

On the Ψ -asymptotic equivalence of the Ψ - bounded solutions of two Lyapunov matrix differential equations

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Abstract. Using Schauder - Tychonoff fixed point theorem and the technique of Kronecker product of matrices, we prove existence results for Ψ -asymptotic equivalence of the Ψ -bounded solutions of two Lyapunov matrix differential equations.

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1 Introduction

The purpose of this paper is to provide sufficient conditions for Ψ -asymptotic equivalence of the Ψ -bounded solutions of two Lyapunov matrix differential equations

$$Z' = A(t)Z + ZB(t) \tag{1}$$

$$Z' = A(t)Z + ZB(t) + F(t, Z). \tag{2}$$

These conditions can be written in terms of fundamental matrices of the matrix differential equations

$$Z' = A(t)Z \tag{3}$$

$$Z' = ZB(t) \tag{4}$$

and of the function F .

Here, Ψ is a matrix function who allows obtaining a mixed asymptotic behavior for the components of solutions of the differential equations.

History of the problem. A classical result in connection with boundedness of solutions of systems of ordinary differential equations

$$x' = A(t)x + f(t, x) \tag{5}$$

was given by Coppel [5]. The problems of Ψ -bounded solutions for systems of ordinary differential equations or for Lyapunov matrix differential equations have been studied by many authors. See, for instance [2], [6], [7], [8], [9], [11], and the references therein.

The results of this paper extend known results of W. A. Coppel [5], F. Brauer and J. S. W. Wong [3], [4], T. G. Hallam [9].

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The main tools used in this paper are Schauder - Tychonoff fixed point theorem and the technique of Kronecker product of matrices (combined with the variation of constants formula), which has been successfully applied in various fields of matrix theory. See, for example, the cited papers and the references cited therein.

2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let \mathbb{R}^d be the Euclidean d - dimensional space. For $x = (x_1, x_2, \dots, x_d)^T \in \mathbb{R}^d$, let $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_d|\}$ be the norm of x (here, T denotes transpose).

Let $\mathcal{M}_{d \times d}$ be the linear space of all real $d \times d$ matrices.

For $A = (a_{ij}) \in \mathcal{M}_{d \times d}$, we define the norm $|A|$ by formula $|A| = \sup_{\|x\| \leq 1} \|Ax\|$. It is well-

known that $|A| = \max_{1 \leq i \leq d} \left\{ \sum_{j=1}^d |a_{ij}| \right\}$.

In the previous equations, we assume that A and B are continuous $d \times d$ matrices on \mathbb{R}_+ and $F : \mathbb{R}_+ \times \mathcal{M}_{d \times d} \rightarrow \mathcal{M}_{d \times d}$ is a continuous function.

A solution of the equation (2) is a continuous differentiable $d \times d$ matrix function satisfying the equation (2) for all $t \in \mathbb{R}_+ = [0, \infty)$.

We consider the fundamental matrices $X(t)$ and $Y(t)$ for the equations (3) and (4) respectively.

Let $\Psi_i : \mathbb{R}_+ \rightarrow (0, \infty)$, $i = 1, 2, \dots, d$, be continuous functions and

$$\Psi = \text{diag} [\Psi_1, \Psi_2, \dots, \Psi_d].$$

Let $P_1, P_2 \in \mathcal{M}_{d \times d}$ be supplementary projections and define

$$\Omega_i(t, s) = \Psi(t)X(t)P_iX^{-1}(s)\Psi^{-1}(s), \text{ for } i = 1, 2.$$

Definition 2.1 ([6], [7]). A vector function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is called Ψ -bounded on \mathbb{R}_+ if $\Psi(t)\varphi(t)$ is bounded on \mathbb{R}_+ (i.e. there exists $m > 0$ such that $\|\Psi(t)\varphi(t)\| \leq m$, for all $t \in \mathbb{R}_+$). Otherwise, the function φ is Ψ -unbounded on \mathbb{R}_+ .

