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Spatial behaviour of thermoelasticity with microtemperatures and microconcentrations

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Abstract. We consider a thermoelastic material with microtemperatures and microconcentrations. The mathematical model is represented by a system of partial differential equations with the coupling of the displacement, temperature, chemical potential, microconcentrations and microtemperatures fields. The processes of heat and mass diffusion play an important role in many engineering applications, such as satellite problems, manufacturing of integrated circuits or oil extractions.

We study the spatial behaviour in a prismatic cylinder occupied by an anisotropic and inhomogeneous material. We impose final prescribed data that are proportional, but not identical, to their initial values. Moreover, we have zero body forces and zero lateral boundary conditions. The spatial behaviour is analysed in terms of some cross-sectional integrals of the solution that depend on the axial variable.

1 Introduction

The processes of heat and mass diffusion play an important role in many engineering applications, such as satellite problems, manufacturing of integrated circuits or oil extractions [1]. In order to show that the microelements have different microtemperatures, R. Grot introduced the concept of microtemperatures [4]. In order to show that the microelements have different concentrations, M. Aouadi, M. Ciarletta and V. Tibullo introduced the concept of microconcentrations [1].

We consider a thermoelastic material with microtemperatures and microconcentrations, as in [1], [2]. The mathematical model is represented by a system of partial differential equations with the coupling of the displacement, temperature, chemical potential, microconcentrations and microtemperatures fields. In the anisotropic case, well-posedness was shown in [1] by means of the semigroup theory of linear operators. Moreover, the asymptotic behaviour of the solution was discussed. In the isotropic case, the problem was analysed from a numerical point of view in [2] by means of the finite element method and the implicit Euler scheme.

The concept of microtemperatures was also used to model thermal effects at the microlevel in dipolar materials, see for example [7] and [8]. Other types of mathematical models for materials with microstructures were studied in [5] and [6].

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We study the spatial behaviour in a prismatic cylinder occupied by an anisotropic and inhomogeneous material, following the approach from [3]. We impose final prescribed data that are proportional, but not identical, to their initial values. Moreover, we have zero body forces and zero lateral boundary conditions. The spatial behaviour is analysed in terms of some cross-sectional integrals of the solution that depend on the axial variable.

2 Preliminaries

We consider a thermoelastic material with microtemperatures and microconcentrations in the three-dimensional euclidean space and we use a fixed system of rectangular axes Ox_i, $i = 1, 2, 3$. The Latin subscripts take the values $1, 2, 3$, Greek subscripts take the values $1, 2$ and repeated indices indicate the use of the Einstein summation convention. Differentiation is represented by a superposed dot in the case of material time derivatives and by a comma followed by a subscript in the case of derivatives with respect to a spatial coordinate.

First, we consider a domain Ω. Then, we restrict our considerations to a cylinder that is filled by an anisotropic and inhomogeneous thermoelastic material with microtemperatures and microconcentrations.

Below, we use the following notations: u_i is the displacement, t_{ij} is the stress tensor, ρ is the density in the reference configuration, f_i is the body force per unit mass, s is the heat source per unit mass, T_0 is the absolute temperature in the reference configuration, S is the microentropy, q_k is the heat flux vector, η_k is the flux vector of mass diffusion, ε_i is the first moment of energy vector, q_{ij} is the first heat flux moment tensor, ς_i is the microheat flux average, μ_i is the first moment of the heat source vector, Ω_i is the first moment of mass diffusion, η_{ij} is the first mass diffusion flux moment tensor, σ_i is the micromass diffusion flux average, T is the absolute temperature, T_i is the microtemperature vector, C_i is the microconcentration vector, P is the particle chemical potential, C is the concentration, $\theta = T - T_0$.

