to \infty$, even if we do not mention it explicitly.

Let $K \subset X$, $A \colon X \to X$, $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ and $f : \mathbb{R}_+ \to X$. With these data, the time-dependent inequality problem we study in this paper is stated as follows.

Problem $\mathcal{P}$. *Find a function $u \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:*

$$(Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X \geq (f(t), v - u(t))_X \quad \text{for all } v \in K. \tag{1.2}$$

Note that here and below when no confusion arises, we use the shorthand notation $\mathcal{S}u(t)$ to represent the value of the function $\mathcal{S}u$ at the point t, i.e., $\mathcal{S}u(t) = (\mathcal{S}u)(t)$.

Our aim in this paper is three fold. The first one is to study the well-posedness of the inequality (1.2) with respect to various Tykhonov triples. This will allow us to deduce various convergence results, which represents our second aim. Finally, our third aim is to illustrate how these results could be applied in the study of variational models of contact. Process of contact between deformable bodies arise in industry and daily life. Their importance in various real word applications made that, in the last decades, a considerable effort has been put into their modeling, analysis and numerical simulations. The literature on this field includes [4, 7, 12, 16] and, more recently, [1, 17, 18].

The rest of the paper is structured as follows. In Section 2 we introduce the concept of well-posedness for the history-dependent variational inequality (1.2) and then we prove its well-posedness with respect to three relevant Tykhonov triples. The proofs are based on arguments of monotonicity, various estimates and the properties of history-dependent operators. We then use these well-posedness results in Section 3. There, we prove various convergence results which show the continuous dependence of the solution with respect to the data. Next, Section 4, we present additional examples, including an example arising in Contact Mechanics. We end this paper with Section 5 in which we present some concluding remarks.

2 Tykhonov well-posedness results

We start by recalling an existence and uniqueness result in the study of Problem $\mathcal{P}$. To this end, we consider the following hypotheses.

$$K \text{ is a nonempty closed convex subset of } X. \tag{2.1}$$

$$\left\{\begin{array}{l} A\colon X \to X \text{ is a strongly monotone Lipschitz continuous operator, i.e.,} \\ \text{there exists } m_A,\ M_A > 0 \text{ such that} \\ \quad \text{(a)}\ \ (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2, \\ \quad \text{(b)}\ \ \|Au - Av\|_X \leq M_A \|u - v\|_X, \\ \quad \text{for all } u,\ v \in X. \end{array}\right. \tag{2.2}$$

$$\left\{\begin{array}{l} \mathcal{S}\colon C(I;X) \to C(I;X) \text{ is a history-dependent operator, i.e.,} \\ \text{for any } \ m \in \mathbb{N},\ \text{there exists } L^m > 0 \text{ such that} \\ \|\mathcal{S}u(t) - \mathcal{S}v(t)\|_X \leq L^m \displaystyle\int_0^t \|u(s) - v(s)\|_X \, ds \\ \text{for all } u,\ v \in C(\mathbb{R}_+;X),\ t \in [0,m]. \end{array}\right. \tag{2.3}$$

$$f \in C(\mathbb{R};X). \tag{2.4}$$

We have the following existence and uniquenss result.

Theorem 2.1 *Assume that* (2.1)–(2.4) *hold. Then, Problem $\mathcal{P}$ has a unique solution $u \in C(I;K)$.*

A proof of Theorem 2.1 can be found in [17, 19]. It is based on standard arguments of elliptic variational inequalities and a fixed point argument for history-dependent operators.

We now move to the concepts of Tykhonov triple and well-posedness of inequality (1.2) with respect to a given Tykhonov triple. These concepts have been introduced in [22] in the framework of an abstract problem on metric spaces. Here we adapt them to the history-dependent variational inequality (1.2) and, to this end, we denote by $\mathcal{X}$ the the set of nonempty subsets of the space $\mathcal{X}$. Moreover, for any set J we use the notation $\mathcal{R}(J)$ for the set of sequences with elements in J.

Definition 2.1 *A Tykhonov triple is a mathematical object of the form* $\mathcal{T} = (I, \Omega, C)$ *where* I *is a given nonempty set,* $\Omega : I \to X$ *is a multivalued mapping and* C *is a nonempty subset of the set* $\mathcal{R}(I)$.

Below in this paper we refer to I as the set of parameters. A typical element of I will be denoted by θ. We refer to the family of sets $\{\Omega(\theta)\}_{\theta\in I}$ as the family of approximating sets and, moreover, we say that C defines the criterion of convergence.

Definition 2.2 *Given a Tykhonov triple* $\mathcal{T} = (I, \Omega, C)$, *a sequence* $\{u_n\} \subset C(\mathbb{R}_+; X)$ *is called a* $\mathcal{T}$*-approximating sequence if there exists a sequence* $\{\theta_n\} \in C$ *such that* $u_n \in \Omega(\theta_n)$ *for each* $n \in \mathbb{N}$.

Note that approximating sequences always exist since, by assumption, $C \neq \emptyset$ and for any sequence $\{\theta_n\} \in C$ and any $n \in \mathbb{N}$, the set $\Omega(\theta_n)$ is not empty.

We proceed with the following definition.

Definition 2.3 *Given a Tykhonov triple* $\mathcal{T} = (I, \Omega, C)$, *Problem* $\mathcal{P}$ *is said to be* $\mathcal{T}$*-well-posed if it has a unique solution and every* $\mathcal{T}$*-approximating sequence converges in the space* $C(\mathbb{R}_+; X)$ *to its solution.*

We now complete Definition 2.3 with the following comments.

First, we remark that the concept of approximating sequence defined above depends on the Tykhonov triple $\mathcal{T}$ and, for this reason, we use the terminology "$\mathcal{T}$-approximating sequence". As a consequence, the concept of well-posedness depends on the Tykhonov triple $\mathcal{T}$ and, therefore, we refer to it as "well-posedness with respect to $\mathcal{T}$" or "$\mathcal{T}$-well-posedness", for short.

Second, assume that Problem $\mathcal{P}$ has a unique solution u, $\mathcal{T}$ is a Tykhonov triple and consider two sets of sequences $\mathcal{R}_{\mathcal{P}}$ and $\mathcal{R}_{\mathcal{T}}$, defined as follows:

$$\mathcal{R}_{\mathcal{P}} = \{\,\{u_n\} \subset C(\mathbb{R}_+; X)\ :\ u_n \to u \ \text{ in }\ C(\mathbb{R}_+; X)\,\}, \tag{2.5}$$

$$\mathcal{R}_{\mathcal{T}} = \{\,\{u_n\} \subset C(\mathbb{R}_+; X)\ :\{u_n\} \text{ is a } \mathcal{T}\text{-approximating sequence}\,\}. \tag{2.6}$$

Then it is easy to see that the well-posedness of a Problem $\mathcal{P}$ with respect to the Tykhonov triple $\mathcal{T}$ is equivalent with the inclusion $\mathcal{R}_{\mathcal{T}} \subset \mathcal{R}_{\mathcal{P}}$, i.e.,

$$\text{Problem } \mathcal{P} \text{ is } \mathcal{T}\text{-well-posed if and only if } \ \mathcal{R}_{\mathcal{T}} \subset \mathcal{R}_{\mathcal{P}}. \tag{2.7}$$

Therefore, the well-posedness of Problem $\mathcal{P}$ with respect to a Tykhonov triple $\mathcal{T}$ provides implicitly some convergence results, since it guarantees the convergence of any $\mathcal{T}$-approximating sequence to the solution of $\mathcal{P}$. The choice of the Tykhonov triple $\mathcal{T}$ is crucial for the analysis of Problem $\mathcal{P}$, since a convenient structure of $\mathcal{T}$ could provide an unified proof for various convergence results, as shown in the next section.