Definition 2.2 ([7]). A matrix function $M : \mathbb{R}_+ \rightarrow \mathcal{M}_{d \times d}$ is Ψ - bounded on \mathbb{R}_+ if the matrix function $\Psi(t)M(t)$ is bounded on \mathbb{R}_+ (i.e. there exists $m > 0$ such that $|\Psi(t)M(t)| \leq m$, for all $t \in \mathbb{R}_+$). Otherwise, the matrix function M is Ψ - unbounded on \mathbb{R}_+ .

Remark 2.1 1. These definitions extend the definition of boundedness of scalar functions.

2. For $\Psi = I_d$, we get the notion of classical boundedness (see [5]).

3. It is easy to see that if Ψ and Ψ^{-1} are bounded on \mathbb{R}_+ , then the Ψ -boundedness is equivalent to the classical boundedness.

Definition 2.3 ([1]) Let $A = (a_{ij}) \in \mathcal{M}_{m \times n}$ and $B = (b_{ij}) \in \mathcal{M}_{p \times q}$. The Kronecker product of A and B , written $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathcal{M}_{mp \times nq}$.

The important rules of calculation of the Kronecker product are given in [1], [10] and Lemma 2.1 [8].

Definition 2.4 ([10]) The map $\mathcal{V}ec : \mathcal{M}_{m \times n} \rightarrow \mathbb{R}^{mn}$, defined by

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \dots, a_{m1}, a_{12}, a_{22}, \dots, a_{m2}, \dots, a_{1n}, a_{2n}, \dots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathcal{M}_{m \times n}$, is called the vectorization operator.

For important properties and rules of calculation of the $\mathcal{V}ec$ operator, see Lemmas 2.2, 2.3, 2.5 in [8].

For the corresponding Kronecker product system associated with (2), see Lemma 2.4 in [8].

The Lemmas 2.6 and 2.7 in [8], play an important role in the proofs of main results of present paper.

3 Main results

The purpose of this section is to give sufficient conditions for Ψ -asymptotic equivalence of the Ψ -bounded solutions of two pairs of matrix differential equations, namely (3)-(6) and (1)-(2). The first result is motivated by a Theorem of Hallam [9].

Theorem 3.1 Suppose that:

1) There are supplementary projections $P_1, P_2 \in \mathcal{M}_{d \times d}$, a continuous function $\varphi : \mathbb{R}_+ \rightarrow (0, \infty)$ that satisfies the condition $\int_0^\infty \varphi(s) ds = \infty$ and a constant $K > 0$ such that the fundamental matrix $X(t)$ for the matrix differential equation (3) satisfies the inequality

$$\int_{t_0}^t \varphi(s) |\Omega_1(t, s)| ds + \int_t^\infty \varphi(s) |\Omega_2(t, s)| ds \leq K$$

for all $t \geq t_0 \geq 0$, where t_0 is sufficiently large;

2) The continuous matrix function $F : \mathbb{R}_+ \times \mathcal{M}_{d \times d} \rightarrow \mathcal{M}_{d \times d}$ satisfies the inequality

$$\varphi^{-1}(t) |\Psi(t)F(t, Z)| \leq \omega(t, |\Psi(t)Z|)$$

for all $t \geq t_0$ and $Z \in \mathcal{M}_{d \times d}$, where $\omega(t, r) : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous nonnegative function and also nondecreasing in r , for each fixed $t \geq t_0$.

Furthermore, let $\gamma_\lambda(t) = \sup_{s \geq t} \omega(s, \lambda)$ and assume that $\lim_{t \rightarrow \infty} \gamma_\lambda(t) = 0$ for each $\lambda \in (0, \infty)$.

Then, corresponding to each Ψ -bounded solution $Z_0(t)$ of (3), there exists a Ψ -bounded solution $Z(t)$ of the matrix differential equation

$$Z' = A(t)Z + F(t, Z). \tag{6}$$

such that

$$\lim_{t \rightarrow \infty} |\Psi(t)(Z(t) - Z_0(t))| = 0. \tag{7}$$

Conversely, to each Ψ -bounded solution $Z(t)$ of (6), there exists a Ψ -bounded solution $Z_0(t)$ of (3) such that (7) holds.

