Radial basis functions for solving the singular boundary integral equation of the compressible fluid flow around obstacles

Luminiţa Grecu[1,*]

[1]Department of Applied Mathematics
University of Craiova

Abstract. Abstract - In this paper we find a numerical solution for the Singular Boundary Integral Equation (SBIE), with sources distribution, of the compressible fluid flow around obstacles. The numerical solution is obtained by using linear boundary elements, radial basis functions for the unknown approximation and the Cauchy Principal Value for the treatment of singularities that appear. The proposed method is implemented into a computer code, made in Mathcad programming language, and, for some particular cases, numerical solutions are found. An analytic checking of the computer code is also made, in order to validate the proposed approach.

1 Introduction

The problem of the compressible fluid flow around obstacles has been studied by many authors over time and finding numerical solutions of great accuracy was one of the most challenging problem that arose. There are different techniques that can be applied for this purpose, in fact the same that are used for finding computational solutions of partial differential equations (PDEs), as for example: the finite element method (FEM), the finite difference method (FDM), the boundary element method (BEM) and other.

Boundary integral equations are equivalent models of the problems to be solved and arise as a consequence of using BEM to solve them. There are many papers describing BEM and its applications as [2], [3], [4]. This method can be applied in this case, by using any of its two main techniques: the direct technique and the indirect technique with sources or vortex distributions. When the direct technique is used, the boundary integral equation is formulated in terms of the unknown boundary functions and their derivatives. In case of the indirect boundary element technique, which consists in considering the superposition of the effect of fictitious functions and their derivatives, the boundary integral equation is formulated in terms of these unknown boundary fictitious functions. These can be, in case of the problem of a compressible fluid flow around obstacles, sources or vortex distributions, depending on the type of fundamental solution chosen for solving the problem.

Both formulations offer the real advantage of the BEM over other numerical methods, the fact that that they reduce the problem dimension by one, because they lead to equivalent boundary formulations for the problem involved. The boundary integral equations are further

*e-mail: lumigrecu@hotmail.com

solved and so, when finding numerical solutions, only the boundaries of the domain need to be discretized. BEM is, therefore, an efficient method which can be used to solve such problems, because it needs less computational effort than FEM, or FDM.

In this paper we consider the boundary integral equation with sources distribution, obtained in [6] by an indirect technique, and, for obtaining its numerical solution, we use radial basis function approximation and linear boundary elements for the boundary discretization.

Radial basis functions (RBFs) represent an efficient tool in approximation theory. They are usually used for interpolating large scatter data, but also in solving PDE, being part of what we call "meshless methods", because radial functions approximation does not need nodes on a structured grid, nor requires a mesh.

A radial basis function is a real valued function which depends only on the distance from its current argument to a fixed point, named center, and sometimes on values of a shape parameter which is used to scale the radial kernel. A set of linear independent radial basis functions is usually used as a base for a certain function space, so for represent any of its functions by a linear combination of radial basis functions. Their properties are transferred to the function they approximate globally, fact that represents a great advantage in case of finding infinite differentiable functions.

There are different types of RBFs, the well-known linear, Gaussian radial basis functions, multiquadric (MQ) and generalized MQ (GMQ) radial basis functions, the inverse quadric or multiquadric radial basis functions (IMQ), which are all infinitely smooth RBFs, and others, less known, such as polyarmonic spline, compact supported, Poisson RBFs, Laguerre-Gaussinan RBFs (generalization of Gaussians), and many other; for more about them see for example [5], [8]. They were first used by Hardy, [16], namely the multiquadric (MQ) ones, in topography for representing surfaces. They are used in many applications, [17], usually to interpolate scattered data [18], to solve PDE [8], [9], and weakly singular integral equations [10], in nonlinear approximations, in neural networks, pattern analysis and recognision and other computer applications, and these are only some of them.

Their use in solving different kind of PDE and the accuracy of solutions they lead to, make them very attractive in the case of function approximation and represent a good alternative to polynomial interpolation, dealing with cases when the latter can not be used.

When solving singular boundary integral equations a difficult problem arises when evaluating singular integrals and near singular integrals, [19]. Special treatments for evaluating these integrals must be considered in order to find good numerical results because improper evaluations of these integrals bring large errors in the numerical solutions. There are different methods that can be used for this purpose: by using special quadrature formulas, orthogonal polynomials, modified shape functions [2], truncated Taylor series, changes of coordinates, and other regularization techniques.

A regularization technique and linear boundary elements were used in [12] to solve the same problem of the compressible fluid flows around obstacles

In the herein paper we apply two methods in order to overpass this difficulty, both taking into account the definition of the Cauchy Principal Value: first is the truncation method, and the second uses a Taylor series expansion.

For better understanding the problem to be solved, we briefly present the boundary integral equation the problem is reduced at by following the ideas in [6], where a more detailed presentation can be found, because it is a singular boundary integral equation (SBIE) and it is important to well understand the unknowns and the variables that appear in it.

2 The singular boundary integral equation

A uniform, steady, potential motion of an ideal inviscid fluid of subsonic velocity $\overline{U_\infty \bar{i}}$, pressure $p_\infty$and density $\rho_\infty$is perturbed by the presence of a fixed body of a known boundary, assumed to be smooth and closed. The objective is to find the perturbed motion, and the fluid action on the body.