Below in this paper we shall use the following notation:

a) an element of I will be denoted by θ when $I \subset \mathbb{R}$, and by $\boldsymbol{\theta} = \{\theta^m\}_m$ when $I \subset \mathcal{R}(\mathbb{R})$;

b) an element of $\mathcal{R}(I)$ will be denoted by $\boldsymbol{\theta} = \{\theta^m\}_m$ (or $\boldsymbol{\theta} = \{\theta^m\}$) when $I \subset \mathbb{R}$, and by $\boldsymbol{\theta} = \{\boldsymbol{\theta}_n\}_n$ (or $\boldsymbol{\theta} = \{\boldsymbol{\theta}_n\}$) with $\boldsymbol{\theta}_n = \{\theta_n^m\}_m$ when $I \subset \mathcal{R}(\mathbb{R})$.

We now construct three relevant triples in the in the study of Problem $\mathcal{P}$.

Example 2.1 *Keep the assumption in Theorem* 2.1 *and take* $\mathcal{T}_1 = (I_1, \Omega_1, C_1)$ *where*

$$I_1 = \mathbb{R}_+, \tag{2.8}$$

$$C_1 = \{\,\{\theta_n\}_n \ :\ \theta_n \in I_1 \ \ \forall\, n \in \mathbb{N}, \quad \theta_n \to 0 \quad \text{as} \quad n \to \infty\,\} \tag{2.9}$$

and, for each $\theta \in I_1$*, the set* $\Omega_1(\theta)$ *is defined as follows:*

$$\left\{\begin{array}{l} u \in \Omega_1(\theta) \text{ if and only if } u \in C(\mathbb{R}_+; K) \text{ and} \\[2mm] (Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X + \theta\|v - u(t)\|_X \\ \quad \geq (f(t), v - u(t))_X \quad \text{for all } v \in K, \ \ t \in \mathbb{R}_+. \end{array}\right. \tag{2.10}$$

Note that for each $\theta \in I_1$ *the solution* u *obtained in Theorem* 2.1 *belongs to* $\Omega_1(\theta)$ *and, therefore,* $\Omega_1(\theta) \neq \emptyset$*. Therefore, according to Definition* 2.1*,* $\mathcal{T}_1$ *is a Tykhonov triple.*

Example 2.2 *Keep the assumption in Theorem* 2.1 *and take* $\mathcal{T}_2 = (I_2, \Omega_2, C_2)$ *where*

$$I_2 = \mathbb{R}_+, \tag{2.11}$$

$$C_2 = \{\,\{\theta_n\}_n \ :\ \theta_n \in I \ \ \forall\, n \in \mathbb{N}, \quad \theta_n \to 0 \quad \text{as} \quad n \to \infty\,\} \tag{2.12}$$

and, for each $\theta \in I_2$*, the set* $\Omega_2(\theta)$ *is defined as follows:*

$$\left\{\begin{array}{l} u \in \Omega_2(\theta) \text{ if and only if } u \in C(\mathbb{R}_+; K) \text{ and} \\[2mm] (Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X + \theta(\|u(t)\|_X + 1)\|v - u(t)\|_X \\ \quad \geq (f(t), v - u(t))_X \quad \text{for all } v \in K, \ \ t \in \mathbb{R}_+. \end{array}\right. \tag{2.13}$$

Note that, again, using Theorem 2.1 *it follows that* $\Omega_2(\theta) \neq \emptyset$*, for each* $\theta \in I_2$*.*

Example 2.3 *Keep the assumption in Theorem* 2.1 *and take* $\mathcal{T}_3 = (I_3, \Omega_3, C_3)$ *where*

$$I_3 = \{\,\boldsymbol{\theta} = \{\theta^m\}_m \ :\ \theta^m \in \mathbb{R}_+ \quad \forall\, m \in \mathbb{N}\,\}, \tag{2.14}$$

$$\begin{aligned} C_3 = \{\,\{\boldsymbol{\theta}_n\}_n \ :\ & \boldsymbol{\theta}_n = \{\theta_n^m\}_m \in I_3 \ \ \forall\, n \in \mathbb{N}, \\ & \theta_n^m \to 0 \quad \text{as} \quad n \to \infty, \quad \forall\, m \in \mathbb{N}\,\} \end{aligned} \tag{2.15}$$

and, for each $\boldsymbol{\theta} = \{\theta^m\}_m \in I_3$*, the set* $\Omega_3(\boldsymbol{\theta})$ *is defined as follows:*

$$\left\{\begin{array}{l} u \in \Omega_3(\boldsymbol{\theta}) \text{ if and only if } u \in C(\mathbb{R}_+; K) \text{ and} \\[2mm] (Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X + \theta^m\|v - u(t)\|_X \\ \quad \geq (f(t), v - u(t))_X \quad \text{for all } v \in K, \ \ t \in [0, m], \ \ m \in \mathbb{N}. \end{array}\right\} \tag{2.16}$$

Note that, again, using Theorem 2.1 *it follows that* $\Omega_3(\boldsymbol{\theta}) \neq \emptyset$*, for each* $\boldsymbol{\theta} \in I_3$*.*

Our main result in this section is the following.

Theorem 2.2 *Assume that* (2.1)–(2.4) *hold. Then Problem* $\mathcal{P}$ *is well-posed with respect to the Tykhonov triples* $\mathcal{T}_1$*,* $\mathcal{T}_2$ *and* $\mathcal{T}_3$ *in Example* 2.1*,* 2.2 *and* 2.3*, respectively.*

Proof. First, we recall that the existence of a unique solution to problem $\mathcal{P}$, needed for the well-posedness of Problem $\mathcal{P}$ with any Tykhonov triple, follows from Theorem 2.1.

