

# A study of a three-dimensional competitive Lotka–Volterra system

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**Abstract.** In this paper we will consider a community of three mutually competing species modeled by the Lotka–Volterra system:

$$\begin{cases} \dot{x}_i &= x_i \left( b_i - \sum_{j=1}^3 a_{ij} x_j \right), \quad i = 1, 2, 3 \end{cases}$$

where  $x_i(t)$  is the population size of the  $i$ -th species at time  $t$ ,  $\dot{x}_i$  denote  $\frac{dx_i}{dt}$  and  $a_{ij}, b_i$  are all strictly positive real numbers.

This system of ordinary differential equations represent a class of Kolmogorov systems. This kind of systems is widely used in the mathematical models for the dynamics of population, like predator-prey models or different models for the spread of diseases.

A qualitative analysis of this Lotka-Volterra system based on dynamical systems theory will be performed, by studying the local behavior in equilibrium points and obtaining local dynamics properties.

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## 1 Introduction

Although the systems of differential equations that represent the competition between species, the so called Lotka-Volterra systems, were introduced by Lotka ([8]) and Volterra ([10], [11]) almost 90 years ago, their qualitative study, from the point of view of dynamical systems, is still of great interest for mathematical research and more.

In the present paper an algebraic study of the dynamics for a competitive Lotka-Volterra system for two and three competing species will be performed. For the system with two competing species, four equilibrium points with positive coordinates and four distinct phase portraits are obtained, exactly as in [1] and [2]. For the three dimensional competitive system, the algebraic study discovers very important results about the dynamics of the competition between three species like in [3], [12] and [13]. Under some conditions this 3-dimensional system of ordinary differential equations has eight equilibrium points with positive coordinates and represent a class of Kolmogorov systems.

The competitive Lotka-Volterra equations are a simple model of the population dynamics of species competing for some common resource. They can be further generalised to include

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trophic interactions. The form is similar to the Lotka-Volterra equations for predation in that the equation for each species has one term for self-interaction and one term for the interaction with other species. In the equations for predation, the base population model is exponential. For the competition equations, the logistic equation is the basis.

The logistic population model, when used by ecologists often takes the following form:

$$\frac{dx}{dt} = rx \left( 1 - \frac{x}{K} \right), \tag{1}$$

where  $x$  is the size of the population at a given time  $t$ ,  $r$  is inherent per-capita growth rate, and  $K$  is the carrying capacity.

Given two populations,  $x_1$  and  $x_2$ , with logistic dynamics, the Lotka-Volterra formulation adds an additional term to account for the species' interactions. Thus the competitive Lotka-Volterra equations are:

$$\begin{cases} \frac{dx_1}{dt} = r_1 x_1 \left( 1 - \frac{x_1 + \alpha_{12} x_2}{K_1} \right) \\ \frac{dx_2}{dt} = r_2 x_2 \left( 1 - \frac{x_2 + \alpha_{21} x_1}{K_2} \right), \end{cases} \tag{2}$$

where  $\alpha_{12}$  represents the effect species 2 has on the population of species 1 and  $\alpha_{21}$  represents the effect species 1 has on the population of species 2. Because this is the competitive version of the model, all interactions must be harmful (competition) and therefore all values of  $\alpha$  are positive. Also, note that each species can have its own growth rate and carrying capacity.

In [1] and [2], I.M. Bomze presented a classification of phase portraits of the two-dimensional Lotka-Volterra system using the dynamics of 3-type replicator. In [12] and [13], M.L. Zeeman used a three-dimensional geometric analysis to obtain a partial classification of the dynamics of three-dimensional competitive Lotka-Volterra systems and obtain 33 type of stable equivalent classes. For 25 of these classes it was obtained that the limit sets are equilibrium points and for the rest of 8 classes it is possible to have Hopf bifurcations and, consequently, periodic orbits.

This model can be generalized to any number of species competing against each other. One can think of the populations  $x_i$  and growth rates  $r_i$  as vectors and the interaction  $\alpha_{ij}$  as a matrix. Then the competitive system for  $n$  species is

$$\left\{ \frac{dx_i}{dt} = r_i x_i \left( 1 - \frac{\sum_{j=1}^n \alpha_{ij} x_j}{K_i} \right) \quad i = 1, \dots, n, \right. \tag{3}$$

or

$$\left\{ \frac{dx_i}{dt} = r_i x_i \left( 1 - \sum_{j=1}^n \alpha_{ij} x_j \right) \quad i = 1, \dots, n. \right. \tag{4}$$

if the carrying capacity is pulled into the interaction matrix.

If we take  $b_i = r_i$  and  $a_{ij} = r_i \alpha_{ij}$ , then the competitive Lotka-Volterra  $n$ -dimensional system is

$$\left\{ \frac{dx_i}{dt} = x_i \left( b_i - \sum_{j=1}^n a_{ij} x_j \right) \quad i = 1, \dots, n. \right. \tag{5}$$

In general, an autonomous system of ordinary differential equations of the form

$$\{\dot{x}_i = x_i \cdot F_i(x_1, \dots, x_n), \quad i = 1, \dots, n, \tag{6}$$

where  $x_i = x_i(t)$ ,  $F_i$  smooth functions, is called **Kolmogorov system**.

If  $F_i$  are polynomials in  $x_j \quad j = 1, \dots, n$ , then this system is a Lotka-Volterra type system which is widely used in modeling interacting species of predator-prey type arising, for

example, in biology, ecology, epidemiology or even in economy. The class of Kolmogorov systems is widely used in the mathematical models for the dynamics of population. So, many predator-prey models or SIR models for spread of disease are particular classes of Kolmogorov systems.

The functions  $F_i$  are called **fitness functions** and for many models  $F_i$  are considered to be affine functions, i.e. of the form

$$F_i(x_1, \dots, x_n) = r_i + \sum_{j=1}^n \alpha_{ij} x_j$$

In [4] a Lotka-Volterra system of type (6) is called **competitive** if  $\frac{\partial F_i}{\partial x_j} \leq 0$ , for all  $i, j = 1, \dots, n$ . The system is called **cooperative** if  $\frac{\partial F_i}{\partial x_j} \geq 0$ , for all  $i, j = 1, \dots, n$ .

Let us remark that the two classes have something in common. Further, taking into account this definition, a Lotka-Volterra  $n$ -dimensional system (5) is competitive if all  $a_{ij}$  are positive.

In [10, 11] Volterra studies dissipative Lotka-Volterra systems as generalizations of the classical predator-prey model. A Lotka-Volterra system with interaction matrix  $a_{ij}$  is called *dissipative*, respectively *conservative*, if there are constants  $d_i > 0$ ,  $i = 1, \dots, n$ , such that the quadratic form  $q(x) = \sum_{i,j=1}^n a_{ij} d_j x_i x_j$  is negative semi-definite, respectively zero.

Note that the meaning of the term dissipative is not strict because dissipative Lotka-Volterra systems include conservative Lotka-Volterra systems. Moreover, we remark that conservative Lotka-Volterra models are in some sense Hamiltonian systems, a fact that was well known and explored by Volterra in [10, 11].

The Lotka-Volterra competition model describes the outcome of competition between two or more species over ecological time. Because one species can competitively exclude another species in ecological time, the competitively-inferior species may increase the range food types that it eats in order to survive.

