On complete decompositions of dihedral groups

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Abstract. Let G be a finite non-abelian group and $B_1, ..., B_t$ be nonempty subsets of G for integer $t \geq 2$. Suppose that $B_1, ..., B_t$ are pairwise disjoint, then $(B_1, ..., B_t)$ is called a complete decomposition of G of order t if the subset product $B_{i1} ... B_{it} = \{b_{i1} ... b_{it} \mid b_{ij} \in B_{ij}, j = 1, 2, ..., t\}$ coincides with G, where $\{B_{i1}, ..., B_{it}\} = \{B_1, ..., B_t\}$ and the B_{ij} are all distinct. Let $D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{n-1} \rangle$ be the dihedral group of order $2n$ for integer $n \geq 3$. In this paper, we shall give the constructions of the complete decompositions of D_{2n} of order t, where $2 \leq t \leq n$.

1 Introduction

Many researchers have conducted their research related to group coverings over the years. There are many types of group coverings and group factorization is one of the group coverings. The first result that related to group factorization was showed in [1] and rediscovered by Davenport in 1935, see [2]. Hajos (see [3] and [4]) proposed a new method called group theoretical equivalent as a solution for Minkowski's geometry problem [5]. He reformulated it into a problem in group covering, and investigated how to factor a finite abelian group into certain subsets. As a result, people started to realize the importance of factoring group into subsets.

Over the years, there are numerous results related to group factorization. Brujin [6] conjectured that if $G = A + B$ is a factorization in which one of the factors has prime order and if G is a finite cyclic group, then either A or B is periodic. Redei [7] showed that if a finite abelian group G contains identity element, given each subsets of G has a prime number of elements and G is a direct product of its subsets, then he found that at least one of the subsets must be a subgroup.

We notice that the exhaustion numbers of subsets of dihedral groups, D_{2n}, had been thoroughly studied, see ([8] and [9]). In this paper, we shall focus on investigating the complete decompositions of D_{2n}. In year 2018, the existence of complete decompositions of cyclic group $\mathbb{Z}_n$ is determined, refer to [10]. By using the complete decompositions of cyclic groups, they found an application in constructing codes over a binary alphabet.

Throughout this paper, we let G be dihedral groups, D_{2n}, for integer $n \geq 3$. It is clear that $D_{2n} = \langle r, s | r^n = s^2 = 1, sr = r^{-1}s \rangle = \{1, r, ..., r^{n-1}, s, rs, ..., r^{n-1}s\}$ is a finite non-abelian group of order $2n$, where $\langle r \rangle = \{1, r, ..., r^{n-1}\}$ is a cyclic subgroup of D_{2n} of order n. Recall that the definition of complete decompositions of G of order t is as follows: Let G

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be a finite non-abelian group and let $B_1, ..., B_t$ be nonempty subsets of G for integer $t \geq 2$. Suppose that $B_1, ..., B_t$ are pairwise disjoint, then $(B_1, ..., B_t)$ is called a complete decomposition of G of order t if the subset product $B_{i1} ... B_{it} = \{b_{i1} ... b_{it} \mid b_{ij} \in B_{ij}, j = 1, 2, ..., t\}$ coincides with G, where $\{B_{i1}, ..., B_{it}\} = \{B_1, ..., B_t\}$ and the B_{ij} are all distinct.

Let $A, B \subseteq D_{2n}$. If $A \cap B = \emptyset$ and either $|A| = 1$ or $|B| = 1$, then $|AB| < 2n$ and hence (A, B) is not a complete decomposition of D_{2n} of order 2. Thus, we consider the subsets A, B such that $|A|, |B| \in \{2,3, ..., 2n-2\}$. In this paper, we first provide some constructions of complete decompositions of D_{2n} of order 2. We extend the results by showing some constructions of complete decompositions of D_{2n} of order t for $t \in \{3,4, ..., n\}$.

2 Constructions of complete decompositions of D_{2n} of order 2

We begin by providing an example of complete decompositions of D_8 of order 2. Let $A = \{1, r, r^2, r^3 s\}$ and $B = \{s, rs, r^2 s, r^3\}$, where $A, B \subseteq D_8$. Then we see that (A, B) is a complete decomposition of D_8 of order 2, since $AB = D_8$.

By using the group relation in D_{2n}, the result below is obvious.

Proposition 1. Let i, n be integers, where $n \geq 3$ and $i \in \{0,1, ..., n-1\}$. If $r^i, s \in D_{2n}$ then $sr^i = r^{n-i} s$.

Proof.
Since $sr = r^{-1} s$, it follows that $sr^2 = srr = r^{-1} sr = r^{-1} r^{-1} s = r^{-2} s$. Thus, by using induction on power i, we obtain $sr^i = r^{-i} s$. Since $r^n = 1$, it follows that $r^{n-i} = r^{-i}$. Thus, we have $sr^i = r^{n-i} s$.

Proposition 2. Let $n \geq 3$ and $A = \{1, r, ..., r^{n-2}, s\}$ and $B = \{r^{n-1}, rs, r^2 s, ..., r^{n-1} s\}$, where $A, B \subseteq D_{2n}$. Then (A, B) is a complete decomposition of D_{2n} of order 2.