In dimensionless variables, the mathematical model of the problem is represented by the following system of PDE:

$$\beta^2 \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \qquad (1)$$
$$\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y} = 0$$

together with the boundary conditions:

$$(1 + U) N_x + V N_y = 0 \ \ on\ C, \qquad \lim_{\infty} (U, V) = 0. \qquad (2)$$

where U, V are the components of the perturbation velocity, $\beta = \sqrt{1 - M^2}$, M being Mach number for the unperturbed motion, N_x, N_y are the components of the normal vector outward the fluid. Changing the coordinates as follows: $x = X,\ y = \beta Y,\ u = \beta U,\ v = V$, the system of equations becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (3)$$

and the boundary condition:

$$(\beta + u) n_x + \beta^2 v n_y = 0 \quad on \quad C. \qquad (4)$$

It is also required that perturbation velocity vanishes at infinity: $\lim_{\infty} (u, v) = 0$.

The boundary integral equation is based on fundamental solution of source type, in fact the solution of system (3), in which the right hand term in first equation is replaced by $\delta (x - x_0, y - y_0)$, which components have the following expressions:

$$u^* (\overline{x}, \overline{x}_0) = \frac{1}{2\pi} \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}\ ,$$
$$v^* (\overline{x}, \overline{x}_0) = \frac{1}{2\pi} \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} \qquad (5)$$

The obstacle's boundary, C, is further assimilated with a continuous distribution of sources, of intensity noted f , presumed to satisfy a Hölder condition, and the expressions for the components of the perturbation velocity, u, v, are first deduced for $Q(\overline{x}_0)$ in the fluid domain. Expressions for the components of the perturbation velocity are then obtained for a regular point $Q_0(\overline{x}_0) \in C$, (referred as $\overline{x}_0 \in C$,), by evaluating the limits in (5) when $\overline{x}_0$ approaches to C:

$$u (\overline{x}_0) = -\frac{1}{2} f (\overline{x}_0) n_x^0 - CPV \int_C f (\overline{x}) u^* (\overline{x}, \overline{x}_0) ds,$$
$$v (\overline{x}_0) = -\frac{1}{2} f (\overline{x}_0) n_y^0 - CPV \int_C f (\overline{x}) v^* (\overline{x}, \overline{x}_0) ds, \qquad (6)$$

Because $\overline{x}_0 \in C$,the integrals that appear in the above relations are in fact Cauchy Principal Values of integrals and are denoted by CPV: $CPV\int\limits_C = \lim\limits_{\varepsilon\to 0}\oint\limits_{C-c}$, where $c = C(Q_0, \varepsilon)$ is a circle with radius ε centered in $Q_0\,(\overline{x}_0)$, which isolates the regular point.

Introducing the components of the perturbation velocity into the boundary condition (4), the following singular boundary integral is obtained.

$$\left(\left(n_x^0\right)^2 + \beta^2\left(n_y^0\right)^2\right) f\left(\overline{x}_0\right) + \frac{1}{\pi} CPV\int\limits_C f\left(\overline{x}\right)\frac{(x - x_0)\, n_x^0 + \beta^2\,(y - y_0)\, n_y^0}{\|\overline{x} - \overline{x}_0\|^2} ds = 2\beta n_x^0. \qquad (7)$$

Starting with this formulation for the 2D problem of the compressible fluid flow around an obstacle, a numerical solution based on linear boundary elements and radial basis functions is found.

3 Solving the singular boundary integral equation by using linear boundary elements and radial basis functions

The above singular boundary integral equation is solved in different papers: by a collocation method in [6], by using linear boundary elements in [11], quadratic in [13], and higher order isoparametric boundary elements in [14], [15]. The isoparametric boundary elements use the same functions for making both approximations, of the unknown and of the geometry of the problem, and obvious, the best numerical results are obtained in case of using higher order boundary elements, because of the better approximation of the geometry and of the unknown function.

For solving the integral equation (7) we consider, in this approach, a boundary mesh with linear boundary elements for approximating the geometry, namely the obstacles boundary, and radial basis functions for the unknown approximation. We so reduce, by discretization, the integral equation to an algebraic system, whose unknowns are given by the radial basis functions weights. The solution of this system, the weights values, is then used to calculate the perturbation velocity and the pressure coefficient on the body.

For a particular case, when the problem has an analytical solution, we make a comparison between the numerical solution obtained and the exact one, in order to check the numerical solution's accuracy, and to validate the proposed method.

3.1 Discretization procedure

We approximate the boundary, C,with a polygonal line, by dividing it into N linear segments, noted $L_i, i = \overline{1, N}$, each of them with endpoints, situated on C, noted $\overline{x}_i, \overline{x}_{i+1}, i = \overline{1, N}$, satisfying relation $\overline{x}_{N+1} = \overline{x}_1$,as contour C is closed.