Generally, the definition of a **competitive Lotka-Volterra system** (5) assumes that all values in the interaction matrix  $a_{ij}$  are strictly positive ([5]). If it is also assumed that the population of any species will increase in the absence of competition unless the population is already at the carrying capacity  $\frac{b_i}{a_{ii}}$ , then it results that  $b_i > 0$  for all  $i$ . This assumption is considered because we consider a **competitive version** of the model and so all interactions must be harmful, with a strong competition between the two species. Because  $a_{ij}$  represents the effect of species  $j$  on the population of species  $i$  and  $a_{ji}$  represents the effect of species  $i$  on the population of species  $j$  (these values do not have to be equal), the mutual competition between the species dictates that  $a_{ij} > 0$  for  $i \neq j$ . In addition, each species is assumed to be self-regulating ( $a_{ii} > 0$ ), and in the absence of other species, to have a positive density independent growth rate constant  $b_i > 0$ , up to the carrying capacity  $\frac{b_i}{a_{ii}}$ .

## 2 Local analysis of the 2D Lotka-Volterra ODE system

Let us consider the two dimensional Lotka-Volterra system

$$\begin{cases} \dot{x}_1 &= x_1 (b_1 - a_{11}x_1 - a_{12}x_2) \\ \dot{x}_2 &= x_2 (b_2 - a_{21}x_1 - a_{22}x_2) \end{cases} \quad (7)$$

for which we suppose that all coefficients  $a_{ij}$  and  $b_i$  are strictly positive.

Next, we restrict our attention to the first quadrant  $\mathbf{R}_+^2$  and we denote the open first quadrant by  $int\mathbf{R}_+^2$ .

Moreover, we assume that the determinant of the matrix  $(a_{ij})$  of the species interaction's coefficients is non zero, i.e.  $d_{1212} = a_{11}a_{22} - a_{12}a_{21} \neq 0$ .

Under these assumptions about the parameters  $a_{ij}$  and  $b_i$  ( $i, j = 1, 2, 3$ ), the system of ordinary differential equations (7) has at most four equilibrium points with positive components as follows. Indeed, from

$$\begin{cases} b_1x_1 - a_{11}x_1^2 - a_{12}x_1x_2 = 0 \\ b_2x_2 - a_{21}x_2x_1 - a_{22}x_2^2 = 0 \end{cases}$$

we obtain the following four equilibrium points  $O(0, 0)$ ,  $E_1(\frac{b_1}{a_{11}}, 0)$ ,  $E_2(0, \frac{b_2}{a_{22}})$  and  $E(\frac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}}, \frac{b_2a_{11}-a_{21}b_1}{a_{11}a_{22}-a_{12}a_{21}})$ .

In this section we will use the following notations:

$$d_{1213} = b_2a_{11} - a_{21}b_1, \quad d_{1223} = a_{12}b_2 - b_1a_{22}, \quad d_{1212} = a_{11}a_{22} - a_{12}a_{21}.$$

**Definition 2.1** An equilibrium point  $E(x_1, x_2)$  is called **proper** if  $x_1 \geq 0$  and  $x_2 \geq 0$ . Else, if at least one of the components  $x_i$  is strictly negative, then the equilibrium  $E$  is called **virtual**.

From practical reasons the virtual equilibrium points are not studied here.

The Jacobi matrix at  $(x_1, x_2)$  is

$$\begin{pmatrix} b_1 - 2a_{11}x_1 - a_{12}x_2 & -a_{21}x_2 \\ -a_{12}x_1 & b_2 - a_{21}x_1 - 2a_{22}x_2 \end{pmatrix}$$

For  $O(0, 0, 0)$ , the Jacobian is  $\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$ , with eigenvalues  $\lambda_1 = b_1, \lambda_2 = b_2$ .

For  $E_1(\frac{b_1}{a_{11}}, 0)$ , the Jacobian is  $\begin{pmatrix} -b_1 & 0 \\ -a_{12}\frac{b_1}{a_{11}} & b_2 - a_{21}\frac{b_1}{a_{11}} \end{pmatrix}$ , with eigenvalues  $\lambda_1 = -b_1$ ,  $\lambda_2 = \frac{b_2a_{11}-a_{21}b_1}{a_{11}} = \frac{d_{1213}}{a_{11}}$ .

For  $E_2(0, \frac{b_2}{a_{22}})$ , the Jacobian is  $\begin{pmatrix} b_1 - a_{12}\frac{b_2}{a_{22}} & -a_{21}\frac{b_2}{a_{22}} \\ 0 & -b_2 \end{pmatrix}$ , with eigenvalues  $\lambda_1 = \frac{b_1a_{22}-a_{12}b_2}{a_{22}} = \frac{-d_{1223}}{a_{22}}, \lambda_2 = -b_2$ .

**Proposition 2.1** a) The origin  $O$  is a repeller (node).

b) For a fixed  $i = 1, 2$ , the equilibrium point  $E_i$  is an attractor (node) iff  $\frac{b_i}{a_{ii}} > \frac{b_j}{a_{ji}}$  for  $j \neq i$ . Else,  $E_i$  is a saddle point.

*Proof.* See the signs of the eigenvalues of each equilibrium point. □

**Corollary 2.1** a)  $E_1$  is an attractor if and only if  $d_{1213} < 0$ . Else,  $E_1$  is a saddle point.

b)  $E_2$  is an attractor if and only if  $d_{1223} > 0$ . Else,  $E_2$  is a saddle point.

The interior equilibrium point is  $E(x_1^*, x_2^*)$ , where  $x_1^* = -\frac{d_{1223}}{d_{1212}}, x_2^* = \frac{d_{1213}}{d_{1212}}$ , has the Jacobian

$$\begin{pmatrix} b_1 + 2a_{11}\frac{d_{1223}}{d_{1212}} - a_{12}\frac{d_{1213}}{d_{1212}} & -a_{21}\frac{d_{1213}}{d_{1212}} \\ a_{12}\frac{d_{1223}}{d_{1212}} & b_2 + a_{21}\frac{d_{1223}}{d_{1212}} - 2a_{22}\frac{d_{1213}}{d_{1212}} \end{pmatrix}$$

with eigenvalues

$$\lambda_1^* = \frac{1}{2} \frac{(b_1 + b_2)d_{1212} + (a_{21} + 2a_{11})d_{1223} - (a_{12} + 2a_{22})d_{1213} + \sqrt{\Delta}}{d_{1212}},$$

$$\lambda_2^* = \frac{1}{2} \frac{(b_1 + b_2)d_{1212} + (a_{21} + 2a_{11})d_{1223} - (a_{12} + 2a_{22})d_{1213} - \sqrt{\Delta}}{d_{1212}},$$

where  $\Delta = (a_{11}d_{1223} - a_{22}d_{1213})^2 + 4d_{1212}d_{1223}d_{1213}$ .

The characteristic polynomial is

$$P(X) = X^2 - \frac{b_2a_{11}(a_{12} - a_{22}) + b_1a_{22}(a_{21} - a_{11})}{d_{1212}}X - \frac{d_{1223}d_{1213}}{d_{1212}}$$

Let us remark that  $\lambda_1^* \lambda_2^* = -\frac{d_{1223}d_{1213}}{d_{1212}} = x_1^* x_2^* d_{1212}$  and then we obtain:

**Proposition 2.2** *If  $E$  is a proper equilibrium point, then  $E$  is an attractor for  $d_{1212} > 0$  or  $E$  is a saddle point for  $d_{1212} < 0$ .*

**Corollary 2.2** *If the determinant of the matrix of the species interactions's coefficients is strictly positive, i.e.  $d_{1212} > 0$ , then we have:*

- a)  $E$  is a virtual equilibrium point if and only if  $E_1$  or  $E_2$  is an attractor.
- b)  $E$  is a proper equilibrium point if and only if  $E_1$  and  $E_2$  are both saddle points.

**Corollary 2.3** *If the determinant of the matrix of the species interactions's coefficients is strictly negative, i.e.  $d_{1212} < 0$ , then we have:*

- a)  $E$  is a virtual equilibrium point if and only if  $E_1$  or  $E_2$  is a saddle point.
- b)  $E$  is a proper equilibrium point if and only if  $E_1$  and  $E_2$  are both attractors.