Proof.
Note that $AB = \{1, r, ..., r^{n-2}, s\}\{r^{n-1}, rs, ..., r^{n-1} s\} = D_{2n}$. Hence, (A, B) is a complete decomposition of D_{2n} of order 2.

In the following, we show that there exists a complete decomposition of D_{2n} of order 2, where $A \cap B = \emptyset$, $|A| \neq |B|$ and $A \cup B = D_{2n}$.

Proposition 3. Let $A = \{1, r^{v+1}, r^{v+2}, ..., r^{n-1}, r^v s, r^{v+1} s, ..., r^{n-1} s\}$ and $B = \{r, r^2, ..., r^v, s, rs, ..., r^{v-1} s\}$ be the subsets of D_{2n}, where $v \in \{1,2, ..., n-2\}$, $|A| = 2n - 2v$ and $|B| = 2v$. Then (A, B) is a complete decomposition of D_{2n} of order 2 for integer $n \geq 3$.

Proof.
Let $L_1 = \{r^{v+1}, r^{v+2}, ..., r^{n-1}\} \subseteq A$ and $L_2 = \{r, r^2, ..., r^v\} \subseteq B$. Note that $L_1 L_2 = \{r^{v+1}, r^{v+2}, ..., r^{n-1}\}\{r, r^2, ..., r^v\} = \{r^{v+2}, r^{v+3}, ..., r^{n-1}\} \cup \{1, r, ..., r^{v-1}\}$.
Note that $1 \in A$, therefore $\{r, r^2, ..., r^v\} \subseteq AB$. It is clear that $\langle r \rangle \setminus (L_1 L_2 \cup \{r^v\}) = \{r^{v+1}\}$. Observe that $r^{v+1} s \in A$ and $s \in B$. Thus, we have $(r^{v+1} s)(s) = r^{v+1} \in AB$. Therefore, $\{r^v, r^{v+1}, ..., r^{v+n-1}\} = \langle r \rangle \subseteq AB$.

It remains to show that $\langle r \rangle s \subseteq AB$. Let $L_3 = \{s, rs, ..., r^{v-1} s\} \subseteq B$. We see that

$$L_1 L_3 = \{r^{v+1}, r^{v+2}, ..., r^{n-1}\}\{s, rs, ..., r^{v-1} s\}$$
$$= \{r^{v+1} s, r^{v+2} s, ..., r^{n-1} s\} \cup \{s, rs, ..., r^{v-2} s\}.$$

Clearly $\langle r \rangle s \setminus L_1 L_3 = \{r^v s, r^{v+n-1} s\}$. Observe that $\{r^v s, r^{v+1} s\} \subseteq A$ and $r \in B$. Therefore, we have $\{r^v s, r^{v+1} s\} r = \{r^{v+n-1} s, r^{v+n} s\} = \{r^v s, r^{v-1} s\} \subseteq AB$. Thus, $\{r^{v-1} s, r^v s, ...,$

$r^{v+n-2}s\} = \langle r \rangle s \subseteq AB$. Hence, we conclude that (A, B) is a complete decomposition of D_{2n} of order 2.

Next, a construction of complete decompositions of D_{2n} of order 2, where $A \cap B = \emptyset$, $|A| \neq |B|$ and $A \cup B \subset D_{2n}$ is shown below.

Proposition 4. Let $n \geq 6$ be an even integer. Let $A = \{1, r, \dots, r^{n-3}, r^{n-2}s, r^{n-1}s\}$ and $B_i = \{r^{n-2}, r^{n-1}, s, rs, \dots, r^{n-3}s\} \setminus \{rs, r^3s, \dots, r^is\}$, where $A, B_j \subseteq D_{2n}, j \in \{1,3, \dots, n-5\}$, $|A| = n$ and $|B| = n - \frac{j+1}{2}$. Then (A, B_j) is a complete decomposition of D_{2n} of order 2.

Proof.
Note that

$$\begin{aligned} B_{n-5} &= \{r^{n-2}, r^{n-1}, s, r^2s, \dots, r^{n-4}s, r^{n-3}s\}, \\ B_{n-7} &= \{r^{n-2}, r^{n-1}, s, r^2s, \dots, r^{n-6}s, r^{n-5}s, r^{n-4}s, r^{n-3}s\}, \\ &\vdots \\ B_1 &= \{r^{n-2}, r^{n-1}, s, r^2s, r^3s, \dots, r^{n-3}s\}, \end{aligned}$$

where $B_{n-5} \subseteq B_{n-7} \subseteq \cdots \subseteq B_1$. Therefore, we shall focus on proving that $AB_{n-5} = D_{2n}$. Let $L_1 = \{1, r, \dots, r^{n-3}\} \subseteq A$ and $L_2 = \{s, r^2s\} \subseteq B_{n-5}$. We compute L_1L_2 as follows:

$$\begin{aligned} L_1L_2 &= \{1, r, \dots, r^{n-3}\}\{s, r^2s\} \\ &= \{s, rs, \dots, r^{n-3}s, r^2s, r^3s, \dots, r^{n-1}s\} = \langle r \rangle s. \end{aligned}$$

