On quaternionic product on some sets of hyperbolas

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Abstract. The aim of this paper is to introduce some products, induced by the quaternionic product, on the set of equilateral hyperbolas. The study is a mix of elements from Euclidean and projective geometry. Some properties of these products are highlighted and are connected with some special numbers as roots or powers of the unit. Then we extend these products in a natural manner to oriented equilateral hyperbolas and to pairs of equilateral hyperbolas, using the algebra of octonions. Further, using an inversion, we extend these products to Bernoulli lemniscates and to Bernoulli q-lemniscates. Finally, using some isomorphisms, we extend these products to conics, and using an inversion, we extend these products to other curves. In order to highlight the importance of these studied products, the paper ends with some applications.

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1 Introduction

Using the well-known product of quaternions we introduce some products on the set of equilateral hyperbolas considered in a projective way and we give some extensions of them. We define a first product, denoted $\odot_c$; then, as the c-square of the unit hyperbola $H(1) : x^2 - y^2 - 1 = 0$ is the degenerate hyperbola $H(0) : x^2 - y^2 = 0$, we introduce a second product, denoted $\odot_{pc}$, to avoid the degeneration. A detailed study of both these products is the content of section 1. By looking to examples as well as to roots/powers of the unit $1 \in \mathbb{R}$ we obtain some remarkable numbers, some of them algebraic but other of difficult nature.

Using an inversion, we extend the above products from the set of equilateral hyperbolas to sets of Bernoulli lemniscates and q-lemniscates, thus some interesting results are obtained.

Then, using a larger set of conics Q_{Γ_0}, we extend the above products and we obtain a field isomorphic to the field of complex numbers. For some particular values of the parameters, we obtain the product of equilateral hyperbolas firstly considered. Moreover, we extend the product on conics from Q_{Γ_0} to other curves.

Starting from the expression of an octonion as a pair of quaternions, we introduce a product of pairs of equilateral hyperbolas. While the products introduced on the set of equilateral hyperbolas are commutative, the considered product of pairs of equilateral hyperbolas is not.

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Finally we propose three applications of the given products: the first two of them regard hyperbolic objects, namely the reduced equilateral hyperbola and hyperbolic matrices, concerning with multi-valued mapsand the third one returns to the Euclidean plane geometry and defines a chain of labels for a given polygon.

The present study is the hyperbolic counter-part of a similar work concerning circles in [5] while a more general Clifford product for EPH-cycles is introduced in [4] and it is a natural continuation of [5] due to the Lambert's and Riccati's analogies between the circle and the equilateral hyperbola, exposed in [2] and published as [3].

2 Quaternionic product of hyperbolas

Let us consider a given equilateral hyperbola H in the Euclidean plane with coordinates (x, y):

$$H : x^2 - y^2 + ax + by + c = 0. \tag{1}$$

We identificate H with a quaternion:

$$q(H) = k + ai + bj + c = (c, a, b, 1) \in \mathbb{R}^4. \tag{2}$$

The quaternion $q(H)$ is pure imaginary if and only if the origin $O(0, 0)$ belongs to H. Let us point out that the given hyperbola is expressed in a *projective* manner since the coefficient of the quadratic part is chosen as being 1. Hence the set of equilateral hyperbolas is a 3-dimensional projective subspace of the 5-dimensional projective space of conics. Our study will be a mix of elements from Euclidean and projective geometry.

For $H_i, i = 1, 2$ given by (a_i, b_i, c_i) we introduce a product of equilateral hyperbolas by:

$$H_1 \odot_c H_2 := q^{-1}(q(H_1) \cdot q(H_2)), \tag{3}$$

and we have:

$$\begin{aligned} q(H_1 \odot_c H_2) = (a_1b_2 - a_2b_1 + c_1 + c_2)k + (b_1 - b_2 + a_1c_2 + a_2c_1)i + \\ + (a_2 - a_1 + b_1c_2 + b_2c_1)j + +(c_1c_2 - 1 - a_1a_2 - b_1b_2) \end{aligned} \tag{4}$$

which gives a commutative expression only for the free term.

We restrict our study to equilateral hyperbolas $H(r)$ already centered in O (in the chosen projecive setting); hence their set is a 1-dimensional projective subspace of the projective spaces considered above. For such a hyperbola we have:

$$H(r) : (a, b, c) = (0, 0, -r) \tag{5}$$

and hence the equation (4) yields:

$$q(H(r_1) \odot_c H(r_2)) = (c_1 + c_2)k + (c_1c_2 - 1) = -(r_1 + r_2)k + (r_1r_2 - 1). \tag{6}$$

From the properties of quaternionic product we have that the above product can be also expressed in matrix product manner:

$$(-r_2, 0, 0, 1) \cdot \begin{pmatrix} -r_1 & 0 & 0 & 1 \\ 0 & -r_1 & 1 & 0 \\ 0 & -1 & -r_1 & 0 \\ -1 & 0 & 0 & -r_1 \end{pmatrix} = (r_1r_2 - 1, 0, 0, -(r_1 + r_2)). \tag{7}$$

Takeing into account the above considerations, we introduce the product law:

$$H(r_1) \odot_c H(r_2) = H(R), \quad R := \frac{r_1 r_2 - 1}{r_1 + r_2}. \tag{8}$$