The equations of motion are [1]

$$
\begin{aligned}
&\rho\ddot{u}_i = t_{ki,k} + \rho f_i,\\
&\rho T_0\dot{S} = q_{i,i} + \rho s,\\
&\dot{C} = \eta_{i,i},\\
&\rho\dot{\varepsilon}_i = q_{ji,j} + q_i - \varsigma_i + \rho\mu_i,\\
&\rho\dot{\Omega}_i = \eta_{ji,j} + \eta_i - \sigma_i.
\end{aligned}
\tag{1}
$$

The constitutive equations of the linear theory of centrosymmetric materials are [1]

$$
\begin{aligned}
&t_{ij} = \alpha_{ijkl}e_{kl} + \gamma_{ij}\theta + \beta_{ij}P,\\
&\rho S = -\gamma_{ij}e_{ij} + c\theta + \kappa P,\\
&C = -\beta_{ij}e_{ij} + \kappa\theta + mP,\\
&\rho\varepsilon_i = -B_{ij}T_j - E_{ij}C_j,\\
&\rho\Omega_i = -D_{ij}C_j - E_{ij}T_j,
\end{aligned}
\tag{2}
$$

where

$$
e_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i})
\tag{3}
$$

and the following symmetries hold true

$$
\alpha_{ijkl} = \alpha_{jikl} = \alpha_{klij}, \gamma_{ij} = \gamma_{ji}, \beta_{ij} = \beta_{ji}.
\tag{4}
$$

Furthermore, we have [1]

$$\begin{aligned} &c\theta^2 + 2\kappa\theta P + mP^2 > 0,\\ &B_{ij}T_iT_j + D_{ij}C_iC_j + 2E_{ij}T_iC_j > 0. \end{aligned} \tag{5}$$

Moreover, we have [1]

$$\begin{aligned} &q_i = k_{ij}\theta_{,j} + K_{ij}T_j,\\ &q_{ij} = -P_{ijkl}T_{l,k},\\ &\varsigma_i = (k_{ij} - \tilde{k}_{ij})\theta_{,j} + (K_{ij} - \tilde{K}_{ij})T_j,\\ &\eta_i = h_{ij}P_{,j} + H_{ij}C_j,\\ &\eta_{ij} = -F_{ijkl}C_{l,k},\\ &\sigma_i = (h_{ij} - \tilde{h}_{ij})P_{,j} + (H_{ij} - \tilde{H}_{ij})C_j \end{aligned} \tag{6}$$

with the following symmetries for the coefficients

$$\begin{aligned} &P_{ijkl} = P_{klij}, F_{ijkl} = F_{klij}, K_{ij} = K_{ji}, H_{ij} = H_{ji}, k_{ij} = k_{ji}, h_{ij} = h_{ji},\\ &\tilde{K}_{ij} = \tilde{K}_{ji}, \tilde{H}_{ij} = \tilde{H}_{ji}, \tilde{k}_{ij} = \tilde{k}_{ji}, \tilde{h}_{ij} = \tilde{h}_{ji}, E_{ij} = E_{ji}, D_{ij} = D_{ji}, B_{ij} = B_{ji}. \end{aligned} \tag{7}$$

Moreover, we denote by

$$\begin{aligned} \bar{K}(s) = &\frac{1}{T_0}k_{ij}\theta_{,i}(s)\theta_{,j}(s) + \left(\frac{1}{T_0}K_{ij} + \tilde{k}_{ij}\right)\theta_{,i}(s)T_j(s) + P_{ijkl}T_{k,l}(s)T_{i,j}(s) + \tilde{K}_{ij}T_i(s)T_j(s) +\\ &+ \tilde{H}_{ij}C_i(s)C_j(s) + (\tilde{h}_{ij} + H_{ij})P_{,i}(s)C_j(s) + h_{ij}P_{,i}(s)P_{,j}(s) - F_{jikl}C_{l,kj}(s)C_i(s) \end{aligned} \tag{8}$$

and we have

$$\bar{K}(s) \geq 0. \tag{9}$$

Furthermore, we assume that ρ and the constitutive coefficients are continuous and bounded functions on $\bar{\Omega}$ and

$$\rho(\mathbf{x}) \geq \rho_0 > 0, c(\mathbf{x}) \geq c_0 > 0, m(\mathbf{x}) \geq m_0 > 0, \tag{10}$$

with ρ_0, c_0 and m_0 positive constants. Moreover, there exists a positive constant α_0 such that

$$\int_\Omega \alpha_{ijkl}e_{kl}e_{ij}dV \geq \alpha_0 \int_\Omega e_{ij}e_{ij}dV. \tag{11}$$