Therefore $\langle r \rangle s \subseteq AB_{n-5}$. Now, we let $L_3 = \{r^{n-2}, r^{n-1}\} \subseteq B_{n-5}$. Then, we compute L_1L_3 as follows:

$$L_1L_3 = \{1, r, \dots, r^{n-3}\}\{r^{n-2}, r^{n-1}\} = \{1, r, \dots, r^{n-4}, r^{n-2}, r^{n-1}\}.$$

Note that the product of $r^{n-1}s \in A$ and $r^2s \in B_{n-5}$ gives us $(r^{n-1}s)(r^2s) = r^{n-3} \in AB_{n-5}$. Therefore, $\langle r \rangle \subseteq AB_{n-5}$. Since $AB_{n-5} = D_{2n}$, we see that (A, B_{n-5}) is a complete decomposition of D_{2n} of order 2. Since $AB_{n-5} = D_{2n}$ and $B_{n-5} \subseteq B_{n-7} \subseteq \cdots \subseteq B_1$, it follows that $AB_{n-5} = AB_{n-7} = \cdots = AB_1 = D_{2n}$. Hence, we conclude that (A, B_j) is a complete decomposition of D_{2n} of order 2 for $j \in \{1,3, \dots, n-5\}$.

3 Constructions of complete decompositions of D_{2n} of order k for $k \in \{3, 4, \dots, n\}$

By using Matlab, we first generate all the possible complete decomposition of D_8 of order 3. We found that there are total of 368 constructions show that (B_1, B_2, B_3) is a complete decomposition of D_8 of order 3. Since dihedral group is a non-abelian group, the product of the subsets might not be the same. Hence, we check the product of the subsets $B_1B_2B_3$ and $B_2B_1B_3$. We notice that there are 17 out of 368 constructions indicate that $B_1B_2B_3 \neq B_2B_1B_3$. To be more precise, Table 1 showed that there are 17 constructions where (B_1, B_2, B_3) is a complete decomposition of D_8 of order 3 but (B_2, B_1, B_3) is not a complete decomposition of D_8 of order 3.

Table 1. List of complete decompositions of D_8 of order 3, where $B_1, B_2, B_3 \subseteq D_8$ and $B_1 B_2 B_3 \neq B_2 B_1 B_3$.

B_1	B_2	B_3
$\{1, s\}$	$\{r, rs\}$	$\{r^2, r^3, r^2 s, r^3 s\}$
$\{1, s\}$	$\{r^3, r^3 s\}$	$\{r, r^2, rs, r^2 s\}$
$\{1, rs\}$	$\{r, r^2 s\}$	$\{r^2, r^3, s, r^3 s\}$
$\{1, rs\}$	$\{r^3, s\}$	$\{r, r^2, r^2 s, r^3 s\}$
$\{1, r^2 s\}$	$\{r, r^3 s\}$	$\{r^2, r^3, s, rs\}$
$\{1, r^2 s\}$	$\{r^3, rs\}$	$\{r, r^2, s, r^3 s\}$
$\{1, r^3 s\}$	$\{r, s\}$	$\{r^2, r^3, rs, r^2 s\}$
$\{1, r^3 s\}$	$\{r^3, r^2 s\}$	$\{r, r^2, s, rs\}$
$\{r, r^2\}$	$\{1, s\}$	$\{r^3, rs, r^2 s, r^3 s\}$
$\{r^2, rs\}$	$\{r, s\}$	$\{1, r^3, r^2 s, r^3 s\}$
$\{r^2, r^2 s\}$	$\{r, rs\}$	$\{1, r^3, s, r^3 s\}$
$\{r^2, r^3 s\}$	$\{r, r^2 s\}$	$\{1, r^3, s, rs\}$
$\{r^2, s\}$	$\{r, r^3 s\}$	$\{1, r^3, rs, r^2 s\}$
$\{r^2, s\}$	$\{r^3, rs\}$	$\{1, r, r^2 s, r^3 s\}$
$\{r^2, rs\}$	$\{r^3, r^2 s\}$	$\{1, r, s, r^3 s\}$
$\{r^2, r^2 s\}$	$\{r^3, r^3 s\}$	$\{1, r, s, rs\}$
$\{r^2, r^3 s\}$	$\{r^3, s\}$	$\{1, r, rs, r^2 s\}$

Therefore, we remark that if $(B_1, B_2, \dots, B_t)$ is a complete decomposition of D_{2n} of order t, it does not imply that $(B_{j_1}, B_{j_2}, \dots, B_{j_t})$ is a complete decomposition of D_{2n} of order t, where $\{B_{j_1}, B_{j_2}, \dots, B_{jt}\} = \{B_1, B_2, \dots, B_t\}$.

Now, we provide a construction of complete decompositions of D_{2n} of order 3 as follow.