Proof. We prove this Theorem by means of the fixed point theorem of Schauder-Tychonoff (Coppel [5], Chapter I, section 2).

Let C_Ψ denote the set of all matrix functions $Z(t)$ which are continuous and Ψ -bounded on $[t_0, \infty)$ and let S_ρ be the subset of those functions $Z(t)$ such that $|Z|_\Psi = \sup_{t \geq t_0} \{|\Psi(t)Z(t)|\} \leq \rho$.

For $Z \in S_{2\rho}$, we define the operator T by

$$(TZ)(t) = Z_0(t) + \int_{t_0}^t X(t)P_1X^{-1}(s)F(s, Z(s))ds - \int_t^\infty X(t)P_2X^{-1}(s)F(s, Z(s))ds \quad (8)$$

for $t \geq t_0$, where $Z_0(t)$ is a Ψ -bounded solution of (3) such that $Z_0 \in S_\rho$ and t_0 is such that $\gamma_{2\rho}(t_0) \leq \rho/K$.

From hypotheses 1) and 2) , TZ exists and is continuous differentiable on \mathbb{R}_+ .

Indeed, for $v \geq t \geq t_0$,

$$\begin{aligned} & \left| \int_t^v X(t)P_2X^{-1}(s)F(s, Z(s))ds \right| \\ &= \left| \Psi^{-1}(t) \int_t^v \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\Psi(s)F(s, Z(s))ds \right| \\ &\leq |\Psi^{-1}(t)| \int_t^v \varphi(s) |\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)| \varphi^{-1}(s) |\Psi(s)F(s, Z(s))| ds \\ &\leq |\Psi^{-1}(t)| \int_t^v \varphi(s) |\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)| \omega(s, |\Psi(s)Z(s)|) ds \\ &\leq |\Psi^{-1}(t)| \int_t^v \varphi(s) |\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)| \omega(s, 2\rho) ds \\ &\leq |\Psi^{-1}(t)| \int_t^v \varphi(s) |\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)| \gamma_\lambda(t) ds \\ &\leq |\Psi^{-1}(t)| \int_t^v \varphi(s) |\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)| \gamma_\lambda(t_0) ds \\ &\leq |\Psi^{-1}(t)| \int_t^v \varphi(s) |\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)| \rho/K ds \\ &\leq \rho/K |\Psi^{-1}(t)| \int_t^v \varphi(s) |\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)| ds. \end{aligned}$$

From the first assumption of Theorem 3.1, it follows that the integral

$$\int_t^\infty X(t)P_2X^{-1}(s)F(s, Z(s))ds$$

is convergent for all $Z \in S_{2\rho}$ and $t \geq t_0$.

From hypotheses, TZ exists and is continuous differentiable on $[t_0, \infty)$.

This operator has the following properties:

- a) T maps $S_{2\rho}$ into itself;

Indeed, for any $Z \in S_{2\rho}$, and for $t \geq t_0$, we have

$$\begin{aligned}
 |\Psi(t)(TZ)(t)| &\leq |\Psi(t)Z_0(t)| \\
 &\quad + \int_{t_0}^t \varphi(s) |\Omega_1(t, s)| \varphi^{-1}(s) |\Psi(s)F(s, Z(s))| ds \\
 &\quad + \int_t^\infty \varphi(s) |\Omega_2(t, s)| \varphi^{-1}(s) |\Psi(s)F(s, Z(s))| ds \\
 &\leq |\Psi(t)Z_0(t)| + \int_{t_0}^t \varphi(s) |\Omega_1(t, s)| \omega(s, |\Psi(s)Z(s)|) ds \\
 &\quad + \int_t^\infty \varphi(s) |\Omega_2(t, s)| \omega(s, |\Psi(s)Z(s)|) ds \\
 &\leq |\Psi(t)Z_0(t)| + \int_{t_0}^t \varphi(s) |\Omega_1(t, s)| \omega(s, 2\rho) ds \\
 &\quad + \int_t^\infty \varphi(s) |\Omega_2(t, s)| \omega(s, 2\rho) ds \\
 &\leq \rho + \int_{t_0}^t \varphi(s) |\Omega_1(t, s)| \gamma_{2\rho}(t_0) ds + \int_t^\infty \varphi(s) |\Omega_2(t, s)| \gamma_{2\rho}(t_0) ds \\
 &\leq \rho + \rho/K \cdot K = 2\rho.
 \end{aligned}$$