Consider now the Tykhonov triple $\mathcal{T}_3$ in Example 2.3 and let $\{u_n\}$ be a $\mathcal{T}_3$-approximating sequence. Then, using Definition 2.2 we deduce that there exists a sequence $\{\boldsymbol{\theta}_n\}_n$ such that $\boldsymbol{\theta}_n = \{\theta_n^m\}_m \in \mathcal{R}(\mathbb{R}_+)$ for all $n \in \mathbb{N}$,

$$\theta_n^m \to 0 \quad \text{as} \quad n \to \infty, \quad \forall\, m \in \mathbb{N}, \tag{2.17}$$

and, moreover, $u_n \in \Omega_3(\boldsymbol{\theta}_n)$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and let $t \in [0, m]$. Then, using the definition (2.16) of the set $\Omega_3(\boldsymbol{\theta}_n)$ it follows that $u_n \in C(\mathbb{R}_+; K)$ and, moreover,

$$\begin{aligned} &(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X + \theta_n^m \|v - u_n(t)\|_X \\ &\qquad \geq (f(t), v - u_n(t))_X \quad \text{for all } v \in K. \end{aligned} \tag{2.18}$$

We now take $v = u(t)$ in (2.18) then $v = u_n(t)$ in (1.2) and add the resulting inequalities to find that

$$\begin{aligned} &(\mathcal{S}u_n(t) - \mathcal{S}u(t), u(t) - u_n(t))_X + \theta_n^m \|u_n(t) - u(t)\|_X \\ &\qquad \geq (Au_n(t) - Au(t), u_n(t) - u(t))_X. \end{aligned}$$

Therefore, using assumption (2.2)(a) we deduce that

$$m_A \|u_n(t) - u(t)\|_X^2 \leq \|\mathcal{S}u_n(t) - \mathcal{S}u(t)\|_X \|u_n(t) - u(t)\|_X + \theta_n^m \|u_n(t) - u(t)\|_X.$$

This inequality combined with condition (2.3) yields

$$\|u_n(t) - u(t)\|_X \leq \frac{\theta_n^m}{m_A} + \frac{L^m}{m_A} \int_0^t \|u_n(s) - u(s)\|_X \, ds$$

and, using the Gronwall argument, it follows that

$$\|u_n(t) - u\|_X \leq \frac{\theta_n^m}{m_A} e^{\frac{L^m}{m_A} t}. \tag{2.19}$$

Inequality (2.19) and convergence (2.17) show that

$$\max_{t \in [0,m]} \|u_n(t) - u(t)\|_X \to 0 \ \text{ as } \ n \to \infty$$

and, since $m \in \mathbb{N}$ is arbitrary, using (1.1) we deduce that

$$u_n \to u \ \text{ in } \ C(\mathbb{R}_+; X) \ \text{ as } \ n \to \infty. \tag{2.20}$$

The convergence (2.20) combined with Definition 2.3 imply that Problem $\mathcal{P}$ is well-posed with respect to the Tykhonov triple $\mathcal{T}_3$ in Example 2.3.

We now focus on the Tykhonov triple $\mathcal{T}_2$ in Example 2.2 and, to this end we use a claim that we state here and proof at the end of this section.

Claim 1. *Let $\{u_n\}$ be a $\mathcal{T}_2$-approximating sequence. Then, for each $m \in \mathbb{N}$ there exists $Z^m > 0$ such that*

$$\|u_n(t)\|_X \leq Z^m \qquad \text{for all } \ t \in [0, m], \ n \in \mathbb{N}. \tag{2.21}$$

Assume now that $\{u_n\}$ is a $\mathcal{T}_2$-approximating sequence. Then, using Definition 2.2 and (2.13) we deduce that there exists a sequence $\{\alpha_n\} \in \mathcal{R}(\mathbb{R}_+)$ such that $\alpha_n \to 0$ and, for any $n \in \mathbb{N}$, the following inequality holds:

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X + \alpha_n(\|u_n(t)\|_X + 1)\|v - u_n(t)\|_X \tag{2.22}$$
$$\geq (f(t), v - u_n(t))_X \quad \text{for all } v \in K, \ \ t \in \mathbb{R}_+.$$

Consider the sequence $\{\boldsymbol{\theta}_n\}$ where

$$\boldsymbol{\theta}_n = \{\theta_n^m\}_m \in \mathcal{R}(\mathbb{R}_+), \quad \theta_n^m = \alpha_n(Z^m + 1) \quad \text{for all } \ m, n \in \mathbb{N}. \tag{2.23}$$

Then, (2.21)–(2.23) imply that

$$(Au_n(t), v - u(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X + \theta_n^m\|v - u_n(t)\|_X$$
$$\geq (f(t), v - u_n(t))_X \quad \text{for all } v \in K, \ m, n \in \mathbb{N}, \ t \in [0, m]$$

and, since $\theta_n^m \to 0$ as $n \to \infty$, for any $m \in \mathbb{N}$, (2.15) and (2.16) imply that $\{u_n\}$ is a $\mathcal{T}_3$-approximating sequence. We now use the notation (2.6) to conclude that

$$\mathcal{R}_{\mathcal{T}_2} \subset \mathcal{R}_{\mathcal{T}_3}. \tag{2.24}$$

On the other hand, using (2.10) and (2.13) it is easy to see that $\Omega_1(\theta) \subset \Omega_2(\theta)$ for any $\theta \geq 0$ and, therefore, (2.6) implies that

$$\mathcal{R}_{\mathcal{T}_1} \subset \mathcal{R}_{\mathcal{T}_2}. \tag{2.25}$$

Finally, since Problem $\mathcal{P}$ is $\mathcal{T}_3$-well-posed, we deduce from (2.7) that

$$\mathcal{R}_{\mathcal{T}_3} \subset \mathcal{R}_{\mathcal{P}}. \tag{2.26}$$

The well-posedness of Problem $\mathcal{P}$ with respect the Tykhonov triples $\mathcal{T}_1$ and $\mathcal{T}_2$ is now a direct consequence of (2.7) and inclusions (2.24)–(2.26). □

We end this section with the proof of the bound (2.21).

Proof of Claim 1. Let $\{u_n\}$ be a $\mathcal{T}_2$-approximating sequence. Then, using Definition 2.2, (2.12) and (2.13), we deduce that there exists a sequence $\{\theta_n\} \in \mathcal{S}(\mathbb{R}_+)$ such that $\theta_n \to 0$ and, for any $n \in \mathbb{N}$, the following inequality holds:

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X + \theta_n(\|u_n(t)\|_X + 1)\|v - u_n(t)\|_X \tag{2.27}$$
$$\geq (f(t), v - u_n(t))_X \quad \text{for all } v \in K, \ \ t \in \mathbb{R}_+.$$

Let $m \in \mathbb{N}$, $t \in [0, m]$, $n \in \mathbb{N}$ and $u_0 \in K$. Then, (2.27) implies that

$$(Au_n(t), u_n(t) - u_0)_X \leq (\mathcal{S}u_n(t), u_0 - u_n(t))_X$$
$$+\theta_n(\|u_n(t)\|_X + 1)\|u_0 - u_n(t)\|_X + (f(t), u_n(t) - u_0)_X.$$

Using now this inequality and assumption (2.2)(a) we write

$$m_A\|u_n(t) - u_0\|_X^2 \leq (Au_n(t), u_n(t) - u_0)_X - (Au_0, u_n(t) - u_0)_X$$
$$\leq (\mathcal{S}u_n(t), u_0 - u_n(t))_X + \theta_n(\|u_n(t)\|_X + 1)\|u_0 - u_n(t)\|_X$$
$$+(f(t), u_n(t) - u_0)_X + (Au_0, u_0 - u_n(t))_X$$

which implies that

$$m_A\|u_n(t)-u_0\|_X \le \|\mathcal{S}u_n(t)\|_X + \theta_n(\|u_n(t)\|_X+1) + \|f(t)\|_X + \|Au_0\|_X$$

and, moreover,

$$\begin{aligned} m_A\|u_n(t)-u_0\|_X &\le \|\mathcal{S}u_n(t)-\mathcal{S}u_0(t)\|_X + \|\mathcal{S}u_0(t)\|_X \\ &+\theta_n(\|u_n(t)-u_0\|_X + \|u_0\|_X+1) + \|f(t)\|_X + \|Au_0\|_X. \end{aligned} \tag{2.28}$$

Note that here and below in this paper we keep the notation u_0 for the constant function $t \mapsto u_0$ for all $t \in \mathbb{R}$ and, therefore, notation $\mathcal{S}u_0$ makes sense.

