On complete decompositions of dihedral groups

Huey Voon Chen*1, and Chang Seng Sin1

1Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science, Universiti Tunku Abdul Rahman, 43000 Kajang, Selangor, Malaysia

Abstract. Let G be a finite non-abelian group and B1, ..., Bt be nonempty subsets of G for integer t ≥ 2. Suppose that B1, ..., Bt are pairwise disjoint, then (B1, ..., Bt) is called a complete decomposition of G of order t if the subset product B1 ... Bt = \{b1 ... bt | bi ∈ Bi, j = 1, 2, ..., t\} coincides with G, where \{B1, ..., Bt\} = \{B1, ..., Bt\} and the Bi are all distinct. Let D2n = ⟨r, s⟩|rn = s2 = 1, rs = srs−1⟩ be the dihedral group of order 2n for integer n ≥ 3. In this paper, we shall give the constructions of the complete decompositions of D2n of order t, where 2 ≤ t ≤ n.

1 Introduction

Many researchers have conducted their research related to group coverings over the years. There are many types of group coverings and group factorization is one of the group coverings. The first result that related to group factorization was showed in [1] and rediscovered by Davenport in 1935, see [2]. Hajos (see [3] and [4]) proposed a new method called group theoretical equivalent as a solution for Minkowski’s geometry problem [5]. He reformulated it into a problem in group covering, and investigated how to factor a finite abelian group into certain subsets. As a result, people started to realize the importance of factoring group into subsets.

Over the years, there are numerous results related to group factorization. Brujin [6] conjectured that if G = A + B is a factorization in which one of the factors has prime order and if G is a finite cyclic group, then either A or B is periodic. Redei [7] showed that if a finite abelian group G contains identity element, given each subsets of G has a prime number of elements and G is a direct product of its subsets, then he found that at least one of the subsets must be a subgroup.

We notice that the exhaustion numbers of subsets of dihedral groups, D2n, had been thoroughly studied, see ([8] and [9]). In this paper, we shall focus on investigating the complete decompositions of D2n. In year 2018, the existence of complete decompositions of cyclic group \( \mathbb{Z}_n \) is determined, refer to [10]. By using the complete decompositions of cyclic groups, they found an application in constructing codes over a binary alphabet.

Throughout this paper, we let G be dihedral groups, D2n, for integer n ≥ 3. It is clear that D2n = ⟨r, s⟩|rn = s2 = 1, rs = r−1s⟩ = \{1, r, ..., r^{n-1}, s, rs, ..., r^{n-1}s\} is a finite non-abelian group of order 2n, where ⟨r⟩ = \{1, r, ..., r^{n-1}\} is a cyclic subgroup of D2n of order n. Recall that the definition of complete decompositions of G of order t is as follows: Let G

*Corresponding author: chenhv@utar.edu.my

© The Authors, published by EDP Sciences. This is an open access article distributed under the terms of the Creative Commons Attribution License 4.0 (http://creativecommons.org/licenses/by/4.0/).
be a finite non-abelian group and let \( B_1, ..., B_t \) be nonempty subsets of \( G \) for integer \( t \geq 2 \). Suppose that \( B_1, ..., B_t \) are pairwise disjoint, then \((B_1, ..., B_t)\) is called a complete decomposition of \( G \) of order \( t \) if the subset product \( B_1 \cdots B_t = \{ b_1 \cdots b_t | b_i \in B_i, j = 1, 2, ..., t \} \) coincides with \( G \), where \( \{B_1, ..., B_t\} = \{B_1, ..., B_t\} \) and the \( B_i \) are all distinct.

Let \( A, B \subseteq D_{2n} \). If \( A \cap B = \emptyset \) and either \(|A| = 1 \) or \(|B| = 1 \), then \(|AB| < 2n \) and hence \((A,B)\) is not a complete decomposition of \( D_{2n} \) of order 2. Thus, we consider the subsets \( A,B \) such that \(|A|,|B| \in \{2,3, ..., 2n-2 \} \). In this paper, we first provide some constructions of complete decompositions of \( D_{2n} \) of order 2. We extend the results by showing some constructions of complete decompositions of \( D_{2n} \) of order \( t \) for \( t \in \{3,4, ..., n \} \).