Imposing (7) to be satisfied in all nodes, $\overline{x}_0 = \overline{x}_k$, $k = \overline{1, N}$, we get a system of equations of the following form:

$$\left(\left(n_x^k\right)^2 + \beta^2\left(n_y^k\right)^2\right) f\left(\overline{x}_k\right) + \frac{1}{\pi}\sum_{i=1}^{N}\int\limits_{L_i} f\left(\overline{x}\right)\frac{(x - x_k)\, n_x^k + \beta^2\,(y - y_k)\, n_y^k}{\|\overline{x} - \overline{x}_k\|^2} ds = 2\beta n_x^k \qquad (8)$$

where n_x^k, n_y^k are the components of the normal vector at $\overline{x}_k$.

For simplifying the writing we eliminate the sign CPV, which denotes the CPV of an integral, but when $\overline{x}_k \in L_i$ we keep in mind that we have the CPV of the integral involved.

For the unknown approximation we use radial basis functions so we consider:

$$f(\overline{x}) = \sum_{j=1}^{N} \lambda_j \varphi_j(\overline{x}) \tag{9}$$

where $\varphi_j(\overline{x})$are the radial basis functions and λ_j are the coefficients to be found.

Some of the advantages of using radial basis function approximation, over the use of classical boundary elements are: using radial basis function which are infinitely smooth, we obtain an approximation of the same type; the approximation is made for the whole domain, uniform, and not only local, as in case of using boundary elements, so no connection conditions between adjacent elements need to be imposed; because the approximation is global we do not need different systems of notations (local, and global); also the assembly stage is eliminated, fact that makes easier the procedure implementation into a computer code; and suitable radial basis functions can also help in eliminating the singularities.

In the herein paper we consider Gauss type radial basis functions, called Gaussians, to solve the SIBE, which have the following expressions:

$$\varphi_j(\overline{x}) = e^{-\alpha^2 \cdot \|\overline{x}-\overline{c}_j\|^2}, \ j = \overline{1, N} \tag{10}$$

Introducing the approximation of f,given by (9), in (8), the discrete form of the SBIE, we get a linear system of equations in unknowns λ_j. After solving this system we can evaluate f not only on the boundary but everywhere in the fluid domain and so we can evaluate the perturbation velocity components.

With the following notation, $q_{jk} = \left(\left(n_x^k\right)^2 + \beta^2\left(n_y^k\right)^2\right)\varphi_j(\overline{x}_k)$, (8) becomes:

$$\sum_{j=1}^{N} \lambda_j q_{jk} + \frac{1}{\pi}\sum_{i=1}^{N}\int_{L_i}\sum_{j=1}^{N}\lambda_j \frac{\varphi_j(\overline{x})\left[(x-x_k)n_x^k + \beta^2(y-y_k)n_y^k\right]}{\|\overline{x}-\overline{x}_k\|^2} ds = 2\beta n_x^k \tag{11}$$

Denoting by

$$c_{jk}^i = \int_{L_i} \varphi_j(\overline{x}) \frac{(x-x_k)\, n_x^k + \beta^2 (y-y_k)\, n_y^k}{\|\overline{x}-\overline{x}_k\|^2} ds\ , \ d_{jk} = \frac{1}{\pi}\sum_{i=1}^{N} c_{jk}^i \tag{12}$$

we obtain the following system of equations, of unknowns the weights, λ_j, $j = \overline{1, N}$:

$$\sum_{j=1}^{N} \lambda_j q_{jk} + \sum_{j=1}^{N} \lambda_j d_{jk} = 2\beta n_x^k, \ k = \overline{1, N} \tag{13}$$

The matrix form of the above system is:

$$A\{\lambda\} = B, \tag{14}$$

where $A = C^T + D^T, C = (q_{jk})_{1\le j,k\le N}$, $D = (d_{jk})_{1\le j,k\le N}$, $\{\lambda\} = (\lambda_1, \lambda_2, ..., \lambda_N)^T$, $B = \left(2\beta n_x^1, 2\beta n_x^2, ..., 2\beta n_x^N\right)^T$.

In order to obtain the numerical solution a computer code will be used, so we further need to get detailed expressions for its coefficients, in fact we need to evaluate all coefficients that appear in terms of nodal coordinates.

3.2 Coefficients evaluation

For evaluating the integrals we first introduce the geometry approximation. For $\overline{x} \in L_i$, so for a boundary element, which we consider to be a segement, the following parametric representation is considered:

$$\overline{x} = \overline{x}_i(1-t) + \overline{x}_{i+1}t, \quad t \in [0,1]\,,\ i = \overline{1,N}.$$

Using the above relations we deduce that $ds = l_i dt$, $l_i = \|\overline{x}_i - \overline{x}_{i+1}\| = \left((x_i - x_{i+1})^2 + (y_i - y_{i+1})^2\right)^{\frac{1}{2}}$,and introducing the geometry approximation into the coefficient expressions (12) we obtain:

$$c^i_{jk} = \int_0^1 \varphi_j(\overline{x}) \frac{\left(n^k_x(x_{i+1}-x_i)+\beta^2 n^k_y(y_{i+1}-y_i)\right)t+(x_i-x_k)n^k_x+\beta^2(y_i-y_k)n^k_y}{a_i t^2+b_{ki}t+c_{ki}} l_i dt,$$
$$a_i = l_i^2, b_{ki} = (x_i - x_k)(x_{i+1} - x_i) + (y_i - y_k)(y_{i+1} - y_i), c_{ki} = \|\overline{x}_k - \overline{x}_i\|^2 \tag{15}$$

Taking into account the going sense on C we have the following relations for the components of the normal in $\overline{x}_k$: $n^k_x = \dfrac{y_{k+1} - y_k}{l_k}$, $n^k_y = \dfrac{x_k - x_{k+1}}{l_k}, \forall k = \overline{1,N}$.