Then we define on the set $M = (0, +\infty)$ a *non-internal* composition law :

$$r_1 \odot_c r_2 := \frac{r_1 r_2 - 1}{r_1 + r_2} < \min\{r_1, r_2\} \tag{9}$$

thus we have:

$$H(r_1) \odot_c H(r_2) = H(r_1 \odot_c r_2). \tag{10}$$

Property 1.1 The product $\odot_c$ is commutative and associative but does not have a neutral element:

$$r_1 \odot_c r_2 \odot_c r_3 = \frac{r_1 r_2 r_3 - (r_1 + r_2 + r_3)}{r_1 r_2 + r_2 r_3 + r_3 r_1 - 1}, \quad r^3_{\odot_c} = \frac{r^3 - 3r}{3r^2 - 1}, \quad r > \frac{1}{\sqrt{3}}. \tag{11}$$

Property 1.2 With $r_i = \tan\varphi_i$ we get:

$$\tan\varphi_1 \odot_c \tan\varphi_2 := -\cot(\varphi_1 + \varphi_2). \tag{12}$$

Property 1.3 Concerning the unit hyperbola $H(1) : x^2 - y^2 = 1$ we have:

$$r \odot_c 1 = \frac{r-1}{r+1} < \min\{1, r\}, \quad \lim_{r \to +\infty} (r \odot_c 1) = 1. \tag{13}$$

In particular, the unit hyperbola is the square root of the degenerate hyperbola, $H(1) \odot_c H(1) = H(0) : x^2 - y^2 = 0$; in fact $[q(H(1))]^2 = (k-1)^2 = -2k$. With a rational $r = \frac{x}{y}$:

$$\frac{x}{y} \odot_c 1 = \frac{x-y}{x+y}. \tag{14}$$

Property 1.4 Concerning the squares we have:

$$r^2_{\odot_c} = \frac{r^2 - 1}{2r} < r, \quad (\tan\varphi)^2_{\odot_c} = -\cot(2\varphi), \quad (r^2_{\odot_c}) \odot_c 1 = \frac{r^2 - 2r - 1}{r^2 + 2r - 1} \tag{15}$$

and the first relation (15) means that $\odot_c$ is a "shrinking" composition. The $\odot_c$-square root of 1 is the number:

$$\sqrt[c]{1} := 1 + \sqrt{2} = 2.4142135... = \tan\frac{3\pi}{8}, \quad (\sqrt[c]{1})^2 - 2\sqrt[c]{1} - 1 = 0 \tag{16}$$

while the $\odot_c$-square root of $\sqrt[c]{1}$ is the number:

$$\sqrt[2c]{1} := 1 + \sqrt{2} + \sqrt{4 + 2\sqrt{2}} = 5.027339..., \quad (\sqrt[2c]{1})^2_{\odot_c} = \sqrt[c]{1}. \tag{17}$$

Let us remark that $\sqrt[c]{1}$ is exactly *the silver ratio* $\Psi := 1 + \sqrt{2}$ and we point out that Ψ is a quadratic Pisot-Vijayaraghavan number considered as solution of:

$$x^2 - 2x - 1 = 0. \tag{18}$$

The conjugate of Ψ with respect to this algebraic equation is:

$$-\Psi^{-1} = 1 - \sqrt{2} = -0.44.... \tag{19}$$

Let us point out that from the point of view of endomorphisms on smooth manifolds the silver mean is treated in [6, p. 16] and a $\odot_c$-fourth order root of unit is called *almost electromagnetic structure* in [7, p. 721]. The continuous fraction of these remarkable numbers are easy to compute with Mathematica:

$$\sqrt[c]{1} = [2; \bar{2}], \quad \sqrt[2c]{1} = [5; 36, 1, 1, 2, 1, 2, 1, 6, ...]. \tag{20}$$

Let us note that the radius involved in the well-known Hopf fibration as the Riemannian submersion $S^3(1) \to S^2(\frac{1}{2})$ are involved in the expression of these two remarcable numbers, thus we intend to investigate this approach in a future study.

Property 1.5 The given square (15) is:

$$r^2_{\odot_c} = \frac{\|q(H(r))\|^2 - 2}{2\sqrt{\|q(H(r))\|^2 - 1}}, \tag{21}$$

where the quaternion (2) has the Euclidean norm:

$$\|q(H(r))\|^2 = 1 + a^2 + b^2 + c^2 = 1 + r^2 \geq 1 \tag{22}$$

We can avoid the degeneration $(H(1))^2_{\odot_c} = H(0)$ by considering a new product $\odot_{pc}$ on $\mathbb{R}^*_+ = (0, +\infty)$:

$$x \odot_{pc} y = \frac{xy + 1}{x + y}. \tag{23}$$

The product $\odot_{pc}$ is commutative with $x \odot_{pc} 1 = 1$ and:

$$x^2_{\odot_{pc}} = \frac{x^2 + 1}{2x}, \quad x \odot_{pc} y \odot_{pc} z = \frac{xyz + x + y + z}{xy + yz + zx + 1}, \quad \tan\varphi_1 \odot_{pc} \tan\varphi_2 = \frac{\cos(\varphi_2 - \varphi_1)}{\sin(\varphi_1 + \varphi_2)}. \tag{24}$$

3 Quaternionic product of oriented hyperbolas

We extend the previous products from hyperbolas to *oriented hyperbolas,* that is pairs $\mathcal{H} := (H, \varepsilon)$ with $\varepsilon \in \{\pm 1\}$.
Property 2.1 Let (H_i, ε_i), $i = \overline{1, 2}$, be two oriented hyperbolas. Then:

$$\mathcal{H}_1 \odot_c \mathcal{H}_2 := (H_1 \odot_c H_2, \varepsilon_1 \cdot \varepsilon_2). \tag{25}$$

All the aspects and properties obtained above for hyperbolas remain also true in the case of oriented hyperbolas.