Next, since $\theta_n \to 0$, for n large enough we can assume that $\theta_n \le \frac{m_A}{2}$ and, therefore (2.28) yields

$$\begin{aligned} \frac{m_A}{2}\|u_n(t)-u_0\|_X &\le \|\mathcal{S}u_n(t)-\mathcal{S}u_0(t)\|_X + \|\mathcal{S}u_0(t)\|_X \\ &+\frac{m_A}{2}(\|u_0\|_X+1) + \|f(t)\|_X + \|Au_0\|_X. \end{aligned} \tag{2.29}$$

Denote

$$F^m = \max_{t\in[0,m]}\Big(\|\mathcal{S}u_0(t)\|_X + \frac{m_A}{2}(\|u_0\|_X+1) + \|f(t)\|_X + \|Au_0\|_X\Big). \tag{2.30}$$

Then, using (2.29), (2.30) and assumption (2.3) we find that

$$\|u_n(t)-u_0\|_X \le \frac{2L^m}{m_A}\int_0^t \|u_n(s)-u_0\|_X\,ds + \frac{2F^m}{m_A}\,.$$

We now use the Gronwall argument to find that

$$\|u_n(t)-u_0\|_X \le \frac{2F^m}{m_A}e^{\frac{2L^m}{m_A}t}.$$

This inequality implys that there exists $Y^m > 0$ which does not depend on n and t such that

$$\|u_n(t)-u_0\|_X \le Y^m. \tag{2.31}$$

We now use (2.31) to obtain the bound (2.21) with $Z^m = Y^m + \|u_0\|_X$, which concludes the proof. □

3 Convergence results

In this section we use the well-posedness of Problem $\mathcal{P}$ with respect to the Tykhonov triples $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ in order to deduce several convergence results in the study of Problem $\mathcal{P}$. These results are presented as corrolaries of Theorem 2.2. Below in this section we keep the assumptions of this theorem, even if we do not mention it explicitely.

Our first convergence result concerns the dependence of the solution with respect to the function f. To this end we consider a sequence of functions $\{f_n\}$ such that, for each $n \in \mathbb{N}$,

$$f_n \in C(\mathbb{R}_+; X). \tag{3.1}$$

Moreover, we consider the following variational problem.

Problem $\mathcal{P}_n^1$. *Find a function $u_n \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:*

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X \tag{3.2}$$
$$\geq (f_n(t), v - u_n(t))_X \quad \text{for all } v \in K.$$

Then, using Theorem 2.1 it follows that Problem $\mathcal{P}_n^1$ has a unique solution, for each $n \in \mathbb{N}$. Assume now that

$$\left\{\begin{array}{l} \text{(a) For each } n \in \mathbb{N} \text{ there exists } \ \theta_n > 0 \text{ such that} \\ \qquad \|f_n(t) - f(t)\|_X \leq \theta_n \quad \text{for all } \ t \in \mathbb{R}_+. \\ \text{(b) } \ \theta_n \to 0 \ \text{ as } n \to \infty. \end{array}\right. \tag{3.3}$$

We have the following convergence result.

Corollary 3.1 *Assume that* (2.1)–(2.4), (3.1) *and* (3.3) *hold. Then, the solution u_n of Problem $\mathcal{P}_n^1$ converges to the solution u of Problem $\mathcal{P}$, i.e.,*

$$u_n \to u \quad \text{in} \quad C(\mathbb{R}_+; X). \tag{3.4}$$

Proof. Consider the sequence $\{\theta_n\}$ provided by assumption (3.3)(a) and let $n \in \mathbb{N}$, $t \in \mathbb{R}_+$. Inequality (3.2) implies that

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X + (f(t) - f_n(t), v - u_n(t))_X$$
$$\geq (f(t), v - u_n(t))_X \quad \text{for all } \ v \in K$$

and, using (3.3) yields

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X + \theta_n \|v - u_n(t)\|_X \tag{3.5}$$
$$\geq (f(t), v - u_n(t))_X \quad \text{for all } \ v \in K.$$

We now combine (3.5) and (2.10) to see that $u_n \in \Omega_1(\theta_n)$ and, therefore, (3.3)(b) implies that $\{u_n\}$ as a $\mathcal{T}_1$-aproximating sequence for Problem $\mathcal{P}$. We now use Theorem 2.2 and Definition 2.3 to deduce the convergence (1.1), which concludes the proof. □

Our second convergence result concerns the dependence of the solution with respect the operator A. To this end we consider a sequence of operators $\{A_n\}$ such that, for each $n \in \mathbb{N}$, the following condition holds.

$$\left\{\begin{array}{l} A_n \colon X \to X \text{ is a strongly monotone Lipschitz continuous operator, i.e.,} \\ \text{there exists } m_n, M_n > 0 \text{ such that} \\ \quad \text{(a) } \ (A_n u - A_n v, u - v)_X \geq m_n \|u - v\|_X^2, \\ \quad \text{(b) } \ \|A_n u - A_n v\|_X \leq M_n \|u - v\|_X, \\ \quad \text{for all } u, \ v \in X. \end{array}\right. \tag{3.6}$$

Moreover, we consider the following variational problem.

Problem $\mathcal{P}_n^2$. *Find a function $u_n \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:*

$$(A_n u_n(t), v - u_n(t))_X + (\mathcal{S} u_n(t), v - u_n(t))_X \tag{3.7}$$
$$\geq (f(t), v - u_n(t))_X \quad \text{for all } v \in K.$$

Then, using Theorem 2.1 it follows that Problem $\mathcal{P}_n$ has a unique solution, for each $n \in \mathbb{N}$. Assume now that

$$\begin{cases} \text{(a) For each } n \in \mathbb{N} \text{ there exists } \ \theta_n > 0 \text{ such that} \\ \qquad \|A_n u - Au\|_X \leq \theta_n(\|u\|_X + 1) \quad \text{for all } \ u \in X. \\ \text{(b) } \ \theta_n \to 0 \ \text{ as } n \to \infty. \end{cases} \tag{3.8}$$

We have the following convergence result.