### 2 Constructions of complete decompositions of \( D_{2n} \) of order 2

We begin by providing an example of complete decompositions of \( D_8 \) of order 2. Let \( A = \{1, r, r^2, r^3 s\} \) and \( B = \{s, rs, r^2 s, r^3\} \), where \( A,B \subseteq D_8 \). Then we see that \((A,B)\) is a complete decomposition of \( D_8 \) of order 2, since \( AB = D_8 \).

By using the group relation in \( D_{2n} \), the result below is obvious.

**Proposition 1.** Let \( i, n \) be integers, where \( n \geq 3 \) and \( i \in \{0,1,...,n-1\} \). If \( r^i, s \in D_{2n} \) then \( sr^i = r^{n-i}s \).

**Proof.** Since \( sr = r^{-1}s \), it follows that \( sr^2 = srr = r^{-1}s = r^{-1}r^{-1}s = r^{-2}s \). Thus, by induction on power \( i \), we obtain \( sr^i = r^{-i}s \). Since \( r^n = 1 \), it follows that \( r^{n-i} = r^{-i} \). Thus, we have \( sr^i = r^{n-i}s \).

**Proposition 2.** Let \( n \geq 3 \) and \( A = \{1, r, ..., r^{n-2}, s\} \) and \( B = \{r^{n-1}, rs, r^2 s, ..., r^{n-1}s\} \), where \( A,B \subseteq D_{2n} \). Then \((A,B)\) is a complete decomposition of \( D_{2n} \) of order 2.

**Proof.** Note that \( AB = \{1, r, ..., r^{n-2}, s\}\{r^{n-1}, rs, ..., r^{n-1}s\} = D_{2n} \). Hence, \((A,B)\) is a complete decomposition of \( D_{2n} \) of order 2.

In the following, we show that there exists a complete decomposition of \( D_{2n} \) of order 2, where \( A \cap B = \emptyset, |A| \neq |B| \) and \( A \cup B = D_{2n} \).

**Proposition 3.** Let \( A = \{1, r^{v+1}, r^{v+2}, ..., r^{n-1}, r^{v}s, r^{v+1}s, ..., r^{n-1}s\} \) and \( B = \{r, r^{2}, ..., r^{v}, s, rs, ..., r^{v-1}s\} \) be the subsets of \( D_{2n} \), where \( v \in \{1,2, ..., n-2\}, |A| = 2n - 2v \) and \(|B| = 2v \). Then \((A,B)\) is a complete decomposition of \( D_{2n} \) of order 2 for integer \( n \geq 3 \).

**Proof.** Let \( L_1 = \{r^{v+1}, r^{v+2}, ..., r^{n-1}\} \subseteq A \) and \( L_2 = \{r, r^2, ..., r^v\} \subseteq B \). Note that \( L_1L_2 = \{r^{v+1}, r^{v+2}, ..., r^{n-1}\}\{r, r^2, ..., r^v\} = \{r^{v+2}, ..., r^{n-1}\} \cup \{r, ..., r^v\} \).

Note that 1 is in \( A \), therefore \( \{r, r^2, ..., r^v\} \subseteq AB \). It is clear that \( \langle r \rangle \setminus (L_1L_2 \cup \{r^v\}) = \{r^{v+1}\} \). Observe that \( r^{v+1}s \in A \) and \( s \in B \). Thus, we have \( (r^{v+1}s)(s) = r^{v+1}e \in AB \). Therefore, \( \langle r, r^{v+1}, ..., r^{v+n-1}\rangle = \langle r \rangle \subseteq AB \).

It remains to show that \( \langle r \rangle s \subseteq AB \). Let \( L_3 = \{s, rs, ..., r^{v-1}s\} \subseteq B \). We see that \( L_1L_3 = \{r^{v+1}, r^{v+2}, ..., r^{n-1}\}\{s, rs, ..., r^{v-1}s\} \equiv \{r^{v+2}s, ..., r^{n-1}s\} \cup \{s, rs, ..., r^{v-2}s\} \).

Clearly \( \langle r \rangle s \setminus L_1L_3 = \{r^v s, r^{v-1+s}\} \subseteq AB \). Observe that \( \{r^v, r^{v+1}s\} \subseteq A \) and \( r \in B \). Therefore, we have \( \{r^{v}s, r^{v+1}s\}r = \{r^{v+n-1}s, r^{v+n}s\} \subseteq AB \). Thus, \( \{r^{v+n-1}s, r^v s, ..., \)
\( r^{v+n-2}\{s \}\) = \( \{r\} \subseteq AB \). Hence, we conclude that \((A, B)\) is a complete decomposition of \( D_{2n} \) of order 2.