Then we obtain that if the interior equilibrium point  $E$  is virtual, then one of the axial equilibrium points is a saddle, while the other is an attractor. So, we recover the well known result that for the two species competitive in a Lotka-Volterra model with no equilibrium point in the open first quadrant, one of the species will go to extinction, while the other population stabilises at its own carrying capacity ([12], [13]).

The types of these four equilibrium points are summarized in the table 1.

	$O$	$E_1$	$E_2$	$E$
$\lambda_1$	$b_1$	$-b_1$	$-a_{12} \left( \frac{b_2}{a_{22}} - \frac{b_1}{a_{12}} \right) = -\frac{d_{1223}}{a_{22}}$	$\lambda_1^*$
$\lambda_2$	$b_2$	$-a_{21} \left( \frac{b_1}{a_{11}} - \frac{b_2}{a_{21}} \right) = \frac{d_{1213}}{a_{11}}$	$-b_2$	$\lambda_2^*$
type	$r$	$a$ or $s$	$a$ or $s$	$a$ or $s$

Table 1: The eigenvalues and types of the equilibrium points of the system (7); the abbreviations a, r and s stand for attractor, repeller and saddle, respectively.

The configuration of the lines  $d_1 : a_{11}x_1 + a_{12}x_2 = b_1$  and  $d_2 : a_{21}x_1 + a_{22}x_2 = b_2$  determines the dynamic behaviour of the system in the first quadrant  $\mathbf{R}_+^2$ .

**Example 2.1** *Let us consider the following two dimensional competitive Lotka-Volterra system:*

$$\begin{cases} \dot{x}_1 = x_1 - 3x_1^2 - x_1x_2 \\ \dot{x}_2 = x_2 - 2x_2x_1 - 2x_2^2 \end{cases} \quad (8)$$

The Jacobi matrix at  $(x_1, x_2)$  is  $\begin{pmatrix} 1 - 6x_1 - x_2 & -2x_2 \\ -x_1 & 1 - 2x_1 - 4x_2 \end{pmatrix}$  and we have four equilibrium points  $O(0, 0)$ ,  $E_1(\frac{1}{3}, 0)$ ,  $E_2(0, \frac{1}{2})$  and  $E(\frac{1}{4}, \frac{1}{4})$  with corresponding eigenvalues  $\lambda_1 = 1$ ,  $\lambda_2 = 1$  for  $O$ ,  $\lambda_1 = -1$ ,  $\lambda_2 = \frac{1}{3}$  for  $E_1$ ,  $\lambda_1 = \frac{1}{2}$ ,  $\lambda_2 = -1$  for  $E_2$  and  $\lambda_1 = -1$ ,  $\lambda_2 = -\frac{1}{4}$  for  $E$ . Therefore the origin  $O$  is a repeller, the axial equilibrium points  $E_1$  and  $E_2$  are both saddle points, while the interior equilibrium point  $E$  is an attractor.

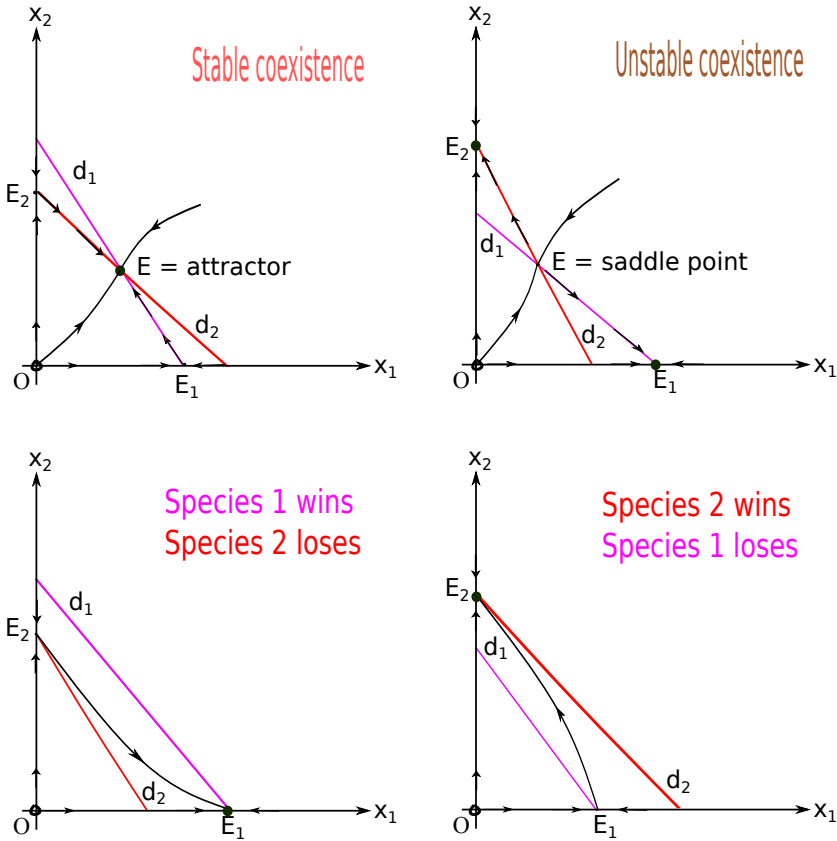


Figure 1: The four distinct phase portraits for 2-dimensional competitive Lotka-Volterra system

**Example 2.2** Let us consider the following two dimensional competitive Lotka-Volterra system:

$$\begin{cases} \dot{x}_1 = \frac{2}{3}x_1 - x_1^2 - x_1x_2 \\ \dot{x}_2 = \frac{3}{2}x_2 - 3x_2x_1 - 2x_2^2 \end{cases} \quad (9)$$

The Jacobi matrix at  $(x_1, x_2)$  is  $\begin{pmatrix} \frac{2}{3} - 2x_1 - x_2 & -3x_2 \\ -x_1 & \frac{3}{2} - 3x_1 - 4x_2 \end{pmatrix}$  and we have four equilibrium points  $O(0, 0)$ ,  $E_1(\frac{2}{3}, 0)$ ,  $E_2(0, \frac{3}{4})$  and  $E(\frac{1}{6}, \frac{1}{2})$  with corresponding eigenvalues  $\lambda_1 = \frac{2}{3}$ ,  $\lambda_2 = \frac{3}{2}$  for  $O$ ,  $\lambda_1 = -\frac{2}{3}$ ,  $\lambda_2 = -\frac{1}{2}$  for  $E_1$ ,  $\lambda_1 = -\frac{1}{12}$ ,  $\lambda_2 = -\frac{3}{2}$  for  $E_2$  and  $\lambda_{1,2} = -\frac{7}{12} \pm \frac{\sqrt{61}}{12}$  for  $E$ . Therefore the origin  $O$  is a repeller, the axial equilibrium points  $E_1$  and  $E_2$  are both attractors, while the interior equilibrium point  $E$  is a saddle point.