Proposition 5. Let $n \geq 3$ and $A_1, A_2, A_3 \subseteq D_{2n}$, where $A_1 = \{1, r\}, A_2 = \{s, rs\}$ and $A_3 = \{r^2, r^3, \dots, r^{n-1}, r^2 s, r^3 s, \dots, r^{n-1} s\}$. Then (A_1, A_2, A_3) is a complete decomposition of D_{2n} of order 3.

Proof.
We first compute $A_1 A_2$ as follows:

$$\begin{aligned} A_1 A_2 &= \{1, r\}\{s, rs\} \\ &= \{s, rs, r^2 s\} \end{aligned}$$

and followed by

$$\begin{aligned} A_1 A_2 A_3 &= \{s, rs, r^2 s\}\{r^2, r^3, \dots, r^{n-1}, r^2 s, r^3 s, \dots, r^{n-1} s\} \\ &= \{r^{n-2} s, r^{n-3} s, \dots, rs, r^{n-2}, r^{n-3}, \dots, r, \\ &\quad r^{n-1} s, r^{n-2} s, \dots, r^2 s, r^{n-1}, r^{n-2}, \dots, r^2, \\ &\quad r^n s, r^{n-1} s, \dots, r^3 s, r^n, r^{n-1}, \dots, r^3\} \\ &= \{1, r, \dots, r^{n-1}, s, rs, \dots, r^{n-1} s\} = D_{2n}. \end{aligned}$$

Hence, (A_1, A_2, A_3) is a complete decomposition of D_{2n} of order 3.

Now, we shall focus on the construction, where $(A_1, A_2, \dots, A_k)$ is a complete decomposition of D_{2n} of order $k \in \{4,5, \dots, n\}$. The constructions of complete decompositions of D_{2n} of order k for $k \in \{4,5, \dots, n\}$ are as follows:

$$\begin{aligned} A_1 &= \{1, r\}; \\ A_2 &= \{s, rs\}; \\ A_i &= \{r^{i-1}, r^{i-1} s\} \text{ for } i \in \{3,4, \dots, k-1\}; \\ A_k &= \{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1} s, r^k s, \dots, r^{n-1} s\}. \end{aligned}$$

Proposition 6. Let $n \geq 7$ be integer. There exists a complete decomposition of D_{2n} of order k for $k \in \{4,5,6,7\}$.

Proof.
Let

$$\begin{aligned} A_1 &= \{1, r\}; \\ A_2 &= \{s, rs\}; \\ A_i &= \{r^{i-1}, r^{i-1}s\}, \qquad \text{for } i \in \{3,4,\dots,k-1\}; \\ A_k &= \{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1}s, r^k s, \dots, r^{n-1}s\}. \end{aligned}$$

where $A_1, A_2, \dots, A_k \subseteq D_{2n}$. We separate the proof into 4 cases for $k \in \{4,5,6,7\}$ respectively. Note that $A_1A_2 = \{1,r\}\{s,rs\} = \{s, rs, r^2s\}$. We compute $A_1A_2A_3$ as follows:

$$A_1A_2A_3 = \{s, rs, r^2s\}\{r^2, r^2s\} = \{1, r^{n-2}, r^{n-1}, s, r^{n-2}s, r^{n-1}s\}. \tag{1}$$

(i) When $k = 4$, we have $A_4 = \{r^3, r^4, \dots, r^{n-1}, r^3s, r^4s, \dots, r^{n-1}s\}$. We compute $A_1A_2A_3A_4$ as follows:

$$\begin{aligned} A_1A_2A_3A_4 &= \{1, r^{n-2}, r^{n-1}, s, r^{n-2}s, r^{n-1}s\}\{r^3, r^4, \dots, r^{n-1}, r^3s, r^4s, \dots, r^{n-1}s\} \\ &= \{1, r, \dots, r^{n-1}, s, rs, \dots, r^{n-1}s\} = D_{2n}. \end{aligned}$$

Hence, (A_1, A_2, A_3, A_4) is a complete decomposition of D_{2n} of order 4.

(ii) When $k = 5$, we have $A_4 = \{r^3, r^3s\}$. From Equation (1), we have $A_1A_2A_3 = \{1, r^{n-2}, r^{n-1}, s, r^{n-2}s, r^{n-1}s\}$. Now, we compute $A_1A_2A_3A_4$ as follows:

$$\begin{aligned} &A_1A_2A_3A_4 \\ &= \{1, r^{n-2}, r^{n-1}, s, r^{n-2}s, r^{n-1}s\}\{r^3, r^3s\} \\ &= \{r, r^2, r^3, r^{n-5}, r^{n-4}, r^{n-3}, rs, r^2s, r^3s, r^{n-5}s, r^{n-4}s, r^{n-3}s\}. \end{aligned} \tag{2}$$

Observe that $A_5 = \{r^4, r^5, \dots, r^{n-1}, r^4s, r^5s, \dots, r^{n-1}s\}$. We compute $A_1A_2A_3A_4A_5$ as follows:

$$\begin{aligned} &A_1A_2A_3A_4A_5 \\ &= \{r, r^2, r^3, r^{n-5}, r^{n-4}, r^{n-3}, rs, r^2s, r^3s, r^{n-5}s, r^{n-4}s, r^{n-3}s\}\{r^4, r^5, \\ &\quad \dots, r^{n-1}, r^4s, r^5s, \dots, r^{n-1}s\} \\ &= \{1, r, \dots, r^{n-1}, s, rs, \dots, r^{n-1}s\} = D_{2n}. \end{aligned}$$

Hence, $(A_1, A_2, A_3, A_4, A_5)$ is a complete decomposition of D_{2n} of order 5.