This proves the assertion.

b). T is continuous, in the sense that if $Z_n \in S_{2\rho}$, ($n = 1, 2, \dots$) and $Z_n \rightarrow Z$ uniformly on every compact subinterval J of $[t_0, \infty)$, then $TZ_n \rightarrow TZ$ uniformly on every compact subinterval J of $[t_0, \infty)$;

Indeed, from (8) we have, for any $t \in J \subset [t_0, \infty)$,

$$\begin{aligned}
 |(TZ_n)(t) - (TZ)(t)| &\leq |\Psi^{-1}(t)| \int_{t_0}^t \varphi(s) \Omega_1(t, s) \varphi^{-1}(s) |\Psi(s) (F(s, Z_n(s)) - F(s, Z(s)))| ds \\
 &\quad + |\Psi^{-1}(t)| \int_t^\infty \varphi(s) |\Omega_2(t, s)| \varphi^{-1}(s) |\Psi(s) (F(s, Z_n(s)) - F(s, Z(s)))| ds.
 \end{aligned} \tag{9}$$

Let $J = [\alpha, \beta]$. For a fixed $\varepsilon > 0$, we choose $t_1 \geq t_0$ sufficiently large ($t_1 \geq \beta$) such that

$$\gamma_{2\rho}(t_1) \leq \varepsilon \left(4K \sup_{t \in J} \{ |\Psi^{-1}(t)| \} \right)^{-1}.$$

Since F, Ψ, φ are continuous and $Z_n \rightarrow Z$ uniformly on $[t_0, t_1]$, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi^{-1}(s) |\Psi(s) (F(s, Z_n(s)) - F(s, Z(s)))| < \varepsilon \left(4K \sup_{t \in J} \{ |\Psi^{-1}(t)| \} \right)^{-1},$$

for $s \in [t_0, t_1]$ and $n \geq n_0$.

Thus, for $t \in J$ and $n \geq n_0$, the first integral term of (9) is strictly smaller than $\varepsilon/4$.

For the second integral term of (9), we write $\int_t^\infty \dots = \int_t^{t_1} \dots + \int_{t_1}^\infty \dots$ and have two cases:

i) For $t \in J$, $n \geq n_0$ and for $s \in [t, t_1]$, similarly, we also deduce that the first integral term is strictly smaller than $\varepsilon/4$.

ii) For $t \in J$, $n \in \mathbb{N}$ and $s \geq t_1$, we have

$$\begin{aligned} & \varphi^{-1}(s) |\Psi(s) (F(s, Z_n(s)) - F(s, Z(s)))| \\ \leq & \varphi^{-1}(s) (|\Psi(s)F(s, Z_n(s))| + |\Psi(s)F(s, Z(s))|) \\ \leq & \omega(s, |\Psi(s)Z_n(s)|) + \omega(s, |\Psi(s)Z(s)|) \\ \leq & 2 \cdot \omega(s, 2\rho) \leq 2 \cdot \gamma_{2\rho}(t_1) \end{aligned}$$

and then, the second integral term is strictly smaller than $\varepsilon/2$.

From (9) and these results, we get the sequence $(TZ_n)_n$ converges uniformly to TZ on compact subintervals of $[t_0, \infty)$.

c) The functions in the image set $TS_{2\rho}$ are echicontinuous and uniformly bounded at every point of every compact subinterval J of $[t_0, \infty)$.