**Example 2.3** Let us consider the following two dimensional competitive Lotka-Volterra system:

$$\begin{cases} \dot{x}_1 = \frac{1}{2}x_1 - 2x_1^2 - x_1x_2 \\ \dot{x}_2 = \frac{1}{3}x_2 - x_2x_1 - x_2^2 \end{cases} \quad (10)$$

The Jacobi matrix at  $(x_1, x_2)$  is  $\begin{pmatrix} \frac{1}{2} - 4x_1 - x_2 & -x_2 \\ -x_1 & \frac{1}{3} - x_1 - 2x_2 \end{pmatrix}$  and we have four equilibrium points  $O(0, 0)$ ,  $E_1(\frac{1}{4}, 0)$ ,  $E_2(0, \frac{1}{3})$  and  $E(\frac{1}{6}, \frac{1}{6})$  with corresponding eigenvalues  $\lambda_1 = \frac{1}{2}$ ,  $\lambda_2 = \frac{1}{3}$  for  $O$ ,  $\lambda_1 = -\frac{1}{2}$ ,  $\lambda_2 = \frac{1}{12}$  for  $E_1$ ,  $\lambda_1 = \frac{1}{6}$ ,  $\lambda_2 = -\frac{1}{3}$  for  $E_2$  and  $\lambda_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{5}}{12}$  for  $E$ . Therefore the origin  $O$  is a repeller, the axial equilibrium points  $E_1$  and  $E_2$  are both saddle points, while the interior equilibrium point  $E$  is an attractor.

**Example 2.4** Let us consider the following two dimensional competitive Lotka-Volterra system:

$$\begin{cases} \dot{x}_1 = 3x_1 - 2x_1^2 - \frac{1}{2}x_1x_2 \\ \dot{x}_2 = x_2 - 2x_2x_1 - x_2^2 \end{cases} \quad (11)$$

The Jacobi matrix at  $(x_1, x_2)$  is  $\begin{pmatrix} 3 - 4x_1 - \frac{1}{2}x_2 & -2x_2 \\ -\frac{1}{2}x_1 & 1 - 2x_1 - 2x_2 \end{pmatrix}$  and we have four equilibrium points  $O(0, 0)$ ,  $E_1(\frac{3}{2}, 0)$ ,  $E_2(0, 1)$  and  $E(\frac{5}{2}, -4)$  with corresponding eigenvalues  $\lambda_1 = 3$ ,  $\lambda_2 = 1$  for  $O$ ,  $\lambda_1 = -3$ ,  $\lambda_2 = -2$  for  $E_1$ ,  $\lambda_1 = \frac{5}{2}$ ,  $\lambda_2 = -1$  for  $E_2$  and  $\lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{41}$  for  $E$ . Therefore the origin  $O$  is a repeller,  $E_1$  is an attractor and  $E_2$  is a saddle point, while the interior equilibrium point  $E$  is virtual attractor. In this case the first species wins and the second species disappears.

**Example 2.5** Let us consider the following two dimensional competitive Lotka-Volterra system:

$$\begin{cases} \dot{x}_1 = x_1 - x_1^2 - 2x_1x_2 \\ \dot{x}_2 = 3x_2 - x_2x_1 - x_2^2 \end{cases} \quad (12)$$

The Jacobi matrix at  $(x_1, x_2)$  is  $\begin{pmatrix} 1 - 2x_1 - 2x_2 & -x_2 \\ -2x_1 & 3 - x_1 - 2x_2 \end{pmatrix}$  and we have four equilibrium points  $O(0, 0)$ ,  $E_1(1, 0)$ ,  $E_2(0, 3)$  and  $E(5, -2)$  with corresponding eigenvalues  $\lambda_1 = 1$ ,  $\lambda_2 = 3$  for  $O$ ,  $\lambda_1 = -1$ ,  $\lambda_2 = 2$  for  $E_1$ ,  $\lambda_1 = -5$ ,  $\lambda_2 = -3$  for  $E_2$  and  $\lambda_{1,2} = -\frac{3}{2} \pm \frac{1}{2}i\sqrt{31}$  for  $E$ . Therefore the origin  $O$  is a repeller,  $E_1$  is a saddle point and  $E_2$  is an attractor, while the interior equilibrium point  $E$  is a virtual focar attractor. In this case the second species wins and the first species disappears.

### 3 Local analysis of the 3D Lotka-Volterra ODE system

In this section we will study a community of three mutually competing species modeled by the following Lotka–Volterra system

$$\begin{cases} \dot{x}_1 = x_1(b_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3) \\ \dot{x}_2 = x_2(b_2 - a_{21}x_1 - a_{22}x_2 - a_{23}x_3) \\ \dot{x}_3 = x_3(b_3 - a_{31}x_1 - a_{32}x_2 - a_{33}x_3) \end{cases} \quad (13)$$

for which we suppose that all coefficients  $a_{ij}$  and  $b_i$  are strictly positive.

Next, we restrict our attention to the closed positive octant  $\mathbf{R}_+^3$  and we denote the open positive octant by  $int\mathbf{R}_+^3$ .

Moreover, we assume that all diagonal second order minors of the matrix  $(a_{ij})$  and the determinant of the matrix of the species interactions's coefficients are non zero, i.e.  $d_{1212} = a_{11}a_{22} - a_{12}a_{21} \neq 0$ ,  $d_{1313} = a_{11}a_{33} - a_{13}a_{31} \neq 0$ ,  $d_{2323} = a_{22}a_{33} - a_{23}a_{32} \neq 0$ ,  $\det(a_{ij}) \neq 0$ .

Under this assumptions about the parameters  $a_{ij}$  and  $b_i$  ( $i, j = 1, 2, 3$ ), the system of ordinary differential equations (13) has at most eight equilibrium points with positive components

as follows. Indeed, from

$$\begin{cases} b_1x_1 - a_{11}x_1^2 - a_{12}x_1x_2 - a_{13}x_1x_3 & = 0 \\ b_2x_2 - a_{21}x_2x_1 - a_{22}x_2^2 - a_{23}x_2x_3 & = 0 \\ b_3x_3 - a_{31}x_1x_3 - a_{32}x_2x_3 - a_{33}x_3^2 & = 0 \end{cases}$$

we obtain the following eight equilibrium points  $O(0, 0, 0)$ ,  $E_1(\frac{b_1}{a_{11}}, 0, 0)$ ,  $E_2(0, \frac{b_2}{a_{22}}, 0)$ ,  $E_3(0, 0, \frac{b_3}{a_{33}})$ ,  $E_{12}(\frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}}, \frac{b_2a_{11}-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}, 0)$ ,  $E_{13}(\frac{b_1a_{33}-b_3a_{13}}{a_{11}a_{33}-a_{13}a_{31}}, 0, \frac{b_3a_{11}-b_1a_{31}}{a_{11}a_{33}-a_{13}a_{31}})$ ,  $E_{23}(0, \frac{b_2a_{33}-b_3a_{23}}{a_{22}a_{33}-a_{23}a_{32}}, \frac{b_3a_{22}-b_2a_{32}}{a_{22}a_{33}-a_{23}a_{32}})$  and  $E(x_1^*, x_2^*, x_3^*)$ , where  $(x_1^*, x_2^*, x_3^*)$  is the unique solution of the linear system  $\{a_{ij}x_j = b_i, \text{ i.e. } x_1^* = \frac{d_1}{d}, x_2^* = \frac{d_2}{d}, x_3^* = \frac{d_3}{d}, \text{ where}$

$$d = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$$

$$d_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = b_1a_{22}a_{33} - b_1a_{23}a_{32} - b_2a_{12}a_{33} + b_2a_{13}a_{32} + b_3a_{12}a_{23} - b_3a_{13}a_{22}$$

$$d_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} = a_{11}b_2a_{33} - a_{11}b_3a_{23} - a_{21}b_1a_{33} + a_{21}a_{13}b_3 + a_{31}b_1a_{23} - a_{31}a_{13}b_2$$

$$d_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} = a_{11}b_3a_{22} - a_{11}b_2a_{32} - a_{21}a_{12}b_3 + a_{21}b_1a_{32} + a_{31}a_{12}b_2 - a_{31}b_1a_{22}$$