Next, a construction of complete decompositions of \( D_{2n} \) of order 2, where \( A \cap B = \emptyset \), \(|A| \neq |B|\) and \( A \cup B \subseteq D_{2n} \) is shown below.

**Proposition 4.** Let \( n \geq 6 \) be an even integer. Let \( A = \{1, r, \ldots, r^{n-3}, r^{n-2}s, r^{n-1}s\} \) and \( B_j = \{r^{n-2}, r^{n-1}, s, rs, \ldots, r^{i-3}s\} \) \( \setminus \{rs, r^3s, \ldots, r^i s\} \), where \( A, B_j \subseteq D_{2n}, j \in \{1, 3, \ldots, n-5\} \), \(|A| = n\) and \(|B| = n - \frac{j+1}{2}\). Then \((A, B_j)\) is a complete decomposition of \( D_{2n} \) of order 2.

**Proof.**

Note that

\[
B_{n-5} = \{r^{n-2}, r^{n-1}, s, r^2s, \ldots, r^{n-4}s, r^{n-3}s\},
B_{n-7} = \{r^{n-2}, r^{n-1}, s, r^2s, \ldots, r^{n-6}s, r^{n-5}s, r^{n-4}s, r^{n-3}s\},
\]

\[
\vdots
\]

\[
B_1 = \{r^{n-2}, r^{n-1}, s, r^2s, r^3s, \ldots, r^{n-3}s\},
\]

where \( B_{n-5} \subseteq B_{n-7} \subseteq \cdots \subseteq B_1 \). Therefore, we shall focus on proving that \( AB_{n-5} = D_{2n} \). Let \( L_1 = \{1, r, \ldots, r^{n-3}\} \subseteq A \) and \( L_2 = \{s, r^2s\} \subseteq B_{n-5} \). We compute \( L_1L_2 \) as follows:

\[
L_1L_2 = \{1, r, \ldots, r^{n-3}\} \{s, r^2s\}
\]

\[
= \{s, rs, \ldots, r^{n-3}s, r^2s, r^3s, \ldots, r^{n-1}s\} = \{r\} \subseteq AB_{n-5}.
\]

Therefore \( \{r\} \subseteq AB_{n-5} \). Now, we let \( L_3 = \{r^{n-2}, r^{n-1}\} \subseteq B_{n-5} \). Then, we compute \( L_1L_3 \) as follows:

\[
L_1L_3 = \{1, r, \ldots, r^{n-3}\} \{r^{n-2}, r^{n-1}\} = \{1, r, \ldots, r^{n-4}, r^{n-2}, r^{n-1}\}.
\]

Note that the product of \( r^{n-1}s \in A \) and \( r^2s \in B_{n-5} \) gives us \( (r^{n-1}s)(r^2s) = r^{n-3} \in AB_{n-5} \).

Therefore, \( \{r\} \subseteq AB_{n-5} \). Since \( AB_{n-5} = D_{2n} \), we see that \((A, B_{n-5})\) is a complete decomposition of \( D_{2n} \) of order 2. Since \( AB_{n-5} = D_{2n} \) and \( B_{n-5} \subseteq B_{n-7} \subseteq \cdots \subseteq B_1 \), it follows that \( AB_{n-5} = AB_{n-7} = \cdots = AB_1 = D_{2n} \). Hence, we conclude that \((A, B_j)\) is a complete decomposition of \( D_{2n} \) of order 2 for \( j \in \{1, 3, \ldots, n-5\} \).

### 3 Constructions of complete decompositions of \( D_{2n} \) of order \( k \) for \( k \in \{3, 4, \ldots, n\} \)

By using Matlab, we first generate all the possible complete decomposition of \( D_8 \) of order 3. We found that there are total of 368 constructions show that \((B_1, B_2, B_3)\) is a complete decomposition of \( D_8 \) of order 3. Since dihedral group is a non-abelian group, the product of the subsets might not be the same. Hence, we check the product of the subsets \( B_1B_2B_3 \) and \( B_2B_1B_3 \). We notice that there are 17 out of 368 constructions indicate that \( B_1B_2B_3 \neq B_2B_1B_3 \). To be more precise, Table 1 showed that there are 17 constructions where \((B_1, B_2, B_3)\) is a complete decomposition of \( D_8 \) of order 3 but \((B_2, B_1, B_3)\) is not a complete decomposition of \( D_8 \) of order 3.
Table 1. List of complete decompositions of $D_6$ of order 3, where $B_1, B_2, B_3 \subseteq D_6$ and $B_1B_2B_3 \neq B_2B_1B_3$.