Indeed, from a), $TS_{2\rho} \subset S_{2\rho}$. This shows the functions in the image set $TS_{2\rho}$ are uniformly bounded at every point of every compact subinterval J of $[t_0, \infty)$.

On the other hand, for $V = TZ$ and for $t \geq t_0$, we have

$$\begin{aligned} V'(t) &= Z'_0(t) + \int_{t_0}^t X'(t)P_1X^{-1}(s)F(s, Z(s))ds + X(t)P_1X^{-1}(t)F(t, Z(t)) \\ &\quad - \int_t^\infty X'(t)P_2X^{-1}(s)F(s, Z(s))ds + X(t)P_2X^{-1}(t)F(t, Z(t)) \\ &= A(t) \left(Z_0(t) + \int_{t_0}^t X(t)P_1X^{-1}(s)F(s, Z(s))ds - \int_t^\infty X(t)P_2X^{-1}(s)F(s, Z(s))ds \right) \\ &\quad + X(t)(P_1 + P_2)X^{-1}(t)F(t, Z(t)) \\ &= A(t)V(t) + F(t, Z(t)). \end{aligned}$$

Since

$$V'(t) = (A(t)\Psi^{-1}(t))(\Psi(t)V(t)) + (\varphi(t)\Psi^{-1}(t))(\varphi^{-1}(t)\Psi(t)F(t, Z(t))),$$

for $t \geq t_0$, and the matrices $A(t)\Psi^{-1}(t)$, $\Psi(t)V(t)$, $\varphi(t)\Psi^{-1}(t)$, $\varphi^{-1}(t)\Psi(t)F(t, Z(t))$ are uniformly bounded on every compact subinterval J of $[t_0, \infty)$, the derivatives of the functions in $TS_{2\rho}$ are uniformly bounded on every compact subinterval J of $[t_0, \infty)$. This shows the functions in $TS_{2\rho}$ are echicontinuous on every compact subinterval J of \mathbb{R}_+ .

Thus, all the conditions of the Schauder - Tychonoff theorem are satisfied. We conclude the operator T has at least one fixed point Z in $S_{2\rho}$. This fixed point Z is clearly a Ψ - bounded solution of (6).

To prove that (7) holds, we use the relation $\lim_{t \rightarrow \infty} |\Psi(t)X(t)P_1| = 0$, which is an immediate consequence of the inequality

$$|\Psi(t)X(t)P_1| \leq Ne^{-K^{-1} \int_{t_0}^t \varphi(s)ds}, \text{ for } t \geq t_0,$$

where N is a positive constant (see Theorem 4, [6]; actually, the proof of this result needs to be modified to include the function φ).

For a fixed $\varepsilon > 0$, we choose $t_2 > t_1$ so that

$$|\Psi(t)X(t)P_1| \int_{t_0}^t |X^{-1}(s)F(s, Z(s))| ds < \frac{\varepsilon}{2}, \text{ for } t \geq t_2.$$

From the choice of t_1 , for $t \geq t_1$ we can write

$$\left| \int_{t_1}^t \Psi(t)X(t)P_1X^{-1}(s)F(s, Z(s))ds - \int_t^\infty \Psi(t)X(t)P_2X^{-1}(s)F(s, Z(s))ds \right| < \frac{\varepsilon}{2}.$$

From the above mentioned results, we obtain

$$|\Psi(t)(Z(t) - Z_0(t))| < \varepsilon, \text{ for } t \geq t_2.$$

This shows that (7) is satisfied.

To prove the last statement of Theorem 3.1, consider a Ψ -bounded solution $Z(t)$ of equation (6). Define

$$Z_0(t) = Z(t) - \int_{t_0}^t X(t)P_1X^{-1}(s)F(s, Z(s))ds + \int_t^\infty X(t)P_2X^{-1}(s)F(s, Z(s))ds, \quad t \geq t_0.$$

With the previous arguments, we can show that $Z_0(t)$ is a Ψ - bounded solution of equation (3) that satisfies (7).