4 On quaternionic product on Bernoulli lemniscates and q-lemniscates

In [1] it is proved, using purely geometrical means, that the image of an equilateral hyperbola with foci F_1 and F_2 by an inversion I_r with respect to the circle centered in O and with radius $r = |OF_1| = |OF_2|$ is a Bernoulli lemniscate with the same foci F_1 and F_2. We start this section with a complex approach in order to achieve easier the extension to the quaternionic approach.

Thus, we will prove the above assertion using complex numbers. Firstly, we associate to every point (x, y) in the Euclidean plane the complex number $z = x+iy \in \mathbb{C}$. As $z' = I_r(z) = \alpha z$

with $\alpha \in \mathbb{R}_+^*$ and $|I_r(z)| \cdot |z| = r^2$ we have $|\alpha z| \cdot |z| = r^2$, so $\alpha = \frac{r^2}{|z|^2} = \frac{r^2}{z \cdot \overline{z}}$ and therefore the equation of the inversion I_r is:

$$z' = I_r(z) = \frac{r^2}{\overline{z}}. \tag{26}$$

The equation:

$$H(a^2) : x^2 - y^2 = a^2, \tag{27}$$

of the equilateral hyperbola with foci $\left(\pm a\sqrt{2}, 0\right)$, taking into account that $x^2 - y^2 = \mathrm{Re}\left(z^2\right) = \frac{z^2 + \overline{z}^2}{2}$, can be written as:

$$H(a^2) : z^2 + \overline{z}^2 = 2a^2. \tag{28}$$

The image of the above equilateral hyperbola by the inversion I_r has the equation $\left(z^2 + \overline{z}^2\right) r^4 = 2a^2 \left(\overline{z}z\right)^2$. Taking into account that $z \cdot \overline{z} = x^2 + y^2$ we obtain $\left(x^2 + y^2\right)^2 = \frac{r^4}{a^2}\left(x^2 - y^2\right)$ which is the equation of a Bernoulli lemniscate with foci $\left(\pm \frac{r^2}{a\sqrt{2}}, 0\right)$.

As the equilateral hyperbola has the foci $\left(\pm a\sqrt{2}, 0\right)$ while the Bernoulli lemniscate has the foci $\left(\pm \frac{r^2}{a\sqrt{2}}, 0\right)$ the foci are conserved by the inversion I_r if and only if $a\sqrt{2} = \frac{r^2}{a\sqrt{2}}$ i.e. $r = a\sqrt{2}$. More exactly, the image by the inversion I_r of the equilateral hyperbola $H\left(\frac{r^2}{2}\right)$ that has the foci $(\pm r, 0)$ is the Bernoulli lemniscate $L(r)$ with the same foci, having the equation $\left(x^2 + y^2\right)^2 = 2r^2\left(x^2 - y^2\right)$.

Let us note that for a Bernoulli lemniscate $L(r)$ with parameter $r > 0$, the distance between the foci is $2r$. We can introduce the products of Bernoulli lemniscates $L(r_1)$ and $L(r_2)$ in the same manner as the products of equilateral hyperbolas:

$$L(r_1) \odot_c L(r_2) := L(r_1 \odot_c r_2), \quad L(r_1) \odot_{pc} L(r_2) := L(r_{1 \odot_{pc} c} r_2).$$

All the properties proved for quaternionic products of equilateral hyperbolas are also true for quaternionic products of Bernoulli lemniscates.

A well-known property of an equilateral hyperbola given by equation (27) is expressed by:

$$||MF_1| - |MF_2|| = 2a = const. \tag{29}$$

and, in the case of a Bernoulli lemniscate, given by the equation:

$$\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right) \tag{30}$$

is expressed by:

$$|MF_1| \cdot |MF_2| = |OF_1| \cdot |OF_2| = \frac{a^2}{2} = const. \tag{31}$$

These properties could be proved using pure geometric or analytic geometry means but we prove them using complex numbers.

Recall that the inversion I_r with $r = a\sqrt{2}$ preserves the foci $F_{1,2}$. In this case, a current point M on the equilateral hyperbola $H\left(\frac{r^2}{2}\right)$, with foci $F_{1,2}\,(\pm r)$ and vertices $V_{1,2}\left(\pm\frac{r}{\sqrt{2}}\right)$, has the property $||MF_1| - |MF_2|| = r\sqrt{2} = const.$ and a current point M on its image by this inversion I_r, the Bernoulli lemniscate $L(r)$ given by the equation $\left(x^2+y^2\right)^2 = 2r^2\left(x^2-y^2\right)$, with the same foci, has the property $|MF_1|\cdot|MF_2| = |OF_1|\cdot|OF_2| = r^2 = const.$

We extend these constructions in a quaternionic setting. First we recall that $M(x,y,z,w) \in \mathbb{R}^4$ has the quaternionic affix $q = x + yi + zj + wk$. We say that the hyperquadric in $\mathbb{R}^4$ defined by the equation:

$$H_q(a^2) : x^2 - y^2 - z^2 - w^2 = a^2 \tag{32}$$

is a *q-equilateral hyperboloid*. We can define its foci as the points $\left(\pm a\sqrt{2}, 0, 0, 0\right)$.