Corollary 3.2 *Assume that* (2.1)–(2.4), (3.6) *and* (3.8) *hold. Then, the solution u_n of Problem $\mathcal{P}_n^2$ converges to the solution u of Problem $\mathcal{P}$, i.e.,* (3.4) *holds.*

Proof. Consider the sequence $\{\theta_n\}$ provided by assumption (3.8)(a) and let $t \in \mathbb{R}_+$, $n \in \mathbb{N}$. We write

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S} u_n(t), v - u_n(t))_X$$
$$= (A_n u_n(t), v - u_n(t))_X + (\mathcal{S} u_n(t), v - u_n(t))_X$$
$$+ (Au_n(t) - A_n u_n(t), v - u_n(t))_X \quad \text{for all } \ v \in K,$$

then we use inequality (3.7) to find that

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S} u_n(t), v - u_n(t))_X$$
$$\geq (f(t), v - u_n(t))_X + (Au_n(t) - A_n u_n(t), v - u_n(t))_X$$
$$\geq (f(t), v - u_n(t))_X - \|Au_n(t) - A_n u_n(t)\|_X \|v - u_n(t)\|_X \quad \text{for all } \ v \in K$$

and, therefore, (3.8)(a) yields

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S} u_n(t), v - u_n(t))_X \tag{3.9}$$
$$+ \theta_n(\|u_n(t)\|_X + 1)\|v - u_n(t)\|_X \geq (f(t), v - u_n(t))_X \quad \text{for all } \ v \in K.$$

We now combine (3.9) and (2.13) to see that $u_n \in \Omega_2(\theta_n)$ and, therefore, (3.3)(b) implies that $\{u_n\}$ as a $\mathcal{T}_2$-aproximating sequence for Problem $\mathcal{P}$. Finally, we use Theorem 2.2 and Definition 2.3 to deduce the convergence (3.4), which concludes the proof. □

Our third convergence result concerns the dependence of the solution with respect to the operator $\mathcal{S}$. To this end we consider a sequence of operators $\{\mathcal{S}_n\}$ such that, for each $n \in \mathbb{N}$, the following condition holds.

$$\begin{cases} \mathcal{S}_n \colon C(I; X) \to C(I; Y) \text{ is a history-dependent operator, i.e.,} \\ \text{for any } \ m \in \mathbb{N}, \text{ there exists } L_n^m > 0 \text{ such that} \\ \|\mathcal{S}_n u(t) - \mathcal{S}_n v(t)\|_X \leq L_n^m \displaystyle\int_0^t \|u(s) - v(s)\|_X \, ds \\ \text{for all } u, \ v \in C(\mathbb{R}_+; X), \ t \in [0, m]. \end{cases} \tag{3.10}$$

Moreover, we consider the following variational problem.

Problem $\mathcal{P}_n^3$. *Find a function $u_n \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:*

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}_n u_n(t), v - u_n(t))_X \tag{3.11}$$
$$\geq (f(t), v - u_n(t))_X \quad \text{for all } v \in K.$$

Then, using Theorem 2.1 it follows that Problem $\mathcal{P}_n^3$ has a unique solution, for each $n \in \mathbb{N}$. Assume now that

$$\left\{ \begin{array}{l} \text{(a) For each } m \in \mathbb{N} \text{ and } n \in \mathbb{N} \text{ there exists } \ \alpha_n^m > 0 \text{ such that} \\ \qquad \|\mathcal{S}_n u(t) - \mathcal{S}u(t)\|_X \leq \alpha_n^m \displaystyle\int_0^t \|u(s)\|_X \, ds \\ \qquad \text{for all } \ u \in C(\mathbb{R}_+; X), \ t \in [0, m]. \\ \text{(b)} \quad \alpha_n^m \to 0 \ \text{ as } \ n \to \infty, \ \text{for each } m \in \mathbb{N}. \end{array} \right. \tag{3.12}$$

We have the following convergence result.

Corollary 3.3 *Assume that* (2.1)–(2.4)*,* (3.10) *and* (3.12) *hold. Then, the solution u_n of Problem $\mathcal{P}_n^3$ converges to the solution u of Problem $\mathcal{P}$, i.e.,* (3.4) *holds.*

Proof. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, $t \in [0, m]$ and let $v \in K$. We write

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X$$
$$= (Au_n(t), v - u_n(t))_X + (\mathcal{S}_n u_n(t), v - u_n(t))_X$$
$$+ (\mathcal{S}u_n(t) - \mathcal{S}_n u_n(t), v - u_n(t))_X,$$

then we use inequality (3.11) to see that

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X$$
$$\geq (f(t), v - u_n(t))_X + (\mathcal{S}u_n(t) - \mathcal{S}_n u_n(t), v - u_n(t))_X.$$

Therefore,

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X$$
$$+ (\mathcal{S}_n u_n(t) - \mathcal{S}u_n(t), v - u_n(t))_X \geq (f(t), v - u_n(t))_X.$$

and, using assumption (3.12)(a), we find that

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X \tag{3.13}$$
$$+ \alpha_n^m \Big(\int_0^t \|u_n(s)\|_X \, ds \Big) \|v - u_n(t)\|_X \geq (f(t), v - u_n(t))_X.$$

We now use a claim that we state here and proof below in this section.

Claim 2. *For each $n \in \mathbb{N}$, let u_n be the solution of Problem $\mathcal{P}_n^3$. Then, for each $m \in \mathbb{N}$ there exists $U^m > 0$ such that*

$$\|u_n(t)\|_X \leq U^m \qquad \text{for all } \ t \in [0, m], \ n \in \mathbb{N}. \tag{3.14}$$

We now use (3.13) and (3.14) to see that

$$(Au_n(t), v - u_n(t))_X + (\mathcal{S}u_n(t), v - u_n(t))_X \tag{3.15}$$
$$+\alpha_n^m U^m m\|v - u_n(t)\|_X \geq (f(t), v - u_n(t))_X.$$

Consider now the sequence $\boldsymbol{\theta}_n = \{\theta_n^m\}_m \in \mathcal{R}(\mathbb{R}_+)$ where

$$\theta_n^m = \alpha_n^m U^m m \tag{3.16}$$

for each $m, n \in \mathbb{N}$. Then (3.15) implies that $u_n \in \Omega_3(\boldsymbol{\theta}_n)$ where, recall, $\Omega_3(\boldsymbol{\theta})$ is the set defined by (2.16), for each $\boldsymbol{\theta} = \{\theta^m\}_m \in \mathcal{R}(\mathbb{R}_+)$. On the other hand, assumptions (3.12)(b) and definition (3.16) imply that $\theta_n^m \to 0$ as $n \to \infty$, for each $m \in \mathbb{N}$. This implies that $\boldsymbol{\theta}_n = \{\theta_n^m\}_m \in C_3$ where C_3 is given by (2.15). It follows from above that the sequence $\{u_n\}$ is a $\mathcal{T}_3$-aproximating sequence for Problem $\mathcal{P}$ where, recall, $\mathcal{T}_3$ is the Tykhonov triple in Example 2.3. We now use Theorem 2.2 and Definition 2.3 to deduce the convergence (3.4), which concludes the proof. □

We now proceed with the proof of the bound (3.14).