3 Spatial behaviour

We introduce the following notations

$$\begin{aligned} \bar{\Gamma}(t) = &\frac{1}{2}\rho\dot{u}_i(t)\dot{u}_i(t) + \frac{c}{2}\theta^2(t) + \frac{m}{2}P(t)^2 + \frac{1}{2}B_{ij}T_i(t)T_j(t) +\\ &+ \frac{1}{2}D_{ij}C_j(t)C_i(t) + \frac{1}{2}\alpha_{ijkl}e_{kl}(t)e_{ij}(t) + \kappa P(t)\theta(t) + E_{ij}C_j(t)T_i(t), \end{aligned} \tag{12}$$

$$\begin{aligned} \Lambda_1(R,t) = -\int_0^t \int_{S_R} e^{-\lambda s}\Big[&t_{ji}(s)n_j(s)\dot{u}_i(s) + \frac{1}{T_0}q_i(s)n_i(s)\theta(s) +\\ &+\eta_i(s)n_i(s)P(s) - q_{ji}(s)n_j(s)T_i(s)\Big]dads, R \geq 0, t \in [0,T]. \end{aligned} \tag{13}$$

Lemma 3.1 *We obtain*

$$\begin{aligned}&\int_\Omega e^{-\lambda t}\bar{\Gamma}(t)dv+\int_0^t\int_\Omega\left[e^{-\lambda s}\lambda\bar{\Gamma}(s)+e^{-\lambda s}\bar{K}(s)\right]dvds=\int_\Omega\bar{\Gamma}(0)dv+\\&+\int_0^t\int_\Omega e^{-\lambda s}\left[\rho f_i(s)\dot{u}_i(s)+\frac{1}{T_0}\rho s(s)\theta(s)-\rho\mu_i(s)T_i(s)\right]dvds+\bar{\Lambda}_1(t),\end{aligned}\tag{14}$$

where in $\bar{\Lambda}_1(t)$ the integrand from relation (13) is evaluated over $\partial\Omega$.

Lemma 3.2 (Properties) *We obtain*

$$\begin{aligned}&\Lambda_1(R_1,t)-\Lambda_1(R_2,t)=-\int_{B(R_1,R_2)}e^{-\lambda t}\bar{\Gamma}(t)dv-\\&-\int_0^t\int_{B(R_1,R_2)}\left[e^{-\lambda s}\lambda\bar{\Gamma}(s)+e^{-\lambda s}\bar{K}(s)\right]dvds,\end{aligned}\tag{15}$$

$$\frac{\partial\Lambda_1}{\partial R}(R,t)=-\int_{S_R}e^{-\lambda t}\bar{\Gamma}(t)dv-\int_0^t\int_{S_R}\left[e^{-\lambda s}\lambda\bar{\Gamma}(s)+e^{-\lambda s}\bar{K}(s)\right]dvds\tag{16}$$

In the sequel, we assume that the region $\Omega\subset\mathbb{R}^3$ is a prismatic cylinder. Moreover, we consider that its bounded uniform cross-section $D\subset\mathbb{R}^2$ has piecewise continuously differentiable boundary ∂D. The region Ω is assumed to be filled with an anisotropic and inhomogeneous thermoelastic material with microtemperatures and microconcentrations. The base of the cylinder contains the origin of the Cartesian coordinate system. The axis of the cylinder is parallel to the positive x_3-axis.

We use the following notation

$$B(z)=\{\mathbf{x}\in\Omega : z\leq x_3\}.\tag{17}$$

Furthermore, $D(x_3)$ is used to show that the respective quantities are considered over the cross-section whose distance from the origin is x_3. The lateral surface of the cylinder is denoted by Π, with $\Pi=\partial D\times[0,L]$, where L is the length of the cylinder.

The constitutive coefficients are given functions that depend on the spatial variable $\mathbf{x}$. The supply terms are assumed to be zero.

We impose the following null lateral boundary conditions

$$\begin{aligned}&t_{\alpha i}n_\alpha\dot{u}_i=0,\quad\frac{1}{T_0}q_\alpha n_\alpha\theta=0,\quad\eta_\alpha n_\alpha P=0,\\&q_{\alpha i}n_\alpha T_i=0,\quad\eta_{\alpha i}n_\alpha C_i=0\end{aligned}\tag{18}$$

for $(\mathbf{x},t)\in\Pi\times[0,T]$ and the base boundary conditions

$$\begin{aligned}&u_i(\mathbf{x},t)=g_i(x_1,x_2,t),\quad\theta(\mathbf{x},t)=h_1(x_1,x_2,t),\quad P(\mathbf{x},t)=h_2(x_1,x_2,t),\\&T_i(\mathbf{x},t)=h_3(x_1,x_2,t),\quad C_i(\mathbf{x},t)=h_4(x_1,x_2,t)\end{aligned}\tag{19}$$

for $(\mathbf{x},t)\in D(0)\times[0,T]$.