Further, we will use the following notations for the minors of the extended matrix  $(a_{ij}; b_i)$ :

$$d_{1224} = -b_1a_{22} + b_2a_{12}, d_{1214} = b_2a_{11} - b_1a_{21}, d_{1334} = -b_1a_{33} + b_3a_{13},$$

$$d_{1314} = b_3a_{11} - b_1a_{31}, d_{2334} = -b_2a_{33} + b_3a_{23}, d_{2324} = b_3a_{22} - b_2a_{32}.$$

Then the equilibrium points in the coordinate planes can be written as follows:

$$E_{12}\left(-\frac{d_{1224}}{d_{1212}}, \frac{d_{1214}}{d_{1212}}, 0\right), E_{13}\left(-\frac{d_{1334}}{d_{1313}}, 0, \frac{d_{1314}}{d_{1313}}\right), E_{23}\left(0, -\frac{d_{2334}}{d_{2323}}, \frac{d_{2324}}{d_{2323}}\right).$$

**Definition 3.1** An equilibrium point  $E(x_1, x_2, x_3)$  is called **proper** if  $x_i \geq 0$ , for all  $i = \overline{1, 3}$ . Else, if at least one of the components  $x_i$  is strictly negative, then the equilibrium  $E$  is called **virtual**.

From practical reasons the virtual equilibrium points are not studied here.

The Jacobi's matrix at the point  $(x_1, x_2, x_3)$  has the form:

$$\begin{pmatrix} b_1 - 2a_{11}x_1 - a_{12}x_2 - a_{13}x_3 & -a_{21}x_2 & -a_{31}x_3 \\ -a_{12}x_1 & b_2 - a_{21}x_1 - 2a_{22}x_2 - a_{23}x_3 & -a_{32}x_3 \\ -a_{13}x_1 & -a_{23}x_2 & b_3 - a_{31}x_1 - a_{32}x_2 - 2a_{33}x_3 \end{pmatrix}$$

For  $O(0, 0, 0)$  the Jacobian matrix is

$$A_0 = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$$

and  $\lambda_i = b_i$  ( $i = 1, 2, 3$ ) are eigenvalues.



For  $E_1(\frac{b_1}{a_{11}}, 0, 0)$ , we have the Jacobian

$$A_1 = \begin{pmatrix} -b_1 & 0 & 0 \\ -a_{12} \frac{b_1}{a_{11}} & b_2 - a_{21} \frac{b_1}{a_{11}} & 0 \\ -a_{13} \frac{b_1}{a_{11}} & 0 & b_3 - a_{31} \frac{b_1}{a_{11}} \end{pmatrix}$$

with eigenvalues

$$\begin{aligned} \lambda_1 &= -b_1 \\ \lambda_2 &= \frac{b_2 a_{11} - a_{21} b_1}{a_{11}} = -a_{21} \left( \frac{b_1}{a_{11}} - \frac{b_2}{a_{21}} \right) \\ \lambda_3 &= \frac{b_3 a_{11} - a_{31} b_1}{a_{11}} = -a_{31} \left( \frac{b_1}{a_{11}} - \frac{b_3}{a_{31}} \right) \end{aligned}$$

For  $E_2(0, \frac{b_2}{a_{22}}, 0)$  we have the Jacobian

$$A_2 = \begin{pmatrix} b_1 - a_{12} \frac{b_2}{a_{22}} & -a_{21} \frac{b_2}{a_{22}} & 0 \\ 0 & -b_2 & 0 \\ 0 & -a_{23} \frac{b_2}{a_{22}} & b_3 - a_{32} \frac{b_2}{a_{22}} \end{pmatrix}$$

with eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{b_1 a_{22} - a_{12} b_2}{a_{22}} = -a_{12} \left( \frac{b_2}{a_{22}} - \frac{b_1}{a_{12}} \right) \\ \lambda_2 &= -b_2 \\ \lambda_3 &= \frac{b_3 a_{22} - a_{32} b_2}{a_{22}} = -a_{32} \left( \frac{b_2}{a_{22}} - \frac{b_3}{a_{32}} \right) \end{aligned}$$

For  $E_3(0, 0, \frac{b_3}{a_{33}})$  we have the Jacobian

$$A_3 = \begin{pmatrix} b_1 - a_{13} \frac{b_3}{a_{33}} & 0 & -a_{31} \frac{b_3}{a_{33}} \\ 0 & b_2 - a_{23} \frac{b_3}{a_{33}} & -a_{32} \frac{b_3}{a_{33}} \\ 0 & 0 & -b_3 \end{pmatrix}$$

with eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{b_1 a_{33} - a_{13} b_3}{a_{33}} = -a_{13} \left( \frac{b_3}{a_{33}} - \frac{b_1}{a_{13}} \right) \\ \lambda_2 &= \frac{b_2 a_{33} - a_{23} b_3}{a_{33}} = -a_{23} \left( \frac{b_3}{a_{33}} - \frac{b_2}{a_{23}} \right) \\ \lambda_3 &= -b_3 \end{aligned}$$

**Proposition 3.1** a) The origin  $O$  is a repeller (node).

b) For a fixed  $i = 1, 2, 3$ , the equilibrium point  $E_i$  is an attractor (node) iff  $\frac{b_i}{a_{ii}} > \frac{b_j}{a_{ji}}$  for all  $j \neq i$ . Else,  $E_i$  is a saddle point.

*Proof.* See the signs of the eigenvalues of each equilibrium point. □

**Corollary 3.1** a)  $E_1$  is an attractor iff  $d_{1214} < 0$  and  $d_{1314} < 0$ . Else,  $E_1$  is a saddle point.

b)  $E_2$  is an attractor iff  $d_{1224} > 0$  and  $d_{2324} < 0$ . Else,  $E_2$  is a saddle point.

c)  $E_3$  is an attractor iff  $d_{1334} > 0$  and  $d_{2334} > 0$ . Else,  $E_3$  is a saddle point.

*Proof.* See the expressions of  $d_{1214}$ ,  $d_{1314}$  and so on. □

The types of first four equilibrium points are summarized in the next table.

For  $E_{12}(x_1, x_2, x_3)$ , where  $x_1 = -\frac{d_{1224}}{d_{1212}}$ ,  $x_2 = \frac{d_{1214}}{d_{1212}}$ ,  $x_3 = 0$ , we obtain the Jacobian

$$A_{12} = \begin{pmatrix} b_1 + 2a_{11} \frac{d_{1224}}{d_{1212}} - a_{12} \frac{d_{1214}}{d_{1212}} & -a_{21} \frac{d_{1214}}{d_{1212}} & 0 \\ a_{12} \frac{d_{1224}}{d_{1212}} & b_2 + a_{21} \frac{d_{1224}}{d_{1212}} - 2a_{22} \frac{d_{1214}}{d_{1212}} & 0 \\ a_{13} \frac{d_{1224}}{d_{1212}} & -a_{23} \frac{d_{1214}}{d_{1212}} & b_3 + a_{31} \frac{d_{1224}}{d_{1212}} - a_{32} \frac{d_{1214}}{d_{1212}} \end{pmatrix}$$

	$O$	$E_1$	$E_2$	$E_3$
$\lambda_1$	$b_1$	$-b_1$	$-a_{12} \left( \frac{b_2}{a_{22}} - \frac{b_1}{a_{12}} \right) = -\frac{d_{1224}}{a_{22}}$	$-a_{13} \left( \frac{b_3}{a_{33}} - \frac{b_1}{a_{13}} \right) = -\frac{d_{1334}}{a_{33}}$
$\lambda_2$	$b_2$	$-a_{21} \left( \frac{b_1}{a_{11}} - \frac{b_2}{a_{21}} \right) = \frac{d_{1214}}{a_{11}}$	$-b_2$	$-a_{23} \left( \frac{b_3}{a_{33}} - \frac{b_2}{a_{23}} \right) = -\frac{d_{2334}}{a_{33}}$
$\lambda_3$	$b_3$	$-a_{31} \left( \frac{b_1}{a_{11}} - \frac{b_3}{a_{31}} \right) = \frac{d_{1314}}{a_{11}}$	$-a_{32} \left( \frac{b_2}{a_{22}} - \frac{b_3}{a_{32}} \right) = \frac{d_{2324}}{a_{22}}$	$-b_3$
type	$r$	$a$ or $s$	$a$ or $s$	$a$ or $s$