After introducing the following notations: $g_{ki} = (y_{k+1} - y_k)(x_{i+1} - x_i) + \beta^2(x_k - x_{k+1})(y_{i+1} - y_i)$, $h_{ki} = (x_i - x_k)(y_{k+1} - y_k) + \beta^2(y_i - y_k)(x_k - x_{k+1})$,we get the coefficients of the system in terms of nodal coordinates:

$$c^i_{jk} = \int_0^1 \varphi_j(\overline{x}) \frac{g_{ki}t + h_{ki}}{a_i t^2 + b_{ki}t + c_{ki}} \frac{l_i}{l_k} dt \tag{16}$$

As we notice the coefficients depend on nodes coordinates and expressions of radial functions involved. In order to obtain the numerical solution a computer code will be used, so we further need to get detailed expressions for its coefficients, in fact we need to evaluate all coefficients that appear in terms of nodal coordinates.

3.3 Gaussian type radail basis functions (Gaussians)

If we take a closer look to the Gaussian radial basis functions given by (10), we see that they depend on the choice of the centers, $\overline{c}_j, j = \overline{1,N}$. In the following paragraphs we consider $\overline{c}_j = \overline{x}_j$. Other situations can also be considered, as for example $\overline{c}_j = \dfrac{\overline{x}_j + \overline{x}_{j+1}}{2}$, or cases when $\overline{c}_j$ lies on the real boundary, not on its discretization. So the centers choice can influence the numerial solution's accuracy, as well as the other parameters they depend on.

Considering $\overline{c}_j = \overline{x}_j$, and using same notations as before, we get the following expressions for the coefficients:

$$c^i_{jk} = \int_0^1 e^{-\alpha^2\cdot\left(a_i t^2 + b_{ji}t + c_{ji}\right)} \frac{g_{ki}t + h_{ki}}{a_i t^2 + b_{ki}t + c_{ki}} \frac{l_i}{l_k} dt, i, j, k \in \{1, 2, ..N\} \tag{17}$$

For $k \neq i$, the coefficients are represented by regular integrals, they present no singularities and can be evaluated with usuall rulles, so we can use the computer for their evalaution, but we get singular integrals when $k = i$, case in which we use the definition of CPV of an integral for their evaluation.

For the evaluation of the integrals with singular kernels we can use different techniques, such as: special quadrature formulas, orthogonal polynomials, modified shape functions, truncated Taylor series, and so on. In this approach we use two methods for their evaluations, namely: the truncation method, which is a simple method which offers good results, as mentioned in [1], in case of singular integrals with integrands that do not oscillate near the singularity, as the integrands in our case, and Taylor series expansions.

Taking into account that $g_{ii} = (y_{i+1} - y_i)(x_{i+1} - x_i)(1 - \beta^2)$, $h_{ii} = 0, c_{ii} = b_{ii} = 0$,and using the truncation method, we can consider that, for a well chosen value of eps, the singular coefficients as:

$$c_{ji}^{i} = \int_{eps}^{1} e^{-\alpha^2\cdot\left(a_i t^2 + b_{ji}t + c_{ji}\right)} \frac{g_{ii}}{a_i t} dt, i, j \in \{1, .., N\} \tag{18}$$

For the second situation we use the expansion of $e^{-\alpha^2\cdot\left(a_i t^2 + b_{ji}t + c_{ji}\right)}$ around 0, and so we can write:

$$e^{-\alpha^2\cdot\left(a_i t^2 + b_{ji}t + c_{ji}\right)} = e^{-c_{ji}\alpha^2} - b_{ji} t e^{-c_{ji}\alpha^2} + \frac{1}{2}\alpha^2 t^2 e^{-c_{ji}\alpha^2}\left(b_{ji}^2\alpha^2 - 2a_i\right) - \\ - \frac{1}{6}\alpha^4 t^3 b_{ji} e^{-c_{ji}\alpha^2}\left(b_{ji}^2\alpha^2 - 6a_i\right) + O(t^4)$$

Taking into account that the domain of integration is [0,1] we limit our approximation to the above expressions. Of course by considering more terms in the Taylor polynomial we get a better approximation. Other techniques, as modified Gauss type quadrature formulas, can be also used. We obtain: $c_{ji}^{i} = \frac{g_{ii}}{a_i}\left[\int_0^1 \frac{e^{-c_{ji}\alpha^2}}{t} dt - b_{ji} e^{-c_{ji}\alpha^2} + \frac{1}{4}\alpha^2 e^{-c_{ji}\alpha^2}\left(b_{ji}^2\alpha^2 - 2a_i\right)\right] - -\frac{g_{ii}}{a_i}\frac{1}{18}\alpha^4 b_{ji} e^{-c_{ji}\alpha^2}\left(b_{ji}^2\alpha^2 - 6a_i\right) + O(t^4)$.