Also we say that:

$$L_q(a^2) : \left(x^2+y^2+z^2+w^2\right)^2 = a^2\left(x^2-y^2-z^2-w^2\right)$$

is the *Bernoulli q-lemniscate* with the points $\left(\pm\frac{a}{\sqrt{2}}, 0, 0, 0\right)$ as foci.

In the following we will show that the names q-equilateral hyperboloid and Bernoulli q-lemniscate are fully justified because the main properties of equilateral hyperbola and Bernoulli lemniscate stated above in complex context are also preserved in quaternionic context.

Proposition 3.1 *The relation (29), specific to a hyperbola, holds also true for a q-equilateral hyperboloid.*

In a similar way with the quaternionic products on equilateral hyperbolas we can introduce the quaternionic products of q-equilateral hyperboloids $H_q(r_1)$ and $H_q(r_2)$:

$$H_q(r_1)\odot_c H_q(r_2) = H_q(r_1 \odot_c r_2), \quad H_q(r_1)\odot_{pc} H_q(r_2) = H_q(r_1 \odot_{pc} r_2).$$

Recall that the equilateral hyperbola (27) can be written in a complex form as (28) and, in an analogous way, the q-equilateral hyperboloid (32) can be written in a quaternionic form as:

$$H_q(a^2) : q^2 + \bar{q}^2 = 2a^2. \tag{33}$$

Analogously to the usual inversion (26) we can define a quaternionic inversion on $\mathbb{R}^4\setminus\{0\}$:

$$q' = I_r(q) = \frac{r^2}{\bar{q}} \tag{34}$$

and it is easy to see that $I_r^2 := I_r \circ I_r = id$ thus, as in the case of the planar inversion, I_r is an involution.

Proposition 3.2 *The image of the q-equilateral hyperboloid (33) by the inversion I_r is a Bernoulli q-lemniscate.*

Since the q-equilateral hyperboloid has the foci $\left(\pm a\sqrt{2}, 0, 0, 0\right)$ while its image by the inversion I_r is the Bernoulli q-lemniscate with the foci $\left(\pm\frac{r^2}{a\sqrt{2}}, 0, 0, 0\right)$, the foci are conserved by the inversion I_r if and only if $a\sqrt{2} = \frac{r^2}{a\sqrt{2}}$, i.e. $r = a\sqrt{2}$. More exactly, the image by the inversion I_r of the q-equilateral hyperboloid that has the foci $(\pm r, 0, 0, 0)$, i.e. having the equation $x^2 - y^2 - z^2 - w^2 = \dfrac{r^2}{2}$ is the Bernoulli q-lemniscate with the same foci, having the equation $\left(x^2 + y^2 + z^2 + w^2\right)^2 = 2r^2\left(x^2 - y^2 - z^2 - w^2\right)$.

Proposition 3.3 *The relation (31), specific to a Bernoulli lemniscate, holds also true for a Bernoulli q-lemniscate.*

In a similar way with the quaternionic products of Bernoulli lemniscates we can introduce two quaternionic products of Bernoulli q-lemniscates: ∠

$$L_q(r_1) \odot_c L_q(r_2) = L_q(r_1 \odot_c r_2), \quad L_q(r_1) \odot_{pc} L_q(r_2) = L_q(r_1 \odot_{pc} r_2).$$

5 On a quaternionic product of conics

We consider the set $Q_{q_0} = \{c + \alpha q_0; c, \alpha \in \mathbb{R}\}$, where $q_0 = ai + bj + dk$ is a pure imaginary quaternion, arbitrarily chosen but fixed, therefore a, b and d are fixed.

Let us consider $q_0 \neq 0$. For every $q_1 = c_1 + \alpha_1 q_0$, $q_2 = c_2 + \alpha_2 q_0 \in Q_{q_0}$ we have:
$q_1 + q_2 = (c_1 + c_2) + (\alpha_1 + \alpha_2) q_0 \in Q_{q_0}$ and
$q_1 \cdot q_2 = c_3 + \alpha_3 q_0 \in Q_{q_0}$, where:

$$c_3 = c_1c_2 - \alpha_1\alpha_2(a^2+b^2+d^2) = c_1c_2 - \alpha_1\alpha_2\Delta_0, \text{with } \Delta_0 = a^2+b^2+d^2, \alpha_3 = \alpha_1c_2 + \alpha_2c_1, \quad (35)$$

thus Q_{q_0} is stable at the sum and multiplication defined this way.

Since $Q_{q_0} \subset Q$ is a vector subspace, generated by $\{1, q_0\}$, we can consider also the additive group structure on Q_{q_0}. Moreover, the quaternionic product induces on $Q^*_{q_0} = Q_{q_0} \setminus \{0\}$ a group structure isomorphic with the multiplicative group on $\mathbb{C}^*$. Therefore, using these two operations, Q_{q_0} is a field isomorphic with the field $\mathbb{C}$. Thus we can identify Q_{q_0} with $\mathbb{R}^2$ and even with $\mathbb{C}$, using the isomorphism given by $f : Q_{q_0} \longrightarrow \mathbb{C}$, where $f(c + \alpha q_0) = c + \alpha\sqrt{\Delta_0} i$, for every $q = c + \alpha q_0 \in Q_{q_0}$.