<table>
<thead>
<tr>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, s}$</td>
<td>${r, rs}$</td>
<td>${r^2, r^3, r^2s, r^3s}$</td>
</tr>
<tr>
<td>${1, s}$</td>
<td>${r^3, r^3s}$</td>
<td>${r, r^2, rs, r^2s}$</td>
</tr>
<tr>
<td>${1, rs}$</td>
<td>${r, r^2s}$</td>
<td>${r^2, r^3, s, r^3s}$</td>
</tr>
<tr>
<td>${1, r^2s}$</td>
<td>${r^3, s}$</td>
<td>${r, r^2, r^2s, r^3s}$</td>
</tr>
<tr>
<td>${1, r^2s}$</td>
<td>${r, r^3s}$</td>
<td>${r^2, r^3, s, rs}$</td>
</tr>
<tr>
<td>${1, r^3s}$</td>
<td>${r^3, rs}$</td>
<td>${r, r^2s, r^3s}$</td>
</tr>
<tr>
<td>${1, r^3s}$</td>
<td>${r, s}$</td>
<td>${r^2, r^3, rs, r^2s}$</td>
</tr>
<tr>
<td>${r, r^2}$</td>
<td>${1, s}$</td>
<td>${r^3, rs, r^2s, r^3s}$</td>
</tr>
<tr>
<td>${r^2, rs}$</td>
<td>${r, s}$</td>
<td>${1, r^3, r^2s, r^3s}$</td>
</tr>
<tr>
<td>${r^2, r^2s}$</td>
<td>${r, rs}$</td>
<td>${1, r^3, s, r^3s}$</td>
</tr>
<tr>
<td>${r^2, r^3s}$</td>
<td>${r, r^2s}$</td>
<td>${1, r^3, s, rs}$</td>
</tr>
<tr>
<td>${r^2, s}$</td>
<td>${r, r^3s}$</td>
<td>${1, r^3s, rs, r^2s}$</td>
</tr>
<tr>
<td>${r^2, rs}$</td>
<td>${r^3, rs}$</td>
<td>${1, r, r^2s, r^3s}$</td>
</tr>
<tr>
<td>${r^2, r^2s}$</td>
<td>${r^3, r^2s}$</td>
<td>${1, r, s, r^3s}$</td>
</tr>
<tr>
<td>${r^2, r^3s}$</td>
<td>${r^3, s}$</td>
<td>${1, r, rs, r^3s}$</td>
</tr>
<tr>
<td>${r, r^2}$</td>
<td>${1, s}$</td>
<td>${r^3, rs, r^2s, r^3s}$</td>
</tr>
<tr>
<td>${r^{n-1}, r^{n-1}s}$</td>
<td>${s, rs, r^2s}$</td>
<td>${1, r^3, r^2s, r^3s}$</td>
</tr>
</tbody>
</table>

Therefore, we remark that if $(B_1, B_2, ..., B_t)$ is a complete decomposition of $D_{2n}$ of order $t$, it does not imply that $(B_{j_1}, B_{j_2}, ..., B_{j_t})$ is a complete decomposition of $D_{2n}$ of order $t$, where $\{B_{j_1}, B_{j_2}, ..., B_{j_t}\} = \{B_1, B_2, ..., B_t\}$.

Now, we provide a construction of complete decompositions of $D_{2n}$ of order 3 as follows.

**Proposition 5.** Let $n \geq 3$ and $A_1, A_2, A_3 \subseteq D_{2n}$, where $A_1 = \{1, r\}, A_2 = \{s, rs\}$ and $A_3 = \{r^2, r^3, ..., r^{n-1}, r^2s, r^3s, ..., r^{n-1}s\}$. Then $(A_1, A_2, A_3)$ is a complete decomposition of $D_{2n}$ of order 3.