Using again the definition of the CPV of an integral we can eliminate the infinite contribution and we can consider that

$$c_{ji}^{i} = \frac{g_{ii}}{a_i}\left[-b_{ji} e^{-c_{ji}\alpha^2} + \frac{1}{4}\alpha^2 e^{-c_{ji}\alpha^2}\left(b_{ji}^2\alpha^2 - 2a_i\right) - \frac{1}{18}\alpha^4 b_{ji} e^{-c_{ji}\alpha^2}\left(b_{ji}^2\alpha^2 - 6a_i\right)\right] \tag{19}$$

The coefficients of the system can now be computed because they depend only on nodes coordinates and on the shape parameter α.We also analyze in this paper how shape parameter, α,influences the accuracy of the numerical solution, in order to see which choice is the best.

We can solve system (14) and find the unknowns of the problem, in fact the weights of radial basis functions. After finding the solution of the system, we proceed to the evaluation of the velocity components and of the local pressure coefficient.

4 Numerical evaluation of the perturbation velocity components

The numerical evaluation of the velocity components is made in the same manner as in case of system's coefficients, by using linear boundary elements for approximating the geometry of the obstacle's boundary. Returning to relations (6) introducing relation (9), in which now the coefficients λ_j are known, and considering $\overline{x}_0 = \overline{x}_k$ we obtain:

$$u(\overline{x}_k) = -\frac{1}{2}\sum_{j=1}^{N}\lambda_j\varphi_j(\overline{x}_k)\,n_x^k - \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_j\int_{L_i}\frac{\varphi_j(\overline{x})(x-x_k)}{\|\overline{x}-\overline{x}_k\|^2}ds,$$
$$v(\overline{x}_k) = -\frac{1}{2}\sum_{j=1}^{N}\lambda_j\varphi_j(\overline{x}_k)\,n_y^k - \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_j\int_{L_i}\varphi_j(\overline{x})\frac{\varphi_j(\overline{x})(y-y_k)}{\|\overline{x}-\overline{x}_k\|^2} \quad (20)$$

Introducing into the above relations the expressions of the radial basis functions, $e^{-\alpha^2\cdot\|\overline{x}-\overline{x}_j\|^2}$, we obtain:

$$u(\overline{x}_k) = -\frac{1}{2}\sum_{j=1}^{N}\lambda_j e^{-\alpha^2\cdot\|\overline{x}_k-\overline{x}_j\|^2}n_x^k - \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_j\int_{L_i}\frac{e^{-\alpha^2\cdot\|\overline{x}-\overline{x}_j\|^2}(x-x_k)}{(x-x_k)^2+(y-y_k)^2}ds,$$
$$v(\overline{x}_k) = -\frac{1}{2}\sum_{j=1}^{N}\lambda_j e^{-\alpha^2\cdot\|\overline{x}_k-\overline{x}_j\|^2}n_y^k - \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_j\int_{L_i}\frac{e^{-\alpha^2\cdot\|\overline{x}-\overline{x}_j\|^2}(y-y_k)}{(x-x_k)^2+(y-y_k)^2}ds$$

and further the expressions depending only on the nodes geometry, for any kind of smooth obstacle. Using the same notations as in (15) we have:

$$u(\overline{x}_k) = -\frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j e^{-\alpha^2\cdot cc_{kj}}(y_{k+1}-y_k)}{l_k} - \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_j m_{jk}^i$$
$$v(\overline{x}_k) = -\frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j e^{-\alpha^2\cdot cc_{kj}}(x_k-x_{k+1})}{l_k} - \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_j n_{jk}^i \quad (21)$$

with

$$m_{jk}^i = \int_0^1 e^{-\alpha^2\left(a_i t^2+bb_{ji}t+cc_{ji}\right)}\frac{(x_{i+1}-x_i)\,t+(x_i-x_k)}{a_i t^2+b_{ki}t+c_{ki}}l_i dt$$
$$n_{jk}^i = \int_0^1 e^{-\alpha^2\left(a_i t^2+bc_{ji}t+cc_{ji}\right)}\frac{(y_{i+1}-y_i)\,t+(y_i-y_k)}{a_i t^2+b_{ki}t+c_{ki}}l_i dt. \quad (22)$$

Singular integrals, m_{ji}^i, n_{ji}^i are evaluated in the same way as systems coefficients.

After the components of the velocity are found, the pressure coefficient can be calculate with the formula:

$$cp = \frac{2}{\gamma M^2}\left\{\left[1+\frac{M^2(\gamma-1)}{2}\left(1-v^2-\left(1+\frac{u}{\beta}\right)^2\right)\right]^{\frac{\gamma}{\gamma-1}}-1\right\}, M \neq 0 \quad (23)$$

where γ represents a constant for a certain fluid, a constant which represents the ratio between the specific heat at constant volume and the specific heat at constant pressure.