Proof of Claim 2. Let $m \in \mathbb{N}$, $t \in [0, m]$, $n \in \mathbb{N}$ and $u_0 \in K$. Then, using (3.11) we find that

$$(Au_n(t), u_n(t) - u_0)_X \leq (\mathcal{S}_n u_n(t), u_0 - u_n(t))_X + (f(t), u_n(t) - u_0)_X.$$

Using this inequality and assumption (2.2)(a) we write

$$m_A\|u_n(t) - v\|_X^2 \leq (Au_n(t), u_n(t) - u_0)_X - (Au_0, u_n(t) - u_0)_X$$
$$\leq (\mathcal{S}_n u_n(t), u_0 - u_n(t))_X + (f(t), u_n(t) - u_0)_X + (Au_0, u_0 - u_n(t))_X$$

which implies that

$$m_A\|u_n(t) - u_0\|_X^2 \leq (\mathcal{S}_n u_n(t) - \mathcal{S}u_n(t), u_0 - u_n(t))_X + (\mathcal{S}u_n(t) - S\, u_0(t), u_0 - u_n(t))_X$$
$$+(\mathcal{S}u_0(t), u_0 - u_n(t))_X + (f(t), u_n(t) - u_0)_X + (Au_0, u_0 - u_n(t))_X$$

and, moreover,

$$m_A\|u_n(t) - u_0\|_X \leq \|\mathcal{S}_n u_n(t) - \mathcal{S}u_n(t)\|_X + \|\mathcal{S}u_n(t) - \mathcal{S}u_0(t)\|_X$$
$$+\|\mathcal{S}u_0(t)\|_X + \|f(t)\|_X + \|Au_0\|_X.$$

We now use assumptions (3.12) and (2.3) to find that

$$m_A\|u_n(t) - u_0\|_X \leq \alpha_n^m \int_0^t \|u_n(s)\|_X ds$$
$$+L^m \int_0^t \|u_n(s) - u_0\|_X\, ds + \|\mathcal{S}u_0(t)\|_X + \|f(t)\|_X + \|Au_0\|_X$$

and, therefore,

$$m_A\|u_n(t) - u_0\|_X \leq (\alpha_n^m + L^m) \int_0^t \|u_n(s) - u_0\|_X ds \tag{3.17}$$
$$+\alpha_n^m \int_0^t \|u_0\|_X ds + \|\mathcal{S}u_0(t)\|_X + \|f(t)\|_X + \|Au_0\|_X.$$

On the other hand, assumption (3.12)(b) shows that for n large enough we can assume that $\alpha_m^n \leq 1$ and using this inequality in (3.17) we find that

$$m_A\|u_n(t) - u_0\|_X \leq (L^m + 1)\int_0^t \|u_n(s) - u_0\|_X ds \tag{3.18}$$
$$+m\,\|u_0\|_X + \|\mathcal{S}u_0(t)\|_X + \|f(t)\|_X + \|Au_0\|_X.$$

Denote

$$G^m = \max_{t\in[0,m]}\Big(m\,\|u_0\|_X + \|\mathcal{S}u_0(t)\|_X + \|f(t)\|_X + \|Au_0\|_X\Big). \tag{3.19}$$

Then, (3.18) and (3.19) imply that

$$\|u_n(t) - u_0\|_X \leq \frac{L_m + 1}{m_A}\int_0^t \|u_n(s) - u_0\|_X\, ds + \frac{G_m}{m_A}\,. \tag{3.20}$$

We now we use the Gronwall argument to find that

$$\|u_n(t) - u_0\|_X \leq \frac{G^m}{m_A}\, e^{\frac{L^m+1}{m_A}\,t}.$$

This inequality implies that there exists $V^m > 0$ which does not depend on n and t such that

$$\|u_n(t) - u_0\|_X \leq V^m. \tag{3.21}$$

We now use (3.21) to obtain the bound (3.14) with $U^m = V^m + \|u_0\|_X$, which concludes the proof. □

We end this section with a convergence result which includes as particular cases the convergence results presented in Corollaries 3.1–3.3. To this end we consider two sequences of operators $\{A_n\}$ and $\{\mathcal{S}_n\}$ and a sequence of functions $\{f_n\}$ such that, for each $n \in \mathbb{N}$, the conditions (3.1), (3.6) and (3.10) hold. Moreover, we consider the following variational problem.

Problem $\mathcal{P}_n^4$. *Find a function $u_n \in C(\mathbb{R}_+; K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:*

$$(A_n u_n(t), v - u_n(t))_X + (\mathcal{S}_n u_n(t), v - u_n(t))_X \tag{3.22}$$
$$\geq (f_n(t), v - u_n(t))_X \quad \text{for all } v \in K.$$

Then, using Theorem 2.1 it follows that Problem $\mathcal{P}_n^4$ has a unique solution, for each $n \in \mathbb{N}$. Assume now that

$$f_n \to f \quad \text{in} \quad C(\mathbb{R}_+; X). \tag{3.23}$$

We have the following convergence result.

Corollary 3.4 *Assume that* (2.1)–(2.4)*,* (3.1)*,* (3.6)*,* (3.8)*,* (3.10)*,* (3.12) *and* (3.23) *hold. Then, the solution u_n of Problem $\mathcal{P}_n^4$ converges to the solution u of Problem $\mathcal{P}$, i.e.,* (3.4) *holds.*

Proof. The proof is similar to that of Corollary 3.3 and, therefore, we skip the details. We restrict ourselves to recall that it is based on the following ingredients: first, we prove that inequality (3.14) still holds where now u_n is the solution to history-dependent inequality (3.22), for each $n \in \mathbb{N}$. Then, we prove that the sequence $\{u_n\}$ is a $\mathcal{T}_3$-approximating sequence for Problem $\mathcal{P}$; finally, we use Theorem 2.2 and Definition 2.3 to deduce the convergence (3.4), which concludes the proof. □

4 Examples

In this section we complete the results in Section 3 with some examples. First, we state that Corollary 3.2 cannot be proved by using the well-posedness of Problem $\mathcal{P}$ with respect to the Tykhonov triple $\mathcal{T}_1$ in Example 2.1. Moreover, Corollary 3.3 cannot be proved by using the well-posedness of Problem $\mathcal{P}$ with respect to the Tykhonov triple $\mathcal{T}_2$ in Example 2.2. An evidence of these statements is provided by the two examples below.

Example 4.1 *Consider Problem $\mathcal{P}$ and the Tykhonov triple $\mathcal{T}_1$ in Example 2.1 in the particular case when $K = X$, $Au = u$ for all $u \in X$, $\mathcal{S} \equiv 0$ and $f(t) = tf_0$ for all $t \in \mathbb{R}_+$, where $f_0 \in X$, $f_0 \neq 0_X$. Note that in this particular case inequalitiy (1.2) becomes*

$$(u(t), v - u(t))_X \geq (f(t), v - u(t))_X \quad \text{for all } v \in X,\ t \in \mathbb{R}_+. \tag{4.1}$$

Assume now that $A_n u = Au + \frac{1}{n}\, u$, for each $n \in \mathbb{N}$. Then, inequality (3.2) becomes

$$(u_n(t) + \frac{1}{n}\, u_n(t), v - u_n(t))_X \geq (f(t), v - u_n(t))_X \quad \text{for all } v \in X,\ t \in \mathbb{R}_+.$$

Next, it is easy to see that condition (3.6) and (3.8) hold and, therefore, Corollary 3.2 guarantees the convergence (3.4).

This convergence can be proved directly. Indeed, the solution of inequalities (4.1) and (4.2) are $u(t) = f(t)$ and $u_n(t) = \frac{n}{n+1} f(t)$, respectively, for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Therefore,

$$\|u_n(t) - u(t)\|_X = \frac{1}{n+1}\|f(t)\|_X,$$

which implies (3.4).