**Proof.**

We first compute $A_1A_2$ as follows:

$A_1A_2 = \{1, r\}\{s, rs\}$

$= \{s, rs, r^2s\}$

and followed by

$A_1A_2A_3 = \{s, rs, r^2s\}\{r^2, r^3, ..., r^{n-1}, r^2s, r^3s, ..., r^{n-1}s\}$

$= \{r^{n-2}s, r^{n-3}s, ..., rs, r^{n-2}, r^{n-3}, ..., r, r^{n-1}s, r^{n-2}s, ..., r^2s, r^{n-1}, r^{n-2}, ..., r^2, r^n, r^{n-1}s, ..., r^3s, r^n, r^{n-1}, ..., r^3\}$

$= \{1, r, ..., r^{n-1}, s, rs, ..., r^{n-1}s\} = D_{2n}$.

Hence, $(A_1, A_2, A_3)$ is a complete decomposition of $D_{2n}$ of order 3.

Now, we shall focus on the construction, where $(A_1, A_2, ..., A_k)$ is a complete decomposition of $D_{2n}$ of order $k \in \{4, 5, ..., n\}$. The constructions of complete decompositions of $D_{2n}$ of order $k$ for $k \in \{4, 5, ..., n\}$ are as follows:

$A_1 = \{1, r\}$;

$A_2 = \{s, rs\}$;

$A_i = \{r^{i-1}, r^{i-1}s\}$ for $i \in \{3, 4, ..., k-1\}$;

$A_k = \{r^{k-1}, r^k, ..., r^{n-1}, r^{k-1}s, r^k s, ..., r^{n-1}s\}$. 

\[\text{ITM Web of Conferences 36, 03001 (2021)}\]
**Proposition 6.** Let $n \geq 7$ be integer. There exists a complete decomposition of $D_{2n}$ of order $k$ for $k \in \{4, 5, 6, 7\}$.

**Proof.**

Let

$$A_1 = \{1, r\};$$
$$A_2 = \{s, rs\};$$
$$A_i = \{r^{i-1}, r^{i-1}s\}, \text{ for } i \in \{3, 4, ..., k - 1\};$$
$$A_k = \{r^{k-1}, r^k, ..., r^{n-1}, r^{k-1}s, r^ks, ..., r^{n-1}s\}. $$

where $A_3, A_2, ..., A_k \subseteq D_{2n}$. We separate the proof into 4 cases for $k \in \{4, 5, 6, 7\}$ respectively. Note that $A_1 A_2 = \{1, r\} \{s, rs\} = \{s, rs, r^2s\}$. We compute $A_1 A_2 A_3$ as follows:

$$A_1 A_2 A_3 = \{s, rs, r^2s\} \{r^2, r^2s\} = \{1, r^{n-2}, r^{n-1}s, r^{n-2}s, r^{n-1}s\}. \quad (1)$$

(i) When $k = 4$, we have $A_4 = \{r^3, r^4, ..., r^{n-1}, r^3s, r^4s, ..., r^{n-1}s\}$. We compute $A_1 A_2 A_3 A_4$ as follows:

$$A_1 A_2 A_3 A_4 = \{1, r^{n-2}, r^{n-1}s, r^{n-2}s, r^{n-1}s\} \{r^3, r^4, ..., r^{n-1}, r^3s, r^4s, ..., r^{n-1}s\}
= \{1, r, ..., r^{n-1}, s, rs, ..., r^{n-1}s\} = D_{2n}.$$  
Hence, $(A_1, A_2, A_3, A_4)$ is a complete decomposition of $D_{2n}$ of order 4.

(ii) When $k = 5$, we have $A_4 = \{r^3, r^3s\}$. From Equation (1), we have $A_1 A_2 A_3 = \{1, r^{n-2}, r^{n-1}s, r^{n-2}s, r^{n-1}s\}$. Now, we compute $A_1 A_2 A_3 A_4 A_5$ as follows:

$$A_1 A_2 A_3 A_4 A_5 = \{1, r^{n-2}, r^{n-1}s, r^{n-2}s, r^{n-1}s\} \{r^3, r^3s\}
= \{r, r^2, r^3, r^{n-5}, r^{n-4}, r^{n-3}, rs, r^2s, r^3s, r^{n-5}s, r^{n-4}s, r^{n-3}s\}. \quad (2)$$