Now we consider a conic Γ in the Euclidean plane given by:

$$\Gamma : x^2 + dy^2 + ax + by + c = 0$$

and we associate to Γ the quaternion $q(\Gamma) = c + ai + bj + dk = (c, a, b, d) \in \mathbb{R}^4$.

Let Γ_1 and Γ_2 be two conics given by:

$$\Gamma_1 : x^2 + \alpha_1 dy^2 + \alpha_1 ax + \alpha_1 by + c_1 = 0, \quad \Gamma_2 : x^2 + \alpha_2 dy^2 + \alpha_2 ax + \alpha_2 by + c_1 = 0,$$

where a, b, $d, c_1, c_2, \alpha_1, \alpha_2 \in \mathbb{R}$. We can associate a conic $\Gamma_4 = \Gamma_1 \oplus \Gamma_2$ corresponding to the sum and $\Gamma_3 = \Gamma_1 \odot_c \Gamma_2$ corresponding to the product of the corresponding quaternions $q_1 = q(\Gamma_1)$ and $q_2 = q(\Gamma_2)$:

$$\Gamma_4 = \Gamma_1 \oplus \Gamma_2 = q^{-1}\left(q(\Gamma_1) + q(\Gamma_2)\right) : x^2 + \alpha_4 dy^2 + \alpha_4 ax + \alpha_4 by + c_4 = 0,$$
$$\Gamma_3 = \Gamma_1 \odot_c \Gamma_2 = q^{-1}\left(q(\Gamma_1) \cdot q(\Gamma_2)\right) : x^2 + \alpha_3 dy^2 + \alpha_3 ax + \alpha_3 by + c_3 = 0,$$

where $\alpha_4 = \alpha_1 + \alpha_2$, $\quad c_4 = c_1 + c_2$ and c_3 and α_3 are given by formulas (35).

Thus, we can consider now the arbitrarily chosen but fixed conic $\Gamma_0 : x^2 + dy^2 + ax + by = 0$ and also the set of associated conics:

$$\mathcal{Q}_{\Gamma_0} = \left\{\Gamma : x^2 + \alpha dy^2 + \alpha ax + \alpha by + c = 0; c, \alpha \in \mathbb{R}\right\}.$$

One can prove that $\odot_c$ is a commutative and associative law and has as neutral element the (imaginary) conic $x^2 + 1 = 0$ (corresponding to $c = 1$, $\alpha = 0$).

Let us note that for a conic $\Gamma \in \mathcal{Q}_{\Gamma_0}$, the corresponding c and α are unique. Of course, a given conic can be seen as belonging to several families, but once the conical family is fixed, the corresponding c and α are unique; therefore the above operations on an arbitrary, but fixed family $\mathcal{Q}_{\Gamma_0}$ are well defined.

Property 4.1 The triple $(\mathcal{Q}_{\Gamma_0}, \oplus, \odot_c)$ is a field isomorphic to the field of complex numbers.

Let us consider (c, α) as parameters in Q_{q_0} or $\mathcal{Q}_{\Gamma_0}$. We have that the product of (c_1, α_1) and (c_2, α_2) corresponds to the parameters $(c_1c_2 - \alpha_1\alpha_2\Delta_0, \alpha_1c_2 + \alpha_2c_1)$, where $\Delta_0 = -q_0^2$. It is easy to see that the product factorizes to the projective space P^1 i.e. we can define $[c_1, \alpha_1]\odot_{c,\Delta_0}[c_2, \alpha_2] = [c_1c_2 - \alpha_1\alpha_2\Delta_0, \alpha_1c_2 + \alpha_2c_1]$ and the neutral element for $\odot_{c,\Delta_0}$ is $[1, 0]$.

Property 4.2 The corresponding group structure is isomorphic with the multiplicative circular group S^1.

Now let us consider $\alpha_1c_2 + \alpha_2c_1$, α_1, $\alpha_2 \neq 0$. We obtain that the product of $[c_1, \alpha_1] = \left[\frac{c_1}{\alpha_1}, 1\right]$ and $[c_2, \alpha_2] = \left[\frac{c_2}{\alpha_2}, 1\right]$ corresponds to $[c_1c_2 - \alpha_1\alpha_2\Delta_0, \alpha_1c_2 + \alpha_2c_1] = \left[\frac{\frac{c_1}{\alpha_1}\frac{c_2}{\alpha_2} - \Delta_0}{\frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2}}, 1\right]$, therefore the product $\odot_c$ defined in the first section comes from the product $\odot_{c,1}$ when $\Delta_0 = 1$ and it is restricted to the classes $\{[r, 1]\,; r > 0\}$.

Thus if $\Delta_0 = 1$ then we can consider restrictions of the product $\odot_{c,1}$ from $P^1 = P_1^1 \cup P_2^1$ to P_1^1 or P_2^1, where $P_1^1 = \{[r, 1]\,; r \in \mathbb{R}\}$ and $P_2^1 = \{[1, r]\,; r \in \mathbb{R}\}$. We have to note that the products restricted to P_1^1 and P_2^1 are partial. Indeed, for example, if $r \in \mathbb{R}$, then $[r, 1] \odot_{c,1} [-r, 1] = \left[-r^2 - 1, 0\right] = [1, 0] \in P_2^1$. One can explain now why $\odot_c$ does not have a neutral element when it is restricted to the classes $\{[r, 1]\,; r > 0\}$ or even to P_1^1, since the neutral element $[1, 0] \in P_2^1$ does not belong to these sets. Notice also that the sum of parameters do not factorize to an additive law in the projective space P^1.