(iii) When $k = 6$, we have $A_5 = \{r^4, r^4s\}$. From Equation (2), we have

$$A_1A_2A_3A_4 = \{r, r^2, r^3, r^{n-5}, r^{n-4}, r^{n-3}, rs, r^2s, r^3s, r^{n-5}s, r^{n-4}s, r^{n-3}s\}.$$

Then, we compute $A_1A_2A_3A_4A_5$ as follows:

$$\begin{aligned} &A_1A_2A_3A_4A_5 \\ &\quad = \{r, r^2, r^3, r^{n-5}, r^{n-4}, r^{n-3}, rs, r^2s, r^3s, r^{n-5}s, r^{n-4}s, r^{n-3}s\}\{r^4, r^4s\} \\ &\quad = \{1, r, r^5, r^6, r^7, r^{n-9}, r^{n-8}, r^{n-7}, r^{n-3}, r^{n-2}, r^{n-1}, \\ &\qquad s, rs, r^5s, r^6s, r^7s, r^{n-9}s, r^{n-8}s, r^{n-7}s, r^{n-3}s, r^{n-2}s, r^{n-1}s\} \end{aligned} \tag{3}$$

Note that $A_6 = \{r^5, r^6, \dots, r^{n-1}, r^5s, r^6s, \dots, r^{n-1}s\}$. Next, we compute $A_1A_2 \dots A_6$ as follows:

$$
\begin{aligned}
&A_1A_2A_3A_4A_5A_6\\
&=\{1,r,r^5,r^6,r^7,r^{n-9},r^{n-8},r^{n-7},r^{n-3},r^{n-2},r^{n-1},\\
&\quad s,rs,r^5s,r^6s,r^7s,r^{n-9}s,r^{n-8}s,r^{n-7}s,r^{n-3}s,r^{n-2}s,r^{n-1}s\}\\
&\quad \{r^5,r^6,\dots,r^{n-1},r^5s,r^6s,\dots,r^{n-1}s\}\\
&=\{1,r,\dots,r^{n-1},s,rs,\dots,r^{n-1}s\}=D_{2n}.
\end{aligned}
$$

Hence, the result holds.

(iv) When $k=7$, we have $A_6=\{r^5,r^5s\}$. By using the similar way, it can be shown that $(A_1,A_2,\dots,A_7)$ is a complete decomposition of D_{2n} of order 7.

The following lemma shall be used in later proof.

Lemma 7. Let $n\geq 8$, $k\in\{8,9,\dots,n\}$ and $A_i=\{r^{i-1},r^{i-1}s\}\subseteq D_{2n}$ for $i\in\{3,4,\dots,k-1\}$. Then $A_3A_4\dots A_{k-1}=\{r^{-c+5},r^{-c+11},r^{-c+13},\dots,r^{c-9},r^{c-7},\dots,r^{c-1}\}\cup\{r^{-c+5}s,\ r^{-c+11}s,\ r^{-c+13}s,\dots,r^{c-9}s,r^{c-7}s,r^{c-1}s\}$, where $c=\frac{(k-2)(k-1)}{2}$.

Proof.
We first show that the base case $k=8$ is true. We compute $A_3A_4A_5A_6A_7$ as follows:

$$
\begin{aligned}
A_3A_4&=\{r^2,r^2s\}\{r^3,r^3s\}=\{r^{-1},r^5,r^{-1}s,r^5s\};\\
A_3A_4A_5&=\{r^{-1},r^5,r^{-1}s,r^5s\}\{r^4,r^4s\}=\{r^{-5},r,r^3,r^9,r^{-5}s,rs,r^3s,r^9s\};\\
A_3A_4A_5A_6&=\{r^{-5},r,r^3,r^9,r^{-5}s,rs,r^3s,r^9s\}\{r^5,r^5s\}\\
&=\{r^{-10},r^{-4},r^{-2},1,r^4,r^6,r^8,r^{14},r^{-10}s,r^{-4}s,r^{-2}s,s,r^4s,r^6s,r^8s,r^{14}s\};
\end{aligned}
$$

$$
\begin{aligned}
&A_3A_4A_5A_6A_7\\
&=\{r^{-10},r^{-4},r^{-2},1,r^4,r^6,r^8,r^{14},r^{-10}s,r^{-4}s,r^{-2}s,s,r^4s,r^6s,r^8s,r^{14}s\}\{r^6,r^6s\}\\
&=\{r^{-16},r^{-10},r^{-8},r^{-6},r^{-4},r^{-2},1,r^2,r^4,r^6,r^8,r^{10},r^{12},r^{14},r^{20}\}\cup\\
&\quad\{r^{-16}s,r^{-10}s,r^{-8}s,r^{-6}s,r^{-4}s,r^{-2}s,s,r^2s,r^4s,r^6s,r^8s,r^{10}s,r^{12}s,r^{14}s,r^{20}s\}.
\end{aligned}
$$