Observe that $A_5 = \{r^4, r^5, ..., r^{n-1}, r^4s, r^5s, ..., r^{n-1}s\}$. We compute $A_1 A_2 A_3 A_4 A_5$ as follows:

$$A_1 A_2 A_3 A_4 A_5
= \{r, r^2, r^3, r^{n-5}, r^{n-4}, r^{n-3}, rs, r^2s, r^3s, r^{n-5}s, r^{n-4}s, r^{n-3}s\} \{r^4, r^5, ..., r^{n-1}, r^4s, r^5s, ..., r^{n-1}s\}
= \{1, r, ..., r^{n-1}, s, rs, ..., r^{n-1}s\} = D_{2n}.$$  
Hence, $(A_1, A_2, A_3, A_4, A_5)$ is a complete decomposition of $D_{2n}$ of order 5.

(iii) When $k = 6$, we have $A_5 = \{r^4, r^4s\}$. From Equation (2), we have

$$A_1 A_2 A_3 A_4 = \{r, r^2, r^3, r^{n-5}, r^{n-4}, r^{n-3}, rs, r^2s, r^3s, r^{n-5}s, r^{n-4}s, r^{n-3}s\}.$$

Then, we compute $A_1 A_2 A_3 A_4 A_5$ as follows:

$$A_1 A_2 A_3 A_4 A_5
= \{r, r^2, r^3, r^{n-5}, r^{n-4}, r^{n-3}, rs, r^2s, r^3s, r^{n-5}s, r^{n-4}s, r^{n-3}s\} \{r^4, r^4s\}
= \{1, r^5, r^6, r^7, r^{n-9}, r^{n-8}, r^{n-7}, r^{n-3}, r^{n-2}, r^{n-1}, s, rs, r^5s, r^6s, r^7s, r^{n-9}s, r^{n-8}s, r^{n-7}s, r^{n-3}s, r^{n-2}s, r^{n-1}s\}. \quad (3)$$

Note that $A_6 = \{r^5, r^6, ..., r^{n-1}, r^5s, r^6s, ..., r^{n-1}s\}$. Next, we compute $A_1 A_2 ... A_6$ as follows:
\[
A_1 A_2 A_3 A_4 A_5 A_6 = \{1, r, r^5, r^6, r^7, r^{n-9}, r^{n-8}, r^{n-7}, r^{n-3}, r^{n-2}, r^{n-1}, s, rs, r^5s, r^6s, r^7s, r^{n-9}s, r^{n-8}s, r^{n-7}s, r^{n-3}s, r^{n-2}s, r^{n-1}s\}
\cup \{r^5, r^6, \ldots, r^{n-1}, r^5s, r^6s, \ldots, r^{n-1}s\} = \{1, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\} = D_{2n}.
\]

Hence, the result holds.

(iv) When \(k = 7\), we have \(A_6 = \{r^5, r^5s\}\). By using the similar way, it can be shown that \((A_1, A_2, \ldots, A_7)\) is a complete decomposition of \(D_{2n}\) of order 7.

The following lemma shall be used in later proof.

**Lemma 7.** Let \(n \geq 8\), \(k \in \{8, 9, \ldots, n\}\) and \(A_i = \{r^{i-1}, r^{i-1}s\} \subseteq D_{2n}\) for \(i \in \{3, 4, \ldots, k - 1\}\). Then \(A_3 A_4 \ldots A_{k-1} = \{r^{-c+4}, r^{-c+11}, r^{-c+13}, \ldots, r^{-c-9}, r^{-c-7}, \ldots, r^{-c-1}\} \cup \{r^{-c+5}s, r^{-c+11}s, r^{-c+13}s, \ldots, r^{-c-9}s, r^{-c-7}s, r^{-c-1}s\}\), where \(c = \frac{(k-2)(k-1)}{2}\).