More particulary, for $\alpha_1 = \alpha_2 = 1$, $a = b = 0$ and $d = -1$, we obtain the composition law associated to the family of equilateral hyperbolas approached in the first section.

Considering now the determinants:

$$\delta = \begin{vmatrix} 1 & 0 \\ 0 & \alpha d \end{vmatrix} = \alpha d, \quad \Delta = \begin{vmatrix} 1 & 0 & \frac{a\alpha}{2} \\ 0 & \alpha d & \frac{b\alpha}{2} \\ \frac{a\alpha}{2} & \frac{b\alpha}{2} & c \end{vmatrix} = -\frac{1}{4}\alpha\left(b^2\alpha - 4cd + a^2d\alpha^2\right).$$

associated to a conic in a family $\mathcal{Q}_{\Gamma_0}$, it is easy to see that the family does not always have only one type of conic. For example, in the case $a = b = 0$, $d = -1$, we have $\delta = \alpha$ and $\Delta = -c$. If $\alpha c \neq 0$ then all the conics are non-degenerated with the center in origin; for $\alpha > 0$ all the conics are hyperbolas; for $\alpha < 0$ all the conics are ellipses; they are all real for $c < 0$ and all imaginary for $c > 0$.

We give now another approach in the case when the conic has a particular form. Thus, to every $x = (c, a, b, \alpha) \in \mathbb{R}^4$ we associate the quaternion $q(x) = c + (\alpha a)\,i + (\alpha b)\,j$ and the corresponding conic P_x given by $x^2 + \alpha ax + \alpha by + c = 0$. For $(a, b) \in \mathbb{R}^2$ we consider $\mathcal{Q}_{(a,b)} = \{q(x) = c + (\alpha a)\,i + (\alpha b)\,j : c, \alpha \in \mathbb{R}\}$ and $\mathcal{P}_{(a,b)} = \{P_x : c, \alpha \in \mathbb{R}\}$.

If $x_1 = (c_1, a, b, \alpha_1)$, $x_2 = (c_2, a, b, \alpha_2) \in \mathbb{R}^4$ then $q(x_1) \cdot q(x_2) = q(x_3)$ where $x_3 = \left(c_1c_2 - \alpha_1\alpha_2\left(a^2 + b^2\right), a, b, \alpha_1c_2 + \alpha_2c_1\right)$. Thus the quaternionic product is a composition law that is internal on $\mathcal{Q}_{(a,b)}$and induces also an internal composition law on $\mathcal{P}_{(a,b)}$ (for every $P_{x_1}, P_{x_2} \in \mathcal{P}_{(a,b)}$ we have $P_{x_1} \odot_{c,\Delta} P_{x_2} = P_{x_3} \in \mathcal{P}_{(a,b)}$, where $\Delta = a^2 + b^2$).

Property 4.3 The quaternionic product on $\mathcal{Q}_{(a,b)}$ and the induced composition law on $\mathcal{P}_{(a,b)}$ are commutative, associative and has as neutral element the quaternion, respectively the conic corresponding to $(1, a, b, 0)$.

6 The quaternionic product on $\mathcal{Q}_{\Gamma_0}$ extended to other curves

Let us consider a more general case, i.e. the following equation $x^2 + dy^2 + c = 0$, $c \neq 0$. We have to analyze two different cases.

◦ If $d < 0$ then we have $d = -\mu^2$, so the equation $x^2 - (\mu y)^2 + c = 0$ is the equation of a hyperbola and can be written as $(z + \overline{z})^2 + \mu^2 (z - \overline{z})^2 + 4c = 0$. Therefore, the image of this hyperbola by the inversion I_r has the equation:

$$2(x^2 - y^2)(1 + \mu^2) + 2(x^2 + y^2)(1 - \mu^2) + \frac{4c}{r^4}\left(x^2 + y^2\right)^2 = 0. \tag{36}$$

For $\mu = \pm 1$ the equation (36) is $\left(x^2 + y^2\right)^2 = \frac{r^4}{-c}\left(x^2 - y^2\right)$, which is the equation of a Bernoulli lemniscate. Let us note that for $c < 0$ we have an usual equation of a Bernoulli lemniscate ; for $c > 0$ the equation can be written as $\left(y^2 + x^2\right)^2 = \frac{r^4}{c}\left(y^2 - x^2\right)$ where $\frac{r^4}{c} > 0$, therefore, with a change of coordinates, we have also an equation of a Bernoulli lemniscate.

For $\mu \neq \pm 1$ the equation (36) is $\left(x^2 + y^2\right)^2 = \frac{r^4}{-c}\left(x^2 - \mu^2 y^2\right)$ which is (with the above discussion for $c < 0$, but also for $c > 0$) the equation of a generalized lemniscate.

Thus, if $d < 0$ for every type of above lemniscates L_1, L_2 and L_3, taking into account $\odot_{c,\Delta}$ introduced in the previous section, we have $L_1 \odot_{c,\Delta} L_2 = L_3$, where $\Delta = \mu^4$.

For $\alpha_1 = \alpha_2 = 1$ and $\mu = \pm 1$ the above product has the same form as the product $\odot_c$ on the family of Bernoulli lemniscates, considered in a previous section.