Nevertheless, we claim that the sequence $\{u_n\}$ is not a $\mathcal{T}_1$- approximating sequence for Problem $\mathcal{P}$. Indeed, arguing by contradiction, assume that $\{u_n\}_n$ is a $\mathcal{T}_1$-approximating sequence. Then, using (2.9) and (2.10) we deduce that there exists a sequence $\{\theta_n\}$ such that $\theta_n \to 0$ and, for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, the following inequality holds:

$$(u_n(t), v - u_n(t))_X + \theta_n\|v - u_n(t)\|_X \geq (f(t), v - u_n(t))_X \quad \text{for all } v \in X. \tag{4.2}$$

We now substitute $u_n(t) = \frac{n}{n+1} f(t)$ in (4.2) then we take $v = f(t)$ in the resulting inequality to deduce that

$$\frac{1}{n+1}\|f(t)\|_X \leq \theta_n.$$

Moreover, since $f(t) = tf_0$ we find that

$$\frac{t}{n+1}\|f_0\|_X \leq \theta_n. \tag{4.3}$$

Recall that this inequality holds for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Thus, taking $t = n + 1$ we deduce that $\|f_0\|_X \leq \theta_n$ for each $n \in \mathbb{N}$ and, using the convergence $\theta_n \to 0$, we find that $f_0 = 0_X$ which is in contradiction with assumption $f_0 \neq 0_X$. We conclude from above that the sequence $\{u_n\}$ is not a $\mathcal{T}_1$- approximating sequence for Problem $\mathcal{P}$, as claimed. This implies the $\mathcal{T}_1$-well-posedness of inequality (1.2) combined with Definition 2.3 cannot be used to prove Corollary 3.2.

Example 4.2 *Consider Problem $\mathcal{P}$ and the Tykhonov triple $\mathcal{T}_2$ in Example 2.2 in the particular case when $K = X$, $Au = u$ for all $u \in X$, $\mathcal{S} \equiv 0$ and $f(t) = f_0$ for all $t \in \mathbb{R}_+$ where $f_0 \neq 0_X$. Note that in this particular case inequalities (1.2) becomes*

$$(u(t), v - u(t))_X \geq (f_0, v - u(t))_X \quad \text{for all } v \in X,\ t \in \mathbb{R}_+. \tag{4.4}$$

Assume now that

$$\mathcal{S}_n u(t) = \alpha_n \int_0^t u(s)\,ds \quad \text{for all } u \in C(\mathbb{R}_+; X),\ t \in \mathbb{R}_+$$

for each $n \in \mathbb{N}$, where $\alpha_n > 0$ and, moreover, $\alpha_n \to 0$ as $n \to \infty$. Then, inequality (3.2) *becomes*

$$(u_n(t) + \alpha_n \int_0^t u(s)\,ds, v - u_n(t))_X \geq (f_0, v - u_n(t))_X \tag{4.5}$$

for all $\boldsymbol{v} \in X$, $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Then, it is easy to see that conditions (3.10) *and* (3.12) *hold with $L_n^m = \alpha_n^m = \alpha_n$ and, therefore, Corollary* 3.2 *guarantees the convergence* (3.4).

This convergence can be proved directly. Indeed, inequality (4.5) *is equivalent with the differential equation*

$$u_n(t) + \alpha_n \int_0^t u_n(s)\,ds = f_0 \quad \text{for all } t \in \mathbb{R}_+$$

and, therefore, its solution is $u_n(t) = e^{-\alpha_n t} f_0$ for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. On the other hand, the solution of inequality (4.4) *is $u(t) = f_0$, for all $t \in \mathbb{R}_+$. Therefore,*

$$\|u_n(t) - u(t)\|_X = (1 - e^{-\alpha_n t})\|f_0\|_X \quad \text{for all } t \in \mathbb{R}_+,$$

which implies (3.4).

Nevertheless, we claim that the sequence $\{u_n\}$ is not a $\mathcal{T}_2$- approximating sequence for Problem $\mathcal{P}$. Indeed, arguing by contradiction, assume that $\{u_n\}$ is a $\mathcal{T}_2$-approximating sequence. Then, using (2.12) *and* (2.13) *we deduce that there exists a sequence $\{\theta_n\}$ such that $\theta_n \to 0$ and, for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$, the following inequality holds:*

$$\begin{aligned}&(u_n(t), v - u_n(t))_X + \theta_n(\|u_n(t)\|_X + 1)\|v - u_n(t)\|_X\\ &\qquad \geq (f_0, v - u_n(t))_X \quad \text{for all } v \in X.\end{aligned}$$

We now substitute $u_n(t) = e^{-\alpha_n t} f_0$ in (4.6) *then we take $v = f_0$ in the resulting inequality to deduce that*

$$(1 - e^{-\alpha_n t})\|f_0\|_X \leq \theta_n(e^{-\alpha_n t}\|f_0\|_X + 1).$$

Recall that this inequality holds for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Thus, taking $t = \frac{1}{\alpha_n}$ we deduce that $(1-e^{-1})\|f_0\|_X \leq \theta_n(e^{-1}+1)$ for each $n \in \mathbb{N}$. We now use the convergence $\theta_n \to 0$ to deduce that $f_0 = 0_X$, which contradicts the assumption $f_0 \neq 0_X$. We conclude from above that the sequence $\{u_n\}_n$ is not a $\mathcal{T}_2$- approximating sequence for Problem $\mathcal{P}$, as claimed. Therefore, the $\mathcal{T}_2$-well-posednes of the inequality (1.2) *cannot be used to prove Corollary* 3.3.

Examples 4.1 and 4.2 show that the choice of the Tykhonov triple plays a crucial role to deduce convergence results in the study of Problem $\mathcal{P}$. Indeed, it follows from above that the Tykhonov triple $\mathcal{T}_1$ contains enough approximating sequences to guarantee the proof of Corollary 3.1 but, on the other hand, it does not contain enough approximating sequences to be used in the proof of Corollary 3.2. Similarly, the Tykhonov triples $\mathcal{T}_2$ contains enough a-pproximating sequences to guarantee the proof of Corollary 3.2 but it does not contain enough approximating sequences to be used in the proof of Corollary 3.3. In addition, the inclusions (2.24) and (2.25) show that the Tykhonov triples $\mathcal{T}_2$ and $\mathcal{T}_3$ can be used in the proof of Corollary 3.1 and the Tykhonov triple $\mathcal{T}_3$ can be used in the proof of Corollary 3.2. It follows from here that, among the Tykhonov triples $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$, the Tykhonov triple $\mathcal{T}_3$ is the most convenient in the analysis of the history-variational inequality (1.2), since it can be used to obtain a large variety of convergence results for this inequality.

We end this section with the following example which arises in Contact Mechanics.