5 Computation results

No numerical method is useful without a computer program which allows a large amount of calculus, thousands of iteration necessary for coming closer to the real solution of the problem. It the following paragraphs we test the boundary element program we built, based on the herein approach, in order to study the numerical solution's accuracy. For this purpose we consider a situation when the problem has an exact solution. In order to test the method proposed in this paper, we consider the case of the circular obstacle and an incompressible fluid flow. In this particular case the problem has an exact solution (see [6] , [7]). The exact solution furnishes the following expressions for the components of the velocity in a point $M(R, \theta)$ situated on the circle $C(O, R)$ (the quantities we use are also dimensionless): $u = -\cos 2\theta,\ v = -\sin 2\theta$.

The pressure coefficient is given by: $C_p = \dfrac{p_1 - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = 1 - \dfrac{V^2}{U_\infty^2}$,where V is the global velocity: $V = U_\infty\left(\bar{i} + \bar{v}\right)$.

So, after the components of the velocity are found, the pressure coefficient can be calculate with the formula: $C_p(\overline{x}_i) = -\left(u(\overline{x}_i)^2 + v(\overline{x}_i)^2\right) - 2u(\overline{x}_i) = -1 + 2\cos 2\theta$

The computer code is made using Mathcad programming language.

Computer code implementation.

Input data**:** boundary geometry, N, the number of nodes used for the discretization, parameters involved in this approach as: M,Mach number (used to obtain β), eps, a small real value for the parameter used in truncation method, shape parameter α, "*singular" a* variable used for chosing the numerical treatment of the integrals with singular kernels.

Main steps:

Step 1. Finding values for nodes coordinates, for componenets of the normal vector inward the obstacels boundary;

Step 2.System coefficients evaluation (matrix A, and B) - made by computing regular integrals, and by using truncation method, or Taylor serie approach, for the case of singular coefficients;

Step 3. Solving the linear system of equations for geting the coefficients of radial basis functions (using a direct method, *lsolve* or *geninv*);

Step 4. Evaluation of the velocity components coefficients.

Step 5. Evaluation of the real values of velocity components

Step 6. Errors calculation

Output data:

The numerical values for the components of the nodal velocities and the local pressure coefficient

The exact values for velocity components and for the local pressure coefficient at the nodal points

Errors values

Graphical representations for the results.

We have used the computer code to find numerical solutions based on the proposed method. We can use it not only for finding different numerical solutions, but also for studying the influence of the singularities treatment, of the shape parameter, the influence of eps, on the accuracy of the numerical solution.

In order to study the influence of the singularities treatments on the accuracy of the numerical solution, we first need to study the influence of eps on accuracy of the numerical solution. First we present a numerical solution obtained for a small number of nodes. In the following pictures eru, erv, Ercp represent the absolute values of the errors that appear in case of the velocity components along Ox, respective along Oy, and for the local pressure coefficient.

For $N = 20, eps = 0.4, \alpha = 1, M = 0$ and the truncation method the results obtained are shown in Fig. 1.

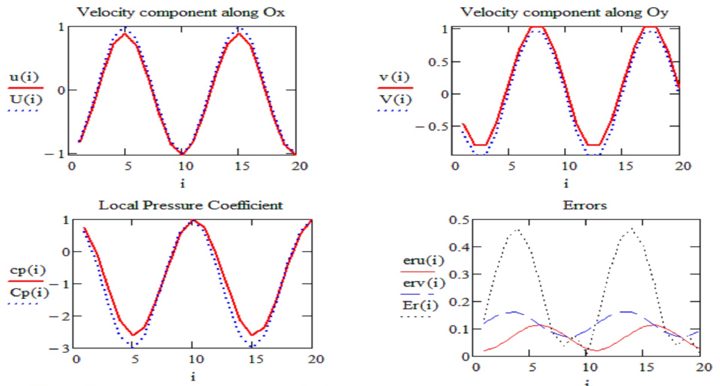

Figure 1. —— exact ······ numerical N=20, α=1, eps=0.4

As we notice the numerical results are in good agreement with the exact ones.

In Fig. 2 there are presented the results obtained for: $N = 20, eps = 0.5, \alpha = 0.1, M = 0$.

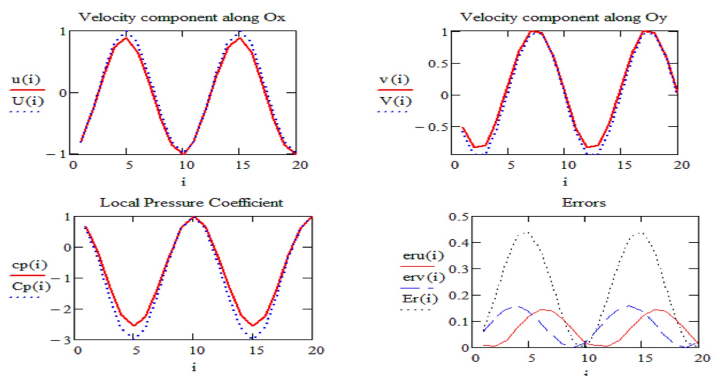

Figure 2 —— exact ······ numerical N=20, α=1, eps=0.5

In this case the best results are obtained when eps is around 0.5.

The numerical results obtained when using Taylor expansion are presented in Fig. 3.

As we can notice when $\alpha = 1$,the numerical solution obtained with truncation method gives better results than that which uses the Taylor expansion used in this approach.

Not only eps influences the numerical solution but also the shape parameter. For the same number of nodes we can modify the value of α in order to study its influence.