◦ If $d > 0$ then we have $d = \mu^2$, so the equation is $x^2 + (\mu y)^2 + c = 0$, which is the equation of an ellipse when $\mu \neq \pm 1$ or of a circle when $\mu = \pm 1$ and can be written as $(z + \overline{z})^2 - \mu^2 (z - \overline{z})^2 + 4c = 0$. Therefore, the image of this curve (ellipse or circle) by the inversion I_r has the equation:

$$2(x^2 - y^2)(1 - \mu^2) + 2(x^2 + y^2)(1 + \mu^2) + \frac{4c}{r^4}\left(x^2 + y^2\right)^2 = 0. \tag{37}$$

For $\mu = \pm 1$ the equation (37) is $\left(x^2 + y^2\right) = \frac{r^4}{-c}$, $x^2 + y^2 \neq 0$, which is the equation of a real circle (for $c < 0$) or an imaginary circle (for $c > 0$); for $x^2 + y^2 = 0$ the circle is degenerated in a point (the origin). Therefore, the image of the circle by the inversion I_r is also a circle.

Using $\odot_{c,\Delta}$ introduced in a previous section, we have $C_1 \odot_{c,\Delta} C_2 = C_3$, where $\Delta = \mu^4$ and C_1, C_2 and C_3 are circles.

For $\alpha_1 = \alpha_2 = 1$ and $\mu = \pm 1$ we obtain the composition law associated to the above family of circles (see [5]).

For $\mu \neq \pm 1$ the equation (37) is $\left(x^2+y^2\right)^2 = \frac{r^4}{-c}\left(x^2+\mu^2 y^2\right)$, which is:
– for $c < 0$, it is the equation of a Booth lemniscate (an oval of Booth with 0 as an isolated point, for $\mu \neq 0$, or a pair of externally tangent circles for $\mu = 0$) or,
– for $c > 0$, it is the equation of a curve degenerated in a double point.

Therefore for $\mu \neq \pm 1$, the image of the ellipse by the inversion I_r is a Booth lemniscate or a curve degenerated in a point.

We have $L_1 \odot_{c,\Delta} L_2 = L_3$, where $\Delta = \mu^4$ and L_1, L_2 and L_3 are lemniscates as above.

Let us note that if $d = 0$ then the equation $x^2 + ax + by + c = 0$ of parabolic type form can be written as: $(z+\bar{z})^2 + 2a(z+\bar{z}) - 2b(z-\bar{z})i + 4c = 0$. Therefore, the image of this curve by the inversion I_r has the equation $r^4 x^2 + r^2(ax+by)\left(x^2+y^2\right) + c\left(x^2+y^2\right)^2 = 0$.

For $a = c = 0$ corresponding to the canonic form of the parabolic type form equation we have $y\left(x^2+y^2\right) = 2\left(\frac{-r^2}{2b}\right)x^2$,
which is the equation of a *cissoid of Diocles*. We have $D_1 \odot_{c,\Delta} D_2 = D_3$, where $\Delta = b^2$ and D_1, D_2, D_3 are cissoids of Diocles.

7 On octonionic product for pairs of hyperbolas

As an octonion $o \in \mathbb{O}$ can be thought as a pair of quaternions $o := (q_1, q_2)$ and their non-associative product is:

$$o_1 \cdot o_2 = (p_1, p_2) \cdot (q_1, q_2) := (p_1 q_1 - \bar{q}_2 p_2, q_2 p_1 + p_2 \bar{q}_1) \tag{38}$$

(with bar for the usual conjugation of quaternions), it follows that a pair of hyperbolas $\mathcal{P} = (H_1, H_2)$ can be considered as an octonion $o(\mathcal{P}) := (q(H_1), q(H_2))$. We define further the product of two pairs of hyperbolas $\mathcal{P}_1$ and $\mathcal{P}_2$ as:

$$\mathcal{P}_1 \odot_o \mathcal{P}_2 = o(\mathcal{P}_1) \cdot o(\mathcal{P}_2). \tag{39}$$

If $H_i = H(r_i)$, $i = \overline{1,4}$, then a long but straightforward computation yields:

$$(r_1, r_2) \odot_o (r_3, r_4) := \left(\frac{r_1 r_3 - r_2 r_4 - 2}{r_1 + r_2 + r_3 - r_4}, \frac{r_1 r_4 + r_2 r_3}{r_1 + r_3 + r_4 - r_2}\right) \tag{40}$$

with the conditions:

$$r_1 + r_2 + r_3 \neq r_4, \quad r_1 + r_3 + r_4 \neq r_2. \tag{41}$$

Let us note that the product $\odot_c$ is commutative, but the quaternionic product and the octonionic product $\odot_o$ are non-commutative.

As in a previous section, we can introduce an octonionic product on pairs of oriented hyperbolas with $(\varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_4)$ on the second slot.

When the first pair or the second pair is $(H(1), H(1))$ or when the unit hyperbola $H(1)$ is on the first or on the second position of the first pair or/and of the second pair of hyperbolas, interesting results are obtained.

8 Applications

As a **first application** of the given product $\odot_c$, we define a 2-valued composition law on $H_e \setminus \{E(1,1)\}$, where H_e is the main branch of the reduced equilateral hyperbola $H_e : xy = 1, \quad x, y \in (0, +\infty)$. Thus, for evey $P_1, P_2 \in H_e \setminus \{E(1,1)\}$ we define:

$$P_1 \odot_c P_2 = \{A, B \in H_e;\ x_A = \max(P_1) \odot_c \max(P_2) = y_B\}, \tag{42}$$

where for every point $P \in H_e$, $\max(P) := \max\{x_P, y_P\} \geq 1$.