Table 2: The eigenvalues and types of first four equilibrium points of the system (13); the abbreviations a, r and s stand for attractor, repeller and saddle, respectively.

with eigenvalues:

$$\lambda_1 = b_3 + \frac{a_{31}d_{1224} - a_{32}d_{1214}}{d_{1212}},$$

$$\lambda_2 = \frac{1}{2} \frac{(b_1 + b_2)d_{1212} + (2a_{11} + a_{21})d_{1224} - (2a_{22} + a_{12})d_{1214} + \sqrt{\Delta_{12}}}{d_{1212}},$$

$$\lambda_3 = \frac{1}{2} \frac{(b_1 + b_2)d_{1212} + (2a_{11} + a_{21})d_{1224} - (2a_{22} + a_{12})d_{1214} - \sqrt{\Delta_{12}}}{d_{1212}},$$

where

$$\Delta_{12} = (a_{11}d_{1224} - a_{22}d_{1214})^2 + 4d_{1212}d_{1214}d_{1224}.$$

The characteristic polynomial is  $P_{12}(X) = X^3 + a_2X^2 + a_1X + a_0$ , where

$$a_2 = -\frac{b_2a_{11}(a_{12} - a_{22}) - b_1a_{22}(a_{11} - a_{21}) + d_3}{d_{1212}},$$

$$a_1 = \frac{d_3(b_2a_{11}(a_{12} - a_{22}) - b_1a_{22}(a_{11} - a_{21}))}{d_{1212}^2} - \frac{d_{1224}d_{1214}}{d_{1212}},$$

$$a_0 = \frac{d_3d_{1224}d_{1214}}{d_{1212}^2}.$$

For  $E_{13}(x_1, x_2, x_3)$ , where  $x_1 = -\frac{d_{1334}}{d_{1313}}$ ,  $x_2 = 0$ ,  $x_3 = \frac{d_{1314}}{d_{1313}}$ , we have the Jacobi matrix

$$A_{13} = \begin{pmatrix} b_1 + 2a_{11} \frac{d_{1334}}{d_{1313}} - a_{13} \frac{d_{1314}}{d_{1313}} & 0 & -a_{31} \frac{d_{1314}}{d_{1313}} \\ a_{12} \frac{d_{1334}}{d_{1313}} & b_2 + a_{21} \frac{d_{1334}}{d_{1313}} - a_{23} \frac{d_{1314}}{d_{1313}} & -a_{32} \frac{d_{1314}}{d_{1313}} \\ a_{13} \frac{d_{1334}}{d_{1313}} & 0 & b_3 + a_{31} \frac{d_{1334}}{d_{1313}} - 2a_{33} \frac{d_{1314}}{d_{1313}} \end{pmatrix}$$

with eigenvalues:

$$\lambda_1 = b_2 + \frac{a_{21}d_{1334} - a_{23}d_{1314}}{d_{1313}},$$

$$\lambda_2 = \frac{1}{2} \frac{(b_1 + b_3)d_{1313} + (2a_{11} + a_{31})d_{1334} - (a_{13} + 2a_{33})d_{1314} + \sqrt{\Delta_{13}}}{d_{1313}},$$

$$\lambda_3 = \frac{1}{2} \frac{(b_1 + b_3)d_{1313} + (2a_{11} + a_{31})d_{1334} - (a_{13} + 2a_{33})d_{1314} - \sqrt{\Delta_{13}}}{d_{1313}},$$

where

$$\Delta_{13} = (a_{11}d_{1334} - a_{33}d_{1314})^2 + 4d_{1313}d_{1334}d_{1314}.$$

The characteristic polynomial is  $P_{13}(X) = X^3 + a_2X^2 + a_1X + a_0$ , where

$$a_2 = -\frac{b_3a_{11}(a_{13} - a_{33}) - b_1a_{33}(a_{11} - a_{31}) + d_2}{d_{1313}},$$

$$a_1 = \frac{d_2(b_3a_{11}(a_{13} - a_{33}) - b_1a_{33}(a_{11} - a_{31}))}{d_{1313}^2} - \frac{d_{1334}d_{1314}}{d_{1313}},$$

$$a_0 = \frac{d_2d_{1334}d_{1314}}{d_{1313}^2}.$$

For  $E_{23}(x_1, x_2, x_3)$ , where  $x_1 = 0$ ,  $x_2 = -\frac{d_{2334}}{d_{2323}}$ ,  $x_3 = \frac{d_{2324}}{d_{2323}}$ , we obtain the Jacobian

$$A_{23} = \begin{pmatrix} b_1 + a_{12}\frac{d_{2334}}{d_{2323}} - a_{13}\frac{d_{2324}}{d_{2323}} & a_{21}\frac{d_{2334}}{d_{2323}} & -a_{31}\frac{d_{2324}}{d_{2323}} \\ 0 & b_2 + 2a_{22}\frac{d_{2334}}{d_{2323}} - a_{23}\frac{d_{2324}}{d_{2323}} & -a_{32}\frac{d_{2324}}{d_{2323}} \\ 0 & a_{23}\frac{d_{2334}}{d_{2323}} & b_3 + a_{32}\frac{d_{2334}}{d_{2323}} - 2a_{33}\frac{d_{2324}}{d_{2323}} \end{pmatrix}$$

with eigenvalues:

$$\lambda_1 = b_1 + \frac{a_{12}d_{2334} - a_{13}d_{2324}}{d_{2323}},$$

$$\lambda_2 = \frac{1}{2} \frac{(b_2 + b_3)d_{2323} + (2a_{22} + a_{32})d_{2334} - (a_{23} + 2a_{33})d_{2324} + \sqrt{\Delta_{23}}}{d_{2323}},$$

$$\lambda_3 = \frac{1}{2} \frac{(b_2 + b_3)d_{2323} + (2a_{22} + a_{32})d_{2334} - (a_{23} + 2a_{33})d_{2324} - \sqrt{\Delta_{23}}}{d_{2323}},$$

where

$$\Delta_{23} = (a_{22}d_{2334} - a_{33}d_{2324})^2 + 4d_{2323}d_{2324}d_{2334}.$$

The characteristic polynomial is  $P_{23}(X) = X^3 + a_2X^2 + a_1X + a_0$ , where

$$a_2 = -\frac{b_3a_{22}(a_{23} - a_{33}) - b_2a_{33}(a_{22} - a_{32}) + d_1}{d_{2323}},$$

$$a_1 = \frac{d_1(b_3a_{22}(a_{23} - a_{33}) - b_2a_{33}(a_{22} - a_{32}))}{d_{2323}^2} - \frac{d_{2334}d_{2324}}{d_{2323}},$$

$$a_0 = \frac{d_1d_{2334}d_{2324}}{d_{2323}^2}.$$

**Corollary 3.2** *If the minors of the matrix of the species interaction's coefficients  $d_{1212}$ ,  $d_{1313}$ ,  $d_{2323}$  are all positive, then we have:*