In Fig. 4 there are presented the numerical solutions obtained for $N = 20, \alpha = 0.01$.

If the shape parameter takes values above 1, the results get worst as the shape parameter grows. If we take smaller values than 0.01, the results do not suffer noticeable modifications. The results in Fig.4 seem to be closer than those obtained for $\alpha = 1$, but still not as good as those obtained with the truncation method, considering 20 nodes for the boundary discretization.

In case of using more nodes the situation changes. In Fig. 5 we present the results, namely the errors that appear for different cases

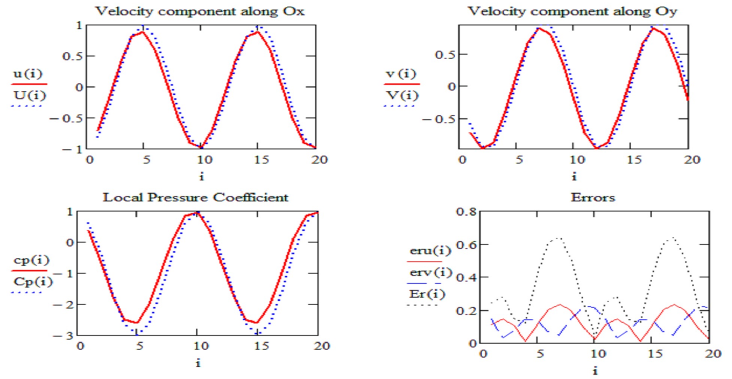

Figure 3 —— exact ····· numerical N=20, α=1, Taylor

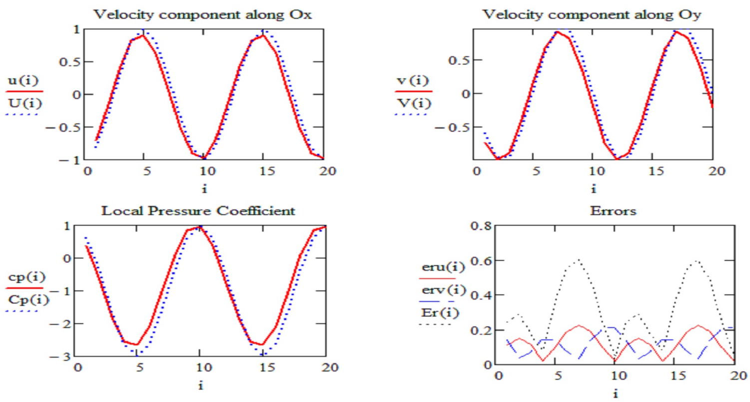

Figure 4. —— exact ····· numerical N=20, α=0.01, Taylor

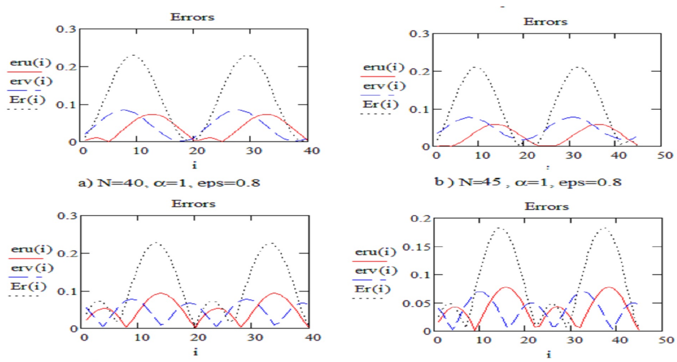

Figure 5.

We observe that in case of using more number of nodes, for the same value for the shape parameter, namely $\alpha = 1$, the numerical results obtained when using Taylor expansion better fit the exact ones.

In Fig. 6 there are presented the results obtained in case of using 50 nodes for the boundary discretization, $\alpha = 0.01$, and Taylor method for evaluating the coefficients arising integrals with singular kernels.

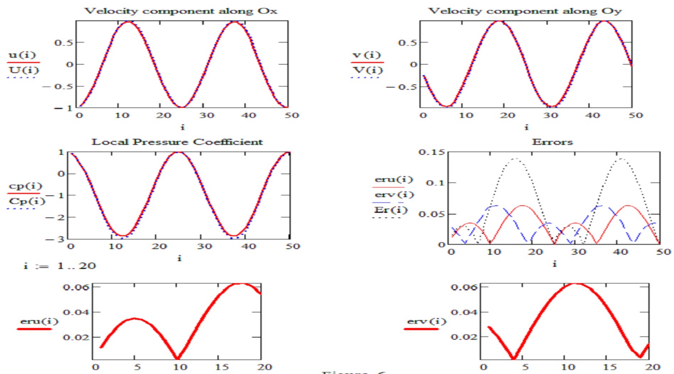

Figure 6

In Fig. 7 we can see, in case when $\alpha = 0.01$ and when Taylor series expansion is used for the treatment of coefficients with singular kernels, how the number of nodes influences the numerical solution's accuracy: more nodes, the better the numerical solution is.