**Proof.**

We first show that the base case \(k = 8\) is true. We compute \(A_3 A_4 A_5 A_6 A_7\) as follows:

\[
A_3 A_4 = \{r^2, r^2s\}\{r^3, r^3s\} = \{r^{-1}, r^5, r^{-1}s, r^5s\};
A_3 A_4 A_5 = \{r^{-1}, r^5, r^{-1}s, r^5s\}\{r^4, r^4s\} = \{r^{-5}, r^3, r^9, r^{-5}s, rs, r^3s, r^9s\};
A_3 A_4 A_5 A_6 = \{r^{-5}, r^3, r^9, r^{-5}s, rs, r^3s, r^9s\}\{r^5, r^5s\} = \{r^{-10}, r^{-4}, r^{-2}, 1, r^4, r^6, r^8, r^{14}, r^{-10}s, r^{-4}s, r^{-2}s, r^4s, r^6s, r^8s, r^{14}s\};
A_3 A_4 A_5 A_6 A_7 = \{r^{-10}, r^{-4}, r^{-2}, 1, r^4, r^6, r^8, r^{14}, r^{-10}s, r^{-4}s, r^{-2}s, r^4s, r^6s, r^8s, r^{14}s\}\{r^6, r^6s\} = \{r^{-16}, r^{-10}, r^{-8}, r^{-6}, r^{-4}, r^{-2}, 1, r^2, r^4, r^6, r^8, r^{10}, r^{12}, r^{14}, r^{20}\} \cup \{r^{-16}s, r^{-10}s, r^{-8}s, r^{-6}s, r^{-4}s, r^{-2}s, r^2s, r^4s, r^6s, r^8s, r^{10}s, r^{12}s, r^{14}s, r^{20}s\}.
\]

Clearly the statement is true when \(k = 8\). Thus the base case holds. Now, we assume that the statement is true when \(k = m\). By assumption, it is clear that

\[
A_3 A_4 \ldots A_{m-1} = \{r^{-\frac{(m-2)(m-1)}{2}}, r^{-\frac{(m-2)(m-1)}{2}+11}, r^{-\frac{(m-2)(m-1)}{2}-1}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+5}, r^{-\frac{(m-2)(m-1)}{2}+6}, r^{-\frac{(m-2)(m-1)}{2}+7}, r^{-\frac{(m-2)(m-1)}{2}+9}, r^{-\frac{(m-2)(m-1)}{2}+10}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+11}, r^{-\frac{(m-2)(m-1)}{2}+12}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+13}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+2n}\} \cup \{r^{-\frac{(m-2)(m-1)}{2}+1}, r^{-\frac{(m-2)(m-1)}{2}+2}, r^{-\frac{(m-2)(m-1)}{2}+3}, r^{-\frac{(m-2)(m-1)}{2}+4}, r^{-\frac{(m-2)(m-1)}{2}+5}, r^{-\frac{(m-2)(m-1)}{2}+6}, r^{-\frac{(m-2)(m-1)}{2}+7}, r^{-\frac{(m-2)(m-1)}{2}+8}, r^{-\frac{(m-2)(m-1)}{2}+9}, r^{-\frac{(m-2)(m-1)}{2}+10}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+11}, r^{-\frac{(m-2)(m-1)}{2}+12}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+14}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+2n}\} = B_1 \cup B_1 s,
\]

where

\[
B_1 = \{r^{-\frac{(m-2)(m-1)}{2}}, r^{-\frac{(m-2)(m-1)}{2}+1}, r^{-\frac{(m-2)(m-1)}{2}+2}, r^{-\frac{(m-2)(m-1)}{2}+3}, r^{-\frac{(m-2)(m-1)}{2}+4}, r^{-\frac{(m-2)(m-1)}{2}+5}, r^{-\frac{(m-2)(m-1)}{2}+6}, r^{-\frac{(m-2)(m-1)}{2}+7}, r^{-\frac{(m-2)(m-1)}{2}+8}, r^{-\frac{(m-2)(m-1)}{2}+9}, r^{-\frac{(m-2)(m-1)}{2}+10}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+11}, r^{-\frac{(m-2)(m-1)}{2}+12}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+14}, \ldots, r^{-\frac{(m-2)(m-1)}{2}+2n}\}.
\]

Next, we show that the statement is also true for \(k = m + 1\). Note that \(A_m = \{r^{m-1}, r^{m-1}s\}\) and \(A_3 A_4 \ldots A_m = B_1 A_m \cup B_1 s A_m\). We compute \(B_1 A_m\) and \(B_1 s A_m\) as follows:
Let $A_1 = \{1, r\}$, $A_2 = \{s, rs\}$, $A_i = \{r^{i-1}, r^{-i-1}s\}$ for $i \in \{3, 4, \ldots, k-1\}$ and $A_k = \{r^{k-1}, r^k, \ldots, r^{-n-1}, r^{k-1}s, r^ks, \ldots, r^{-n-1}s\}$ be the subsets of $D_{2n}$. We first compute $A_1 A_2 = \{1, r\} \{s, rs\} = \{s, rs, r^2s\}$.