- a) *If  $E_1$  or  $E_2$  is an attractor, then  $E_{12}$  is a virtual equilibrium point;*
  - b) *If  $E_1$  or  $E_3$  is an attractor, then  $E_{13}$  is a virtual equilibrium point;*
  - c) *If  $E_2$  or  $E_3$  is an attractor, then  $E_{23}$  is a virtual equilibrium point;*
- If all equilibria  $E_i$ ,  $i = 1, 2, 3$ , are attractors, then  $E_{12}$ ,  $E_{13}$  and  $E_{23}$  are virtual equilibrium points.*

**Corollary 3.3** *If the minors of the matrix of the species interaction's coefficients  $d_{1212}$ ,  $d_{1313}$ ,  $d_{2323}$  are all positive, then we have:*

- a) *If  $E_{12}$  is a proper equilibrium point, then  $E_1$  and  $E_2$  are both saddle points;*
  - b) *If  $E_{13}$  is a proper equilibrium point, then  $E_1$  and  $E_3$  are both saddle points;*
  - c) *If  $E_{23}$  is a proper equilibrium point, then  $E_2$  and  $E_3$  are both saddle points.*
- If  $E_{12}$ ,  $E_{13}$  and  $E_{23}$  are proper equilibrium points, then all equilibria  $E_i$ ,  $i = 1, 2, 3$  are saddle points.*

**Corollary 3.4** *If the minors of the matrix of the species interaction's coefficients  $d_{1212}$ ,  $d_{1313}$ ,  $d_{2323}$  are all negative, then we have:*

- a) *If  $E_1$  and  $E_2$  are both attractors, then  $E_{12}$  is a proper equilibrium point;*
  - b) *If  $E_1$  and  $E_3$  are both attractors, then  $E_{13}$  is a proper equilibrium point;*
  - c) *If  $E_2$  and  $E_3$  are both attractors, then  $E_{23}$  is a proper equilibrium point;*
- If all equilibria  $E_i$ ,  $i = 1, 2, 3$ , are attractors, then  $E_{12}$ ,  $E_{13}$  and  $E_{23}$  are proper equilibrium points.*

**Corollary 3.5** *If the minors of the matrix of the species interaction's coefficients  $d_{1212}$ ,  $d_{1313}$ ,  $d_{2323}$  are all negative, then we have:*

- a) *If  $E_{12}$  is a virtual equilibrium point, then  $E_1$  or  $E_2$  is a saddle point;*
  - b) *If  $E_{13}$  is a virtual equilibrium point, then  $E_1$  or  $E_3$  is a saddle point;*
  - c) *If  $E_{23}$  is a virtual equilibrium point, then  $E_2$  or  $E_3$  is a saddle point.*
- If  $E_{12}$ ,  $E_{13}$  and  $E_{23}$  are virtual equilibrium points, then at least two of equilibria  $E_i$ ,  $i = 1, 2, 3$  are saddle points.*

The interior equilibrium point  $E\left(\frac{d_1}{d}, \frac{d_2}{d}, \frac{d_3}{d}\right)$  has the Jacobian matrix

$$A = \begin{pmatrix} b_1 - 2a_{11}\frac{d_1}{d} - a_{12}\frac{d_2}{d} - a_{13}\frac{d_3}{d} & -a_{21}\frac{d_2}{d} & -a_{31}\frac{d_3}{d} \\ -a_{12}\frac{d_1}{d} & b_2 - a_{21}\frac{d_1}{d} - 2a_{22}\frac{d_2}{d} - a_{23}\frac{d_3}{d} & -a_{32}\frac{d_3}{d} \\ -a_{13}\frac{d_1}{d} & -a_{23}\frac{d_2}{d} & b_3 - a_{31}\frac{d_1}{d} - a_{32}\frac{d_2}{d} - 2a_{33}\frac{d_3}{d} \end{pmatrix}$$

For this last four equilibrium points the study is more than complicated and we will illustrate the dynamical behaviour of the system using the following examples.

**Example 3.1** *Let us consider the following three dimensional competitive Lotka-Volterra system:*

$$\begin{cases} \dot{x}_1 = x_1 - x_1^2 - 2x_1x_2 - 2x_1x_3 \\ \dot{x}_2 = 2x_2 - x_2x_1 - x_2^2 - x_2x_3 \\ \dot{x}_3 = 3x_3 - x_1x_3 - 3x_2x_3 - x_3^2 \end{cases} \quad (14)$$

The Jacobi matrix at  $(x_1, x_2, x_3)$  is

$$\begin{pmatrix} 1 - 2x_1 - 2x_2 - 2x_3 & -x_2 & -x_3 \\ -2x_1 & 2 - x_1 - 2x_2 - x_3 & -3x_3 \\ -2x_1 & -x_2 & 3 - x_1 - 3x_2 - 2x_3 \end{pmatrix}$$

and we have the following equilibria:  $O(0, 0, 0)$  repeller with eigenvalues 1, 2, 3;  $E_1(1, 0, 0)$  saddle point with eigenvalues  $-1, 1, 2$ ;  $E_2(0, 2, 0)$  attractor with eigenvalues:  $-3, -3, -2$ ;  $E_3(0, 0, 3)$  attractor with eigenvalues  $-5, -1, -3$ ;  $E_{12}(3, -1, 0)$  virtual saddle point with eigenvalues  $3, -1 + i\sqrt{2}, -1 - i\sqrt{2}$ ;  $E_{13}(5, 0, -2)$  virtual attractor with eigenvalues  $-1, -\frac{3}{2} + \frac{1}{2}i\sqrt{31}, -\frac{3}{2} - \frac{1}{2}i\sqrt{31}$ ;  $E_{23}(0, \frac{1}{2}, \frac{3}{2})$  attractor with eigenvalues:  $-3, -1 + \frac{1}{2}\sqrt{10}, -1 - \frac{1}{2}\sqrt{10}$  and the interior equilibrium point  $E(1, \frac{1}{2}, -\frac{3}{2})$  which is virtual repeller and has the eigenvalues

$$\lambda_1 = \frac{1}{6} \frac{\left(162 + 6\sqrt{1479}\right)^{\frac{2}{3}} - 30 + 12\sqrt[3]{\left(162 + 6\sqrt{1479}\right)}}{\sqrt[3]{\left(162 + 6\sqrt{1479}\right)}} > 0,$$

$$\lambda_2, \lambda_3 \in \mathbf{C} \setminus \mathbf{R}, \operatorname{Re}\lambda_{2,3} = -\frac{1}{12} \frac{\left(162 + 6\sqrt{1479}\right)^{\frac{2}{3}} - 30 - 24\sqrt[3]{\left(162 + 6\sqrt{1479}\right)}}{\sqrt[3]{\left(162 + 6\sqrt{1479}\right)}} > 0.$$

**Example 3.2** Let us consider the following three dimensional competitive Lotka-Volterra system:

$$\begin{cases} \dot{x}_1 = x_1 - x_1^2 - x_1x_2 - 2x_1x_3 \\ \dot{x}_2 = x_2 - 2x_2x_1 - 3x_2^2 - x_2x_3 \\ \dot{x}_3 = x_3 - x_1x_3 - 2x_2x_3 - 3x_3^2 \end{cases} \quad (15)$$