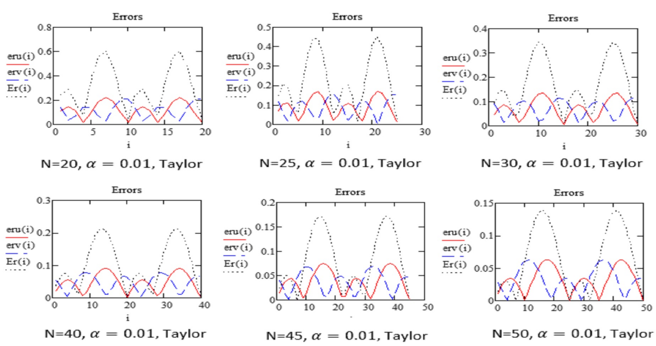

Figure 7. Number of nodes influence, on numerical solution's accuracy

The numerical results are in good agreement with the exact ones and this thing validates in fact the computer code and the method proposed in the herein paper.

As we notice the numerical solutions obtained by using radial basis function approximation and linear boundary element present a high accuracy as reported to the computational cost, because good results can be obtained for small number of nodes on the boundary, but there are many factors which influence the numerical solution's accuracy, one of them being the shape parameter. When Gaussians are used for solving the SBIE of the compressible fluid flow around obstacles and when the centers are same as the nodes used for the boundary discretization, we also notice that the lower the value of shape parameter, the better the numerical solution is. For obtaining good numerical results small values for the shape parameter must be used, $\alpha \leq 1$. Therefore, by choosing $\alpha = 0.01$, $\alpha = 0.1$, or $\alpha = 1$, the numerical results are satisfactory. A further study can be made in order to find the best value for the shape parameter, or how it influences the numerical solutions' accuracy. Finding the best value for the shape parameter is a problem of great importance, still under current investigation, as its value influences a lot the numerical solution's accuracy.

References

[1] H. M. Antia, *Numerical methods for scientists and engineers* (Birkhauser, Basel, 2002) 864 pp.

[2] M. Bonne, *Boundary integral equation methods for solids and fluids* (John Wiley and Sons, New York, 1999) 412 pp.

[3] C.A. Brebbia, *Boundary element techniques in engineering* (Butterworths, London, 1980).

[4] C.A. Brebbia, J.C.F.Telles, L.C. Wobel, *Boundary element theory and application in engineering* (Springer-Verlag, Berlin, 1984) 478 pp.

[5] M. Buhmann, *Radial Basis Functions: Theory and Implementations (Cambridge Monographs on Applied and Computational Mathematics, pp. I-Vi)* (Cambridge University Press, 2003) 272 pp.

[6] L. Dragoş, *Metode matematice în aerodinamică* (*Mathematical Methods in Aerodinamics)* (Ed. Academiei Române, Bucharest, 2000) 560 pp.

[7] L. Dragoş, *Mecanica Fluidelor Vol.1 Teoria Generală, Fluidul Ideal Incompresibil (Fluid Mechanics Vol.1. General Theory, The Ideal Incompressible Fluid)* (Ed. Academiei Române, Bucharest, 1999).

[8] G. E. Fasshauer *Approximation Solution of PDE* (Singapore, World Scientific Publisher, 2007) 520 pp.

[9] C. Franke, R. Schaback, *Solving partial differential equations by collocation using radial basis functions*, Applied Mathematics and Computation **93**(1), 73-82 (July 1998).

[10] E.A. Galperin, E.J. Kansa, *Applications of Global Optimization and Radial Basis Functions to Numerical Solutions of Weakly Singular Volterra Integral Equations*, Computers and Mathematics with Applications **43**, 491-499 (2002).

[11] L. Grecu, *A Solution of the BIE of the Theory of the Infinite Span Airfoil in Subsonic Flow with Linear Boundary Elements*, Analele Universităţii Bucureşti, Matematică **LII**(2), 181-188 (2003).

[12] L. Grecu, *Linear boundary elements for solving the nonsingular boundary integral equation of the fluid flow around obstacles*, Acta universitatis Apulensis, Special Issue ICTAMI **2011**, 501-512 (2011).

[13] L. Grecu, *A solution of the boundary integral equation of the 2D fluid flow around bodies with quadratic isoparametric boundary elements*, ROMAI J. **2**(2), 81–87 (2006).

[14] L. Grecu, *A solution with cubic boundary elements for the compressible fluid flow around obstacle*s, Boundary Value Problems **2013**(78) (2013).

[15] L. Grecu, An improved numerical solution of thesingular boundary integral equation of thecompressible fluid flow around obstacles using modified shape functions, Boundary Value Problems **2015**(35) (2015).

[16] R. L. Hardy, *Multiquadric equations of topography and other irregular surfaces*, Journal of Geophisical Reserach **76**(8), 1905-1915 (1971).

[17] R. L. Hardy, *Theory and applications of the multiquadric-biharmonic method: 20 years of discovery*, Computers and Mathematics with Applications **19**(8-9), 163-208 (1990).

[18] E. J. Kansa, *Multiquadrics - a scattered data approximation scheme with applications to computational fluid dynamics I: Surface approximations and partial derivative estimates*, Computers and Mathematics with Applications **19**(8-9), 127-145 (1990).

[19] I. K., Lifanov, *Singular integral equations and discrete vortices* (VSP, Utrecht, TheNetherlands, 1996) 475 pp.