Example 4.3 *Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2,3)$ with smooth boundary divided into theree measurable disjoints parts Γ_1, Γ_2 and Γ_3 such that meas $\Gamma_1 > 0$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, by " $\cdot$ " the inner product on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ and by $\mathcal{L}(\mathbb{S}^d)$ the space of linear continuous operators from $\mathbb{S}^d$ to $\mathbb{S}^d$. Moreover, we consider the space*

$$X = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1 \,\}$$

and, for every $\boldsymbol{v} \in X$, we denote by $\boldsymbol{\varepsilon}(\boldsymbol{v})$ and v_ν the symmetric part of the gradient of $\boldsymbol{v}$ and the normal component of $\boldsymbol{v}$ to Γ, respectively. Then, it is well known that X is a Hilbert space endowed with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_X = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \text{ for all } \ \boldsymbol{u},\ \boldsymbol{v} \in X.$$

Consider a Lipschitz continuous strongly monotone operator $\mathcal{F} : \mathbb{S}^d \to \mathbb{S}^d$, an operator $\mathcal{B} \in C(\mathbb{R}_+; \mathcal{L}(\mathbb{S}^d))$ and two functions $\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d)$, $\boldsymbol{f}_2 \in C(\mathbb{R}_+, L^2(\Gamma_2)^d)$. With these data we define the set K, the operators $A : X \to X$, $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to X$ by equalities

$$K = \{\, \boldsymbol{v} \in X : \ v_\nu \le 0 \ \text{ on } \ \Gamma_3 \,\},$$

$$(A\boldsymbol{u},\boldsymbol{v})_X = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \text{for all } \boldsymbol{u},\ \boldsymbol{v} \in X,$$

$$(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v})_X = \int_\Omega \Big(\int_0^t \mathcal{B}(t-s)(\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \Big) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx$$
$$\text{for all } \ \boldsymbol{u} \in C(\mathbb{R}_+; X),\ \boldsymbol{v} \in X, t \in \mathbb{R}_+,$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_X = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da$$
$$\text{for all } \boldsymbol{u},\ \boldsymbol{v} \in X, t \in \mathbb{R}_+.$$

Moreover, we consider the problem of finding a function $\boldsymbol{u} \in C(\mathbb{R}_+K)$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:

$$(A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_X + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_X \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_X \quad \text{for all } \boldsymbol{v} \in K. \tag{4.6}$$

This problem represents the variational formulation of a mathematical model which describes the equilibrium of a viscoelastic body in frictionless contact with a rigid foundation, under the action of body forces of density $\boldsymbol{f}_0$ and surface traction of density $\boldsymbol{f}_2$. Here Ω represents the reference configuration of the body, $\boldsymbol{u}$ denotes the displacement field, K is the set of admissible displacement fields, $\mathcal{F}$ represents the elasticity operator and $\mathcal{B}$ is the relaxation tensor. Contact models which lead to inequality problems of the form (4.6) *have been considered in the books [1, 7, 17]. There, besides their variational analysis, the reader can find the classical formulation of the models, including the mechanical assumptions which lead to their construction.*

All the results presented in Sections 2 *and* 3 *can be applied in the study of inequality* (4.6). *In particular, Theorem* 2.1 *states its unique solvability and Corollary* 3.4 *provides the continuous dependence of the solution with respect to the elasticity operator, the relaxation tensor and the densities of body forces and surface tractions. Consider now a sequence of relaxation tensors $\{\mathcal{B}_n\} \subset C(\mathbb{R}_+; \mathcal{L}(\mathbb{S}^d))$ and, for each $n \in \mathbb{N}$, let $\mathcal{S}_n : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$*

be the operator given by

$$(\mathcal{S}_n\boldsymbol{u}(t),\boldsymbol{v})_X = \int_\Omega \Big(\int_0^t \mathcal{B}_n(t-s)(\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds\Big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx$$
$$\text{for all } \boldsymbol{u}\in C(\mathbb{R}_+;X),\ \boldsymbol{v}\in X,\ t\in\mathbb{R}_+.$$

Moreover, denote by $\boldsymbol{u}_n \in C(\mathbb{R}_+K)$ *the solution of the inequality*

$$(A\boldsymbol{u}_n(t),\boldsymbol{v}-\boldsymbol{u}_n(t))_X + (\mathcal{S}_n\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}_n(t))_X$$
$$\geq (\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}_n(t))_X \quad \text{for all } \boldsymbol{v}\in K,\ t\in\mathbb{R}_+.$$

The convergence result in Corollary 3.3 *shows that if* $\mathcal{B}_n \to 0$ *in* $C(\mathbb{R}_+;\mathcal{L}(\mathbb{S}^d))$ *then* $\boldsymbol{u}_n \to \boldsymbol{u}\in C(\mathbb{R}_+;X)$ *where* $\boldsymbol{u}\in C(\mathbb{R}_+K)$ *is the solution of the inequality*

$$(A\boldsymbol{u}_n(t),\boldsymbol{v}-\boldsymbol{u}_n(t))_X \geq (\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}_n(t))_X \quad \text{for all } \boldsymbol{v}\in K,\ t\in\mathbb{R}_+. \tag{4.7}$$

Note that inequality (4.7) *represents a time-dependent version of the well-known Signorini problem which models the equilibrium of an elastic body in frictionless contact with a rigid foundation, under the action of external forces of densities* $\boldsymbol{f}_0$ *and* $\boldsymbol{f}_2$*. We conclude from above that the solution of the Signorini problem* (4.7) *can be approached by the solution of a viscoelastic contact problem* (4.6) *for a "sufficiently small" relaxation tensor. This result is important from mechanical point of view since it establish the link between two different models of contact.* □

5 Conclusions

We considered the history-dependent variational inequality (1.2) for which we recalled an existence and uniqueness result. Then, we introduced the concept of well-posedness with respect to a given Tykhonov triple $\mathcal{T}$, the so called $\mathcal{T}$-well posedness. By definition, the $\mathcal{T}$-well-posedness of inequality (1.2) means the convergence of the $\mathcal{T}$-approximating sequences to its solution u. This property suggested us to adopt the following strategy in proving convergence results: given a sequence $\{u_n\}$, we are looking for an appropriate Tykhonov triple $\mathcal{T}$ such that the history-dependent inequality (1.2) is $\mathcal{T}$-well posed and, moreover, $\{u_n\}$ is a $\mathcal{T}$- approximating sequence. Then, using the well-posedness of $\mathcal{P}$ with respect $\mathcal{T}$ we deduce that $\{u_n\}$ converges to u.

We used the above strategy to obtain de continuous dependence of the solution of inequality (1.2) with respect to the operators A, $\mathcal{S}$ and the function f. Moreover, we presented an example arising in Contact Mechanics and provided some mechanical interpretation of these convergence results. We underline that a continuous dependence result of the solution of inequality (1.2) with respect the set of constraints K can also be obtained by using similar arguments. Moreover, the results presented here can be extended to the class of history-dependent variational inequalities considered in [19], as well as to the class of history-dependent hemivariational inequalities considered in [18].

The examples presented in this paper underlie the importance of the choice of the Tykhonov triple $\mathcal{T}$ in establishing convergence results of the form $u_n \to u$ by using the above strategy. Indeed, more the associated set $\mathcal{R}_\mathcal{T}$ of $\mathcal{T}$-approximating sequences is large, more the inclusion $\{u_n\} \in \mathcal{R}_\mathcal{T}$ has the chance to be valid. Nevertheless, this choice has to fulfil a expenditure condition which arise from the fact that more the set $\mathcal{R}_\mathcal{T}$ is large, more it is difficult to prove that the history-dependent inequality (1.2) is $\mathcal{T}$-well posed. In fact, a compromise policy between the two aims ("$\{u_n\} \in \mathcal{R}_\mathcal{T}$ " and "$\mathcal{R}_\mathcal{T}$ as small as possible") has to be found and the relative importance of each criterion with respect to the other depends on the sequence $\{u_n\}$ we consider.

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