The proof of Theorem 3.1 is complete. □

Remark 3.1 *If we set*

$$Z = \begin{pmatrix} z_1 & z_1 & \cdots & z_1 \\ z_2 & z_2 & \cdots & z_2 \\ \cdots & \cdots & \cdots & \cdots \\ z_d & z_d & \cdots & z_d \end{pmatrix}, \quad F(t, Z) = \begin{pmatrix} f_1(t, z) & f_1(t, z) & \cdots & f_1(t, z) \\ f_2(t, z) & f_2(t, z) & \cdots & f_2(t, z) \\ \cdots & \cdots & \cdots & \cdots \\ f_d(t, z) & f_d(t, z) & \cdots & f_d(t, z) \end{pmatrix},$$

we get a version of Theorem 3.1 for systems of differential equations. In addition, putting $\Psi = \text{diag} [\psi, \psi, \dots, \psi]$, where $\psi : \mathbb{R}_+ \rightarrow (0, \infty)$ is a continuous function, equation (6) becomes equation (2) from [9]. Thus, Theorem 3.1 generalizes Theorem 1 in [9], in two directions: from systems of differential equations to matrix differential equations and the introduction of the matrix function Ψ which allows obtaining a mixed asymptotic behavior for the components of solutions of the above equations. In addition, the function φ satisfies the better condition $\int_0^\infty \varphi(s)ds = \infty$.

The goal of the next theorem is to obtain a new result in connection with Ψ -asymptotic equivalence of the Ψ -bounded solutions of two Lyapunov matrix differential equations, namely (1) and (2).

Theorem 3.2 *Suppose that:*

1) *There exist supplementary projections $P_1, P_2 \in M_{d \times d}$, a continuous function $\varphi : \mathbb{R}_+ \rightarrow (0, \infty)$ that satisfies the condition $\int_0^\infty \varphi(s)ds = \infty$ and a constant $K > 0$ such that the fundamental matrices $X(t)$ and $Y(t)$ for the linear matrix differential equations (3) and (4) respectively, satisfy the inequality*

$$\int_{t_0}^t \varphi(s) \left| \left(Y^T(t)(Y^T)^{-1}(s) \right) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \right) \right| ds + \int_t^\infty \varphi(s) \left| \left(Y^T(t)(Y^T)^{-1}(s) \right) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \right) \right| ds \leq K \tag{10}$$

for all $t \geq t_0 \geq 0$, where t_0 is large enough;

2) *The continuous matrix function $F : \mathbb{R}_+ \times M_{d \times d} \rightarrow M_{d \times d}$ satisfies the inequality*

$$\varphi^{-1}(t) |\Psi(t)F(t, Z)| \leq \omega(t, |\Psi(t)Z|)$$

for all $t \geq t_0$ and $Z \in M_{d \times d}$, where $\omega(t, r) : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function and is nondecreasing in r , for each fixed $t \geq t_0$.

Furthermore, let $\gamma_\lambda(t) = \sup_{s \geq t} \omega(s, \lambda)$ and assume that $\lim_{t \rightarrow \infty} \gamma_\lambda(t) = 0$ for each $\lambda \in (0, \infty)$.

Then, corresponding to each Ψ -bounded solution $Z_0(t)$ of (1), there exists a Ψ -bounded solution $Z(t)$ of Lyapunov matrix differential equation (2) such that

$$\lim_{t \rightarrow \infty} |\Psi(t)(Z(t) - Z_0(t))| = 0. \tag{11}$$

Conversely, to each Ψ -bounded solution $Z(t)$ of (2), there exists a Ψ -bounded solution $Z_0(t)$ of (1) such that (11) holds.

Proof. We will use the version of Theorem 3.1 for systems of differential equations and some results from [8].