Clearly the statement is true when $k=8$. Thus the base case holds. Now, we assume that the statement is true when $k=m$. By assumption, it is clear that

$$
\begin{aligned}
&A_3A_4\dots A_{m-1}\\
&=\{r^{-\frac{(m-2)(m-1)}{2}+5},r^{-\frac{(m-2)(m-1)}{2}+11},r^{-\frac{(m-2)(m-1)}{2}+13},\dots,\\
&\quad r^{\frac{(m-2)(m-1)}{2}-9},r^{\frac{(m-2)(m-1)}{2}-7},r^{\frac{(m-2)(m-1)}{2}-1}\}\cup\\
&\quad\{r^{-\frac{(m-2)(m-1)}{2}+5}s,r^{-\frac{(m-2)(m-1)}{2}+11}s,r^{-\frac{(m-2)(m-1)}{2}+13}s,\dots,\\
&\quad r^{\frac{(m-2)(m-1)}{2}-9}s,r^{\frac{(m-2)(m-1)}{2}-7}s,r^{\frac{(m-2)(m-1)}{2}-1}s\}\\
&=B_1\cup B_1s,
\end{aligned}
$$

where

$$
\begin{aligned}
B_1=\{&r^{-\frac{(m-2)(m-1)}{2}+5},r^{-\frac{(m-2)(m-1)}{2}+11},r^{-\frac{(m-2)(m-1)}{2}+13},\dots,\\
&r^{\frac{(m-2)(m-1)}{2}-9},r^{\frac{(m-2)(m-1)}{2}-7},r^{\frac{(m-2)(m-1)}{2}-1}\}.
\end{aligned}
$$

Next, we show that the statement is also true for $k=m+1$. Note that $A_m=\{r^{m-1},r^{m-1}s\}$ and $A_3A_4\dots A_m=B_1A_m\cup B_1sA_m$. We compute B_1A_m and B_1sA_m as follows:

$$\begin{aligned}B_1A_m &= \{r^{-\frac{(m-2)(m-1)}{2}+5}, r^{-\frac{(m-2)(m-1)}{2}+11}, r^{-\frac{(m-2)(m-1)}{2}+13}, \dots, \\ &\quad r^{\frac{(m-2)(m-1)}{2}-9}, r^{\frac{(m-2)(m-1)}{2}-7}, r^{\frac{(m-2)(m-1)}{2}-1}\}\{r^{m-1}, r^{m-1}s\} \\ &= \{r^{-\frac{(m-2)(m-1)}{2}+5+(m-1)}, r^{-\frac{(m-2)(m-1)}{2}+11+(m-1)}, r^{-\frac{(m-2)(m-1)}{2}+13+(m-1)}, \dots, \\ &\quad r^{\frac{(m-2)(m-1)}{2}-9+(m-1)}, r^{\frac{(m-2)(m-1)}{2}-7+(m-1)}, r^{\frac{(m-2)(m-1)}{2}-1+(m-1)}\}\{1, s\} \\ &= C_1 \cup C_1s\end{aligned}$$

$$\begin{aligned}B_1sA_m &= \{r^{-\frac{(m-2)(m-1)}{2}+5}s, r^{-\frac{(m-2)(m-1)}{2}+11}s, r^{-\frac{(m-2)(m-1)}{2}+13}s, \dots, \\ &\quad r^{\frac{(m-2)(m-1)}{2}-9}s, r^{\frac{(m-2)(m-1)}{2}-7}s, r^{\frac{(m-2)(m-1)}{2}-1}s\}\{r^{m-1}, r^{m-1}s\} \\ &= \{r^{-\frac{(m-2)(m-1)}{2}+5-(m-1)}, r^{-\frac{(m-2)(m-1)}{2}+11-(m-1)}, r^{-\frac{(m-2)(m-1)}{2}+13-(m-1)}, \dots, \\ &\quad r^{\frac{(m-2)(m-1)}{2}-9-(m-1)}, r^{\frac{(m-2)(m-1)}{2}-7-(m-1)}, r^{\frac{(m-2)(m-1)}{2}-1-(m-1)}\}\{1, s\} \\ &= D_1 \cup D_1s,\end{aligned}$$

where

$$\begin{aligned}C_1 &= \{r^{-\frac{(m-2)(m-1)}{2}+5+(m-1)}, r^{-\frac{(m-2)(m-1)}{2}+11+(m-1)}, r^{-\frac{(m-2)(m-1)}{2}+13+(m-1)}, \dots, \\ &\quad r^{\frac{(m-2)(m-1)}{2}-9+(m-1)}, r^{\frac{(m-2)(m-1)}{2}-7+(m-1)}, r^{\frac{(m-2)(m-1)}{2}-1+(m-1)}\} \\ &= \{r^{-\frac{m(m-1)}{2}+2m+3}, r^{-\frac{m(m-1)}{2}+2m+9}, \dots, r^{-\frac{m(m-1)}{2}+2m+11}, \dots, r^{\frac{m(m-1)}{2}-9}, r^{\frac{m(m-1)}{2}-7}, \dots, r^{\frac{m(m-1)}{2}-1}\},\end{aligned}$$