The Jacobi matrix at  $(x_1, x_2, x_3)$  is

$$\begin{pmatrix} 1 - 2x_1 - x_2 - 2x_3 & -2x_2 & -x_3 \\ -x_1 & 1 - 2x_1 - 6x_2 - x_3 & -2x_3 \\ -2x_1 & -x_2 & 1 - x_1 - 2x_2 - 6x_3 \end{pmatrix}$$

and we have the following equilibria:  $O(0, 0, 0)$  repeller with eigenvalues  $1, 1, 1$ ;  $E_1(1, 0, 0)$  non-hyperbolic attractor with eigenvalues  $-1, -1, 0$ ;  $E_2(0, \frac{1}{3}, 0)$  saddle point with eigenvalues  $\frac{2}{3}, -1, \frac{1}{3}$ ;  $E_3(0, 0, \frac{1}{3})$  saddle point with eigenvalues  $\frac{1}{3}, \frac{2}{3}, -1$ ;  $E_{12}(2, -1, 0)$  virtual saddle point with eigenvalues  $1, 2, -1$ ;  $E_{13} = E_1(1, 0, 0)$  non-hyperbolic attractor with eigenvalues  $-1, -1, 0$ ;  $E_{23}(0, \frac{2}{7}, \frac{1}{7})$  saddle point with eigenvalues  $\frac{3}{7}, -1, -\frac{2}{7}$  and the interior equilibrium point  $E(\frac{3}{4}, -\frac{1}{4}, \frac{1}{4})$  which is a virtual saddle point and has the eigenvalues  $\lambda_1 = -1, \lambda_2 = \frac{1}{8} + \frac{1}{8}\sqrt{13}, \lambda_3 = \frac{1}{8} - \frac{1}{8}\sqrt{13}$ .

**Example 3.3** Let us consider the following three dimensional competitive Lotka-Volterra system:

$$\begin{cases} \dot{x}_1 = x_1 - x_1^2 - \frac{1}{2}x_1x_2 - \frac{1}{3}x_1x_3 \\ \dot{x}_2 = x_2 - \frac{1}{2}x_2x_1 - x_2^2 - \frac{1}{4}x_2x_3 \\ \dot{x}_3 = x_3 - \frac{1}{3}x_1x_3 - \frac{1}{4}x_2x_3 - x_3^2 \end{cases} \quad (16)$$

The Jacobi matrix at  $(x_1, x_2, x_3)$  is

$$\begin{pmatrix} 1 - 2x_1 - \frac{1}{2}x_2 - \frac{1}{3}x_3 & -\frac{1}{2}x_2 & -\frac{1}{3}x_3 \\ -\frac{1}{2}x_1 & 1 - \frac{1}{2}x_1 - 2x_2 - \frac{1}{4}x_3 & -\frac{1}{4}x_3 \\ -\frac{1}{3}x_1 & -\frac{1}{4}x_2 & 1 - \frac{1}{3}x_1 - \frac{1}{4}x_2 - 2x_3 \end{pmatrix}$$

and we have the following equilibria:  $O(0, 0, 0)$  repeller with eigenvalues  $1, 1, 1$ ;  $E_1(1, 0, 0)$  saddle point with eigenvalues  $-1, \frac{1}{2}, \frac{2}{3}$ ;  $E_2(0, 1, 0)$  saddle point with eigenvalues  $\frac{1}{2}, -1, \frac{3}{4}$ ;  $E_3(0, 0, 1)$  saddle point with eigenvalues  $\frac{2}{3}, \frac{3}{4}, -1$ ;  $E_{12}(\frac{2}{3}, \frac{2}{3}, 0)$  saddle point with eigenvalues  $\frac{11}{18}, -\frac{1}{3}, -1$ ;  $E_{13}(\frac{3}{4}, 0, \frac{3}{4})$  saddle point with eigenvalues  $-\frac{1}{2}, \frac{7}{16}, -1$ ;  $E_{23}(0, \frac{4}{5}, \frac{4}{5})$  saddle point with eigenvalues  $\frac{1}{3}, -1, -\frac{2}{3}$  and the interior equilibrium point  $E(\frac{9}{19}, \frac{56}{95}, \frac{66}{95})$  which is an attractor with eigenvalues  $\lambda_1 = -1, \lambda_2 = -\frac{36}{95} + \frac{1}{95}\sqrt{141}, \lambda_3 = -\frac{36}{95} - \frac{1}{95}\sqrt{141}$ .

In the paper [3], P. van den Driessche and M. L. Zeeman proved the so called **conjecture of Zeeman**: If three species interacting by a competitive Lotka-Volterra system and each species can resist invasion from any others two species at his carrying capacity, then there can be no coexistence of the three species and two of the species are driven to extinction. It is also proved that if none of the species can resist invasion from either of the others, then there is stable coexistence of at least two of the species.

Therefore, using this algebraic study of the dynamical behaviour of the competitive Lotka-Volterra system, it was recovered this very interesting result. More exactly, if all axial equilibria  $E_i, i = 1, 2, 3$ , are local attractors, then almost all trajectories converge to one of this equilibrium points. Moreover, if the system has an equilibrium point in the interior of the positive octant  $\mathbf{R}_+^3$ , then this equilibria is globally asymptotically stable, representing stable coexistence of all three species. Otherwise, there is a globally asymptotically stable equilibrium point in one of the coordinate planes of  $\mathbf{R}_+^3$ , representing stable coexistence of two of the species.

Taking into account these considerations, we can conclude that in example 3.1 we have a coexistence of the species 2 and 3, in example 3.2 only the species 1 resist, species 2 and 3 disappear, and in example 3.3 we have a stable coexistence of all three species.

## 4 Conclusions

In this paper we have made a complete study of the dynamics of the competitive Lotka–Volterra system with two species, together with illustrative examples. For the 3-dimensional competitive system the approach is more complicated and we obtain only partial results, for particular systems. It would be very interesting to obtain a complete study for the competitive Lotka–Volterra systems with three species or more.

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## References

- [1] I.M. Bomze, *Lotka-Volterra equation and replicator dynamics: A two-dimensional classification*, Biological Cybernetics. Springer Science and Business Media LLC. **48** (3), 201211 (1983).
- [2] I.M. Bomze, *Lotka-Volterra equation and replicator dynamics: new issues in classification*, Biological Cybernetics. Springer Science and Business Media LLC. **72** (5), 447453 (1995).
- [3] P. van den Driessche and M. L. Zeeman, *Three-dimensional competitive Lotka-Volterra systems with no periodic orbits*, SIAM J. Appl. Math., **58**, 227-234 (1998).
- [4] M.W. Hirsch, *Systems of Differential Equations which are Competitive or Cooperative. I: Limit Sets*, SIAM J. Math. Anal. **13** (2), 423439 (1982).
- [5] M.W. Hirsch, *Systems of Differential Equations that are Competitive or Cooperative II: Convergence Almost Everywhere*, SIAM J. Math. Anal. **16** (3), 423439 (1985).
- [6] M.W. Hirsch, *Systems of differential equations that are competitive or cooperative. III: Competing species*, Nonlinearity, **1**, 51-71 (1988).
- [7] M.W. Hirsch, *Systems of Differential Equations That are Competitive or Cooperative. IV: Structural Stability in Three-Dimensional System*, SIAM J. Math. Anal. **21** (5), 12251234 (1990).
- [8] A.J. Lotka, *Elements of Physical Biology* (Williams-Wilkins, Baltimore, 1925) 460 pages.
- [9] S. Smale, *On the differential equations of species in competitio*, Journal of Mathematical Biology. Springer Science and Business Media LLC. **3** (1), 5-7 (1976).
- [10] V. Volterra, *Leçon sur la Theorie Mathématique de la lutte pour la vie* (Gauthier-Villars, Paris, 1931) 226 pages.
- [11] V. Volterra, *Principles de biologie mathématique*, Acta Biother. **3**, 1-36 (1937).
- [12] M.L. Zeeman, *Hopf bifurcations in competitive three dimensional Lotka-Volterra systems*, Dynam. Stability Systems, **8**, 189-217 (1993).
- [13] M.L. Zeeman, *Extinction in competitive Lotka-Volterra systems*, Proc. of American Math. Soc., **123** (1) 87-96 (1995).