We impose the following final conditions at time T

$$\begin{aligned}&u_i(x,T)=\mu_1u_i(x,0),\quad\dot{u}_i(x,T)=\mu_1'\dot{u}_i(x,0),\\&\theta(x,T)=\mu_2\theta(x,0),\quad P(x,T)=\mu_3P(x,0),\\&T_i(x,T)=\mu_4T_i(x,0),\quad C_i(x,T)=\mu_5C_i(x,0)\end{aligned}\tag{20}$$

for $x \in B$. In the relations above, n_α are the components of the unit outward normal on Π, $g_i(x_1, x_2, t)$, $h_1(x_1, x_2, t)$, $h_2(x_1, x_2, t)$, $h_3(x_1, x_2, t)$, $h_4(x_1, x_2, t)$ are given differentiable functions that are compatible with the initial/final data and the lateral boundary conditions.

Moreover, we assume that the following given parameters satisfy these conditions

$$|\mu_1| > 1, |\mu_2| > 1, |\mu_3| > 1, |\mu_4| > 1, |\mu_5| > 1, |\mu_1'| > 1. \tag{21}$$

We will study the spatial behaviour of the solution $\{u_i, \theta, P, T_i, C_i\}(\mathbf{x}, t)$ of the problem defined by the evolution equations (1) with $f_i = 0$, $s = 0$, $\mu_i = 0$, the constitutive equations (2) with the geometrical relations (3), the lateral boundary conditions (18), the base boundary conditions (19) and the initial-final conditions (20).

We define the following function that depends on the axial variable

$$\begin{aligned} I(x_3) = \int_0^T \int_{D(x_3,\tau)} e^{-\lambda\tau} \Big[& t_{3i}(s)\dot{u}_i(s) + \frac{1}{T_0} q_3(s)\theta(s) + \eta_3(s)P(s) - \\ & -q_{3i}(s)T_i(s) - \eta_{3i}(s)C_i(s) \Big] \, da d\tau, \; x_3 \in [0, L], \end{aligned} \tag{22}$$

with λ a positive parameter. The notation $D(x_3, \tau)$ is used to show that the integrand is evaluated at time τ over the cross-section of the cylinder whose distance from the origin is x_3.

Lemma 3.3 (Properties) *Let $I(x_3)$ be defined as in (22). Therefore*

(i) if assumptions (5), (9), (10) and (11) hold true, then $I(x_3)$ is a non-decreasing function with respect to x_3 on $[0, L]$;

(ii) $I(x_3)$ satisfies the following differential inequality

$$\lambda \varepsilon_1 |I(x_3)| \le \frac{dI}{dx_3}, \tag{23}$$

where

$$\varepsilon_1^2 = \rho_0 \frac{1}{(1+\varepsilon)\mu_M}, \tag{24}$$

with

$$\varepsilon = \frac{2}{m\mu_M} \left(1 - c_0 T_0^2 \frac{M_2}{M_1 m_0}\right)^{-1} \left(\beta_{ij}^2 - m c_0 T_0^2 \frac{M_2}{M_1 m_0 c} \gamma_{ij}^2\right); \tag{25}$$

Theorem 3.1 *(i) If $I(x_3) \le 0$, for all $x_3 \in [0, L]$ then*

$$0 \le -I(x_3) \le -I(0) e^{-\lambda \varepsilon_1 x_3}, \quad \forall x_3 \in [0, L]. \tag{26}$$

(ii) If there exists $x_3^* \in [0, L]$ such that $I(x_3^*) > 0$, then the following growth estimate holds

$$I(x_3) \ge I(x_3^*) e^{\lambda \varepsilon_1 (x_3 - x_3^*)}, \quad \forall x_3 \in [x_3^*, L]. \tag{27}$$

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