$$\begin{aligned}D_1 &= \{r^{-\frac{(m-2)(m-1)}{2}+5-(m-1)}, r^{-\frac{(m-2)(m-1)}{2}+11-(m-1)}, r^{-\frac{(m-2)(m-1)}{2}+13-(m-1)}, \dots, \\ &\quad r^{\frac{(m-2)(m-1)}{2}-9-(m-1)}, r^{\frac{(m-2)(m-1)}{2}-7-(m-1)}, r^{\frac{(m-2)(m-1)}{2}-1-(m-1)}\} \\ &= \{r^{-\frac{m(m-1)}{2}+5}, r^{-\frac{m(m-1)}{2}+11}, r^{-\frac{m(m-1)}{2}+13}, \dots, r^{\frac{m(m-1)}{2}-9}, r^{\frac{m(m-1)}{2}-7}, r^{\frac{m(m-1)}{2}-1}\}.\end{aligned}$$

Thus, we have

$$C_1 \cup D_1 = \{r^{-\frac{m(m-1)}{2}+5}, r^{-\frac{m(m-1)}{2}+11}, r^{-\frac{m(m-1)}{2}+13}, \dots\ r^{\frac{m(m-1)}{2}-9}, r^{\frac{m(m-1)}{2}-7}, r^{\frac{m(m-1)}{2}-1}\},$$

$$C_1s \cup D_1s = \{r^{-\frac{m(m-1)}{2}+5}s, r^{-\frac{m(m-1)}{2}+11}s, r^{-\frac{m(m-1)}{2}+13}s, \dots, r^{\frac{m(m-1)}{2}-9}s, r^{\frac{m(m-1)}{2}-7}s, r^{\frac{m(m-1)}{2}-1}s\}.$$

Note that $A_3A_4 \dots A_m = C_1 \cup D_1 \cup C_1s \cup D_1s$. Therefore, the statement is also true for $k = m+1$. Hence, by mathematical induction, the statement is true for $k \in \{8, 9, \dots, n\}$.

Theorem 8. Let $n \geq 8$. There exists a complete decomposition of D_{2n} of order k for $k \in \{8, 9, \dots, n\}$.

Proof.
Let $A_1 = \{1, r\}$, $A_2 = \{s, rs\}$, $A_i = \{r^{i-1}, r^{i-1}s\}$ for $i \in \{3,4, \dots, k-1\}$ and $A_k = \{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1}s, r^ks, \dots, r^{n-1}s\}$ be the subsets of D_{2n}. We first compute

$$A_1A_2 = \{1, r\}\{s, rs\} = \{s, rs, r^2s\}.$$

Let $c = \frac{(k-2)(k-1)}{2}$. By Lemma 7, we have

$$\begin{aligned}A_3A_4 \dots A_{k-1} &= \{r^{-c+5}, r^{-c+11}, r^{-c+13}, \dots, r^{c-9}, r^{c-7}, r^{c-1}\} \cup \\ &\quad \{r^{-c+5}s, r^{-c+11}s, r^{-c+13}s, \dots, r^{c-9}s, r^{c-7}s, r^{c-1}s\} \\ &= L_1 \cup L_2,\end{aligned}$$

where

$$L_1 = \{r^{-c+5}, r^{-c+11}, r^{-c+13}, \dots, r^{c-9}, r^{c-7}, r^{c-1}\}$$

and

$$L_2 = \{r^{-c+5}s, r^{-c+11}s, r^{-c+13}s, \dots, r^{c-9}s, r^{c-7}s, r^{c-1}s\}.$$

Now, we compute $A_1A_2L_1 \subseteq A_1A_2 \dots A_{k-1}$ and $A_1A_2L_2 \subseteq A_1A_2 \dots A_{k-1}$ as follows:

$$
\begin{aligned}
A_1A_2L_1 &= \{s, rs, r^2s\}\{r^{-c+5}, r^{-c+11}, r^{-c+13}, \dots, r^{c-9}, r^{c-7}, r^{c-1}\} \\
&= \{r^{n-c+1}s, r^{n-c+2}s, r^{n-c+3}s\} \cup \{r^{n-c+7}s, r^{n-c+8}s, \dots, r^{n+c-9}s\} \cup \\
&\quad \{r^{n+c-5}s, r^{n+c-4}s, r^{n+c-3}s\} \\
&= P_1 \cup P_2 \cup P_3; \\
A_1A_2L_2 &= \{s, rs, r^2s\}\{r^{-c+5}s, r^{-c+11}s, r^{-c+13}s, \dots, r^{c-9}s, r^{c-7}s, r^{c-1}s\} \\
&= \{r^{n-c+1}, r^{n-c+2}, r^{n-c+3}\} \cup \{r^{n-c+7}, r^{n-c+8}, \dots, r^{n+c-9}\} \cup \\
&\quad \{r^{n+c-5}, r^{n+c-4}, r^{n+c-3}\} \\
&= M_1 \cup M_2 \cup M_3,
\end{aligned}
$$