From Lemma 2.6, [8], we know that $Z(t)$ is a Ψ -bounded solution on \mathbb{R}_+ of (2) iff $z(t) = \mathcal{V}ec(Z(t))$ is a $I \otimes \Psi(t)$ -bounded solution of the corresponding Kronecker product system associated with (2), i.e. the system

$$z' = (I \otimes A(t) + B^T(t) \otimes I)z + f(t, z), \quad t \geq t_0 \tag{12}$$

where $f(t, z) = \mathcal{V}ec(F(t, Z))$.

We verify the hypotheses of Theorem 3.1 (version for systems of differential equations).

a) From Lemma 2.7, [8], we know that $Y^T(t) \otimes X(t)$ is a fundamental matrix for the homogeneous system associated to (12), i.e. the system

$$z' = (I \otimes A(t) + B^T(t) \otimes I)z. \tag{13}$$

From Lemmas 2.1, 2.3, [8], we obtain that hypothesis 1) in Theorem 3.2 implies hypothesis 1) in Theorem 3.1 (with $I \otimes \Psi(t)$ in the role of $\Psi(t)$ and $I \otimes P_i$ in role of P_i).

b) Similar arguments to those above show that the from the hypothesis 2) in Theorem 3.2 it follows that all other hypotheses of Theorem 3.1 hold as well.

Let $Z_0(t)$ be a Ψ -bounded solution of (1). Then, $z_0(t) = \mathcal{V}ec(Z_0(t))$ is a $I \otimes \Psi(t)$ -bounded solution of (13). From Theorem 3.1 (the version for systems), there exists a $I \otimes \Psi(t)$ -bounded solution $z(t)$ of (12) with the property that

$$\lim_{t \rightarrow \infty} |(I \otimes \Psi(t))(z(t) - z_0(t))| = 0. \tag{14}$$

From Lemmas 2.5, 2.6, [8], we obtain that (11) holds, where $Z(t) = \mathcal{V}ec^{-1}(z(t))$ is a $\Psi(t)$ -bounded solution of (2).

For the last statement of Theorem 3.2, let $Z(t)$ a $\Psi(t)$ -bounded solution of (2). Then, $z(t) = \mathcal{V}ec(Z(t))$ is a $I \otimes \Psi(t)$ -bounded solution of (12). From Theorem 3.1 (version for systems), there exists a $I \otimes \Psi(t)$ -bounded solution $z_0(t)$ of (13) such that (14) holds. From Lemmas 2.5, 2.6, [8], we obtain that (11) holds, where $Z_0(t) = \mathcal{V}ec^{-1}(z_0(t))$ is a $\Psi(t)$ -bounded solution of (1). The proof is now complete. \square

Remark 3.2 *If the condition (10) is not satisfied, then the conclusion of Theorem 3.2 does not hold. This is shown by the next simple Example obtained after an example due to O. Perron [12].*

Example 3.1 *In equation (2) consider*

$$A(t) = \begin{pmatrix} \sin \ln(t+1) + \cos \ln(t+1) & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$F(t, Z) = \begin{pmatrix} 0 & be^{-\frac{1}{2}(t+1)} \\ 0 & 0 \end{pmatrix} Z,$$

where $t \in \mathbb{R}_+$, $Z \in \mathcal{M}_{2 \times 2}$ and $b \in \mathbb{R}$, $b \neq 0$.

In addition, consider

$$\Psi(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{1}{2}(t+1)} \end{pmatrix}.$$

Then the condition 2) of Theorem 3.2 is satisfied. Indeed, for $t \in \mathbb{R}_+$, $Z \in \mathcal{M}_{2 \times 2}$, we have

$$\begin{aligned} |\Psi(t)F(t, Z)| &= \left| \Psi(t) \begin{pmatrix} 0 & be^{-\frac{1}{2}(t+1)} \\ 0 & 0 \end{pmatrix} Z \right| \\ &= \left| \Psi(t) \begin{pmatrix} 0 & be^{-\frac{1}{2}(t+1)} \\ 0 & 0 \end{pmatrix} \Psi^{-1}(t) \Psi(t) Z \right| \\ &\leq \left| \Psi(t) \begin{pmatrix} 0 & be^{-\frac{1}{2}(t+1)} \\ 0 & 0 \end{pmatrix} \Psi^{-1}(t) \right| |\Psi(t)Z| \\ &= \left| \begin{pmatrix} 0 & be^{-(t+1)} \\ 0 & 0 \end{pmatrix} \right| |\Psi(t)Z| \\ &= |b| e^{-(t+1)} |\Psi(t)Z| \end{aligned}$$