If $\max(P_1) = r_1$ and $\max(P_2) = r_2$ then the product has the explicit form:

$$P_1 \odot_c P_2 = \left\{ A\left(\frac{r_1 r_2 - 1}{r_1 + r_2}, \frac{r_1 + r_2}{r_1 r_2 - 1}\right), B\left(\frac{r_1 + r_2}{r_1 r_2 - 1}, \frac{r_1 r_2 - 1}{r_1 + r_2}\right) \right\}. \tag{43}$$

As, for example, $\left(2, \frac{1}{2}\right) \odot_c \left(3, \frac{1}{3}\right) = \left(\frac{1}{2}, 2\right) \odot_c \left(\frac{1}{3}, 3\right) = \left(2, \frac{1}{2}\right) \odot_c \left(\frac{1}{3}, 3\right) = \left(\frac{1}{2}, 2\right) \odot_c \left(3, \frac{1}{3}\right) = \{E(1,1)\}$, it follows that the point $E(1,1) \in H_e$ belongs to the image of this composition law; more general, E is obtained when $r_2 = \frac{r_1+1}{r_1-1}$, but it is easy to see that the pair of points is not unique.

Replacing the product $\odot_c$with the product $\odot_{pc}$, we can define (in an analogous way) another 2-valued composition law on the main branch of H_e.

A **second application** is also about a multi-valued product that can be introduced on the set of hyperbolic matrices. A matrix $\gamma \in SL_2(\mathbb{R})$ is called *hyperbolic* if its eigenvalues are real and distinct and we denote by $SL_2^H(\mathbb{R})$ their set. Since the characteristic polynomial of an arbitrary γ is:

$$f_\gamma(x) = x^2 - tr(\gamma)x + \det(\gamma) = x^2 - tr(\gamma)x + 1, \tag{44}$$

it follows that $\gamma \in SL_2^H(\mathbb{R})$ if and only if $|tr(\gamma)| > 2$ and then its eigenvalues are reciprocal numbers. Using the product $\odot_c$we introduce a product on $SL_2^H(\mathbb{R})$:

$$\gamma_1 \odot_c \gamma_2 = \left\{\gamma \in SL_2^H(\mathbb{R}); e(\gamma) = e(\gamma_1) \odot_c e(\gamma_2)\right\}, \tag{45}$$

where $e(\gamma)$ is the eigenvalue of γ which is larger than 1.

In an analogous way, replacing the product $\odot_c$ by the product $\odot_{pc}$ product, we introduce another product on $SL_2^H(\mathbb{R})$.

Following the approach of Section 4 from [5], we obtain expressions for $\odot_c$ and for $\odot_{pc}$ respectively in terms of trace and/or determinant of the corresponding matrices.

In the **third application**, we associate to every vertex P_i, $i = \overline{1,n}$, of a polygon $\mathcal{P} = P_1...P_n$ a p-number, where $p \in \{c, pc\}$, and then we associate a p-chain to $\mathcal{P}$. We define the p-number of the vertex P_i as $p_i := l_{i-1} \odot_p l_i$,where $l_i = \|P_i P_{i+1}\| \in M = (0, +\infty)$ is the Euclidian length.

For example, the right triangle ΔABC with legs $\|AB\| = 3$ and $\|AC\| = 4$ has the c-chain $c(\Delta ABC) := (c_A, c_B, c_C) = \left(\frac{11}{7}, \frac{7}{4}, \frac{19}{9}\right)$, because $c_A = l_{\|CA\|} \odot_c l_{\|AB\|} = 4 \odot_c 3 = \frac{11}{7}$, $c_B = \frac{7}{4}$, $c_C = \frac{19}{9}$and the pc-chain $pc(\Delta ABC) := (pc_A, pc_B, pc_C) = \left(\frac{13}{7}, 2, \frac{7}{3}\right)$, because $pc_A = l_{\|CA\|} \odot_{pc} l_{\|AB\|} = 4 \odot_{pc} 3 = \dfrac{13}{7}$, $pc_B = 2$, $pc_C = \dfrac{7}{3}$.

It is easy to see that a (regular) polygon with sides of length 1 has a vanishing c-chain $(0, ..., 0)$, as $c_i = 1 \odot_c 1 = 0$, $i = \overline{1,n}$, and a constant pc-chain $(1, ..., 1)$, as $pc_i = 1 \odot_{pc} 1 = 1$, $i = \overline{1,n}$. .

Conversely, knowing the c-chain or the pc-chain of a polygon $\mathcal{P}$ we can deduce the length of some of its sides.

More precisely, if a polygon $\mathcal{P}$ has $n = 2k+1$ sides and a vanishing c-chain then $\mathcal{P}$ has all its sides of length 1, and if a polygon $\mathcal{P}$ has $n = 2k$ sides and a vanishing c-chain then $\mathcal{P}$ has the odd sides of a lenght l and the even sides of the lenght $\frac{1}{l}$. If a polygon $\mathcal{P}$ has a constant pc-chain $(1, ..., 1)$ and $n = 2k+1$ sides then $\mathcal{P}$ has at least $k+1$sides of length 1, and if $\mathcal{P}$ has $n = 2k$ sides then $\mathcal{P}$ has at least k sides of length 1. Thus knowing the c-chain or the pc-chain of a polygon $\mathcal{P}$ we know a set of relations involving the length of its sides.

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