where

$$
\begin{aligned}
P_1 &= \{r^{n-c+1}s, r^{n-c+2}s, r^{n-c+3}s\}, \\
P_2 &= \{r^{n-c+7}s, r^{n-c+8}s, \dots, r^{n+c-9}s\}, \\
P_3 &= \{r^{n+c-5}s, r^{n+c-4}s, r^{n+c-3}s\}, \\
M_1 &= \{r^{n-c+1}, r^{n-c+2}, r^{n-c+3}\}, \\
M_2 &= \{r^{n-c+7}, r^{n-c+8}, \dots, r^{n+c-9}\}, \\
M_3 &= \{r^{n+c-5}, r^{n+c-4}, r^{n+c-3}\}.
\end{aligned}
$$

Now, we compute $P_1A_k, P_2A_k, \dots, P_6A_k \subseteq A_1A_2 \dots A_k$ as follows:

$$
\begin{aligned}
&P_1A_k \\
&= \{r^{n-c+1}s, r^{n-c+2}s, r^{n-c+3}s\}\{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1}s, r^ks, \dots, r^{n-1}s\} \\
&= \{r^{n-c+2}, r^{n-c+3}, \dots, r^{2n-c-k+4}\} \cup \{r^{n-c+2}s, r^{n-c+3}s, \dots, r^{2n-c-k+4}s\} \\
&= R_1;
\end{aligned}
$$

$$
\begin{aligned}
&P_2A_k \\
&= \{r^{n-c+7}s, r^{n-c+8}s, r^{n+c+9}s\}\{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1}s, r^ks, \dots, r^{n-1}s\} \\
&= \{r^{n-c+8}, r^{n-c+9}, \dots, r^{2n+c-k-8}\} \cup \{r^{n-c+8}s, r^{n-c+9}s, \dots, r^{2n+c-k-8}s\} \\
&= R_2;
\end{aligned}
$$

$$
\begin{aligned}
&P_3A_k \\
&= \{r^{n+c-5}s, r^{n+c-4}s, r^{n+c-3}s\}\{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1}s, r^ks, \dots, r^{n-1}s\} \\
&= \{r^{n+c-4}, r^{n+c-3}, \dots, r^{2n+c-k-2}\} \cup \{r^{n+c-4}s, r^{n+c-3}s, \dots, r^{2n+c-k-2}s\} \\
&= R_3;
\end{aligned}
$$

$$
\begin{aligned}
&M_1A_k \\
&= \{r^{n-c+1}, r^{n-c+2}, r^{n-c+3}\}\{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1}s, r^ks, \dots, r^{n-1}s\} \\
&= \{r^{n-c+k}, r^{n-c+k+1}, \dots, r^{2n-c+2}\} \cup \{r^{n-c+k}s, r^{n-c+k+1}s, \dots, r^{2n-c+2}s\} \\
&= R_4;
\end{aligned}
$$

$$
\begin{aligned}
&M_2A_k \\
&= \{r^{n-c+7}, r^{n-c+8}, \dots, r^{n+c-9}\}\{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1}s, r^ks, \dots, r^{n-1}s\} \\
&= \{r^{n-c+k+6}, r^{n-c+k+7}, \dots, r^{2n+c-10} \cup \{r^{n-c+k+6}s, r^{n-c+k+7}s, \dots, r^{2n+c-10}s\} \\
&= R_5;
\end{aligned}
$$

$$
\begin{aligned}
&M_3A_k \\
&= \{r^{n+c-5}, r^{n+c-4}, r^{n+c-3}\}\{r^{k-1}, r^k, \dots, r^{n-1}, r^{k-1}s, r^ks, \dots, r^{n-1}s\} \\
&= \{r^{n+c-k-6}, r^{n+c-k-5}, \dots, r^{2n+c-4}\} \cup \{r^{n+c-k-6}s, r^{n+c-k-5}s, \dots, r^{2n+c-4}s\} \\
&= R_6.
\end{aligned}
$$

Note that $A_1A_2 \dots A_k = R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup R_6$. Observe that

$$
R_2 = \{r^{n-c+8}, r^{n-c+9}, \dots, r^{2n+c-k-8}\} \cup \{r^{n-c+8}s, r^{n-c+9}s, \dots, r^{2n+c-k-8}s\},
$$

where

$$\begin{aligned}|\{r^{n-c+8}, r^{n-c+9}, \dots, r^{2n+c-k-8}\}| &= |\{r^{n-c+8}s, r^{n-c+9}s, \dots, r^{2n+c-k-8}s\}| \\ &= n + 2c - k - 15 \\ &= n + k^2 - 2k - 15 \geq n\end{aligned}$$

for $k \in \{8, 9, \dots, n\}$. Therefore, we have $R_2 = Q_{2^n}$. Hence, there exists a complete decomposition of D_{2n} of order k for $k \in \{8, 9, \dots, n\}$.

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