and $\omega(t, \lambda) = |b| e^{-(t+1)} \lambda$ and $\gamma_\lambda(t) = |b| e^{-(t+1)} \lambda$ satisfy the conditions of this Theorem.

If condition 1) of Theorem 3.2 is satisfied, then the conclusion of Theorem 3.2 holds. In particular, corresponding to each Ψ -bounded solution $Z_0(t)$ of (1), there exists a Ψ -bounded solution $Z(t)$ of Lyapunov matrix differential equation (2) such that (11) holds.

We find the general solutions of (1) and (2) in a particular case considered here.

Eq. (1) becomes

$$Z' = \begin{pmatrix} \sin \ln(t+1) + \cos \ln(t+1) - 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} Z.$$

A fundamental matrix for this equation is

$$X(t) = \begin{pmatrix} e^{(t+1)[\sin \ln(t+1)-1]} & 0 \\ 0 & e^{-\frac{1}{2}(t+1)} \end{pmatrix}.$$

Then, the general solution of equation is $Z_{g_0} = X(t)C$, where C is a real 2×2 constant matrix.

Eq. (2) becomes

$$Z' = \begin{pmatrix} \sin \ln(t+1) + \cos \ln(t+1) - 1 & be^{-\frac{1}{2}(t+1)} \\ 0 & -\frac{1}{2} \end{pmatrix} Z.$$

A fundamental matrix for this equation is

$$Y(t) = \begin{pmatrix} f(t) & g(t) \\ e^{-\frac{1}{2}(t+1)} & 0 \end{pmatrix},$$

where

$$f(t) = be^{(t+1)[\sin \ln(t+1)-1]} \cdot \int_1^{t+1} e^{-s \sin \ln s} ds$$

and

$$g(t) = e^{(t+1)[\sin \ln(t+1)-1]},$$

for $t \geq 0$.

The general solution of this equation is $Z_g = Y(t)C$, where C is a real 2×2 constant matrix.

Now, we consider the particular solution

$$Z_0(t) = X(t) \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ce^{-\frac{1}{2}(t+1)} & 0 \end{pmatrix}$$

of (1), where $c \neq 0$. This solution is Ψ -bounded on \mathbb{R}_+ .

We search $Z(t)$ of the form

$$Z(t) = Y(t) \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} c_1 f(t) + c_3 g(t) & c_2 f(t) + c_4 g(t) \\ c_1 e^{-\frac{1}{2}(t+1)} & c_2 e^{-\frac{1}{2}(t+1)} \end{pmatrix}$$

and then,

$$\Psi(t)Z(t) = \begin{pmatrix} c_1 f(t) + c_3 g(t) & c_2 f(t) + c_4 g(t) \\ c_1 & c_2 \end{pmatrix}.$$

Since $f(t)$ is unbounded (see in [5], pp. 71) and $g(t)$ is bounded on \mathbb{R}_+ , the solution $Z(t)$ is Ψ -bounded on \mathbb{R}_+ iff $c_1 = c_2 = 0$. In this case,

$$\Psi(t)(Z(t) - Z_0(t)) = \begin{pmatrix} c_3 g(t) & c_4 g(t) \\ c & 0 \end{pmatrix}.$$

Because $c \neq 0$, it is impossible to make $\lim_{t \rightarrow \infty} |\Psi(t)(Z(t) - Z_0(t))| = 0$. This proves the assertion.

Remark 3.3 This Example shows the importance of hypotheses with regard to equation $Z' = A(t)Z + ZB(t)$ in Theorem 3.2.

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