Let $c = \frac{(k-2)(k-1)}{2}$. By Lemma 7, we have

$$A_3 A_4 \ldots A_{k-1} = \{r^{-c+5}, r^{-c+11}, r^{-c+13}, \ldots, r^{-c-9}, r^{-c-7}, r^{-c-1}\} \cup \{r^{-c+5}s, r^{-c+11}s, r^{-c+13}s, \ldots, r^{-c-9}s, r^{-c-7}s, r^{-c-1}s\} = L_1 \cup L_2,$$

where

$$L_1 = \{r^{-c+5}, r^{-c+11}, r^{-c+13}, \ldots, r^{-c-9}, r^{-c-7}, r^{-c-1}\}$$

and

$$L_2 = \{r^{-c+5}s, r^{-c+11}s, r^{-c+13}s, \ldots, r^{-c-9}s, r^{-c-7}s, r^{-c-1}s\}.$$

Now, we compute $A_1 A_2 L_1 \subseteq A_1 A_2 \ldots A_{k-1}$ and $A_1 A_2 L_2 \subseteq A_1 A_2 \ldots A_{k-1}$ as follows:
Now, we compute \( P_1 A_k, P_2 A_k, \ldots, P_6 A_k \subseteq A_1 A_2 \ldots A_k \) as follows:

\[
P_1 A_k = \{ r^{n-c+1} S, r^{n-c+2} S, r^{n-c+3} S \} \{ r^k - 1, r^k, \ldots, r^{n-1}, r^{k-1} S, r^k S, \ldots, r^n S \}
\]

\[
= \{ r^{n-c+2} S, r^{n-c+3} S, \ldots, r^{2n-c-k+4} S \} \cup \{ r^{n-c+2} S, r^{n-c+3} S, \ldots, r^{2n-c-k+4} S \}
\]

\[= R_1 \];

\[
P_2 A_k = \{ r^{n-c+7} S, r^{n-c+8} S, \ldots, r^{n+c+9} S \} \{ r^k - 1, r^k, \ldots, r^{n-1}, r^{k-1} S, r^k S, \ldots, r^n S \}
\]

\[
= \{ r^{n-c+6} S, r^{n-c+9} S, \ldots, r^{2n+c-k-8} S \} \cup \{ r^{n-c+6} S, r^{n-c+9} S, \ldots, r^{2n+c-k-8} S \}
\]

\[= R_2 \];

\[
P_3 A_k = \{ r^{n+c-5} S, r^{n+c-4} S, r^{n+c-3} S \} \{ r^k - 1, r^k, \ldots, r^{n-1}, r^{k-1} S, r^k S, \ldots, r^n S \}
\]

\[
= \{ r^{n+c-4} S, r^{n+c-3} S, \ldots, r^{2n+c-k-2} S \} \cup \{ r^{n+c-4} S, r^{n+c-3} S, \ldots, r^{2n+c-k-2} S \}
\]

\[= R_3 \];

\[
M_1 A_k = \{ r^{n-c+1} S, r^{n-c+2} S, r^{n-c+3} S \} \{ r^k - 1, r^k, \ldots, r^{n-1}, r^{k-1} S, r^k S, \ldots, r^n S \}
\]

\[
= \{ r^{n-c+k} S, r^{n-c+k+1} S, \ldots, r^{2n-c+2} S \} \cup \{ r^{n-c+k} S, r^{n-c+k+1} S, \ldots, r^{2n-c+2} S \}
\]

\[= R_4 \];

\[
M_2 A_k = \{ r^{n-c+7} S, r^{n-c+8} S, \ldots, r^{n+c-9} S \} \{ r^k - 1, r^k, \ldots, r^{n-1}, r^{k-1} S, r^k S, \ldots, r^n S \}
\]

\[
= \{ r^{n-c+k+6} S, r^{n-c+k+7} S, \ldots, r^{2n+c-10} S \} \cup \{ r^{n-c+k+6} S, r^{n-c+k+7} S, \ldots, r^{2n+c-10} S \}
\]

\[= R_5 \];

\[
M_3 A_k = \{ r^{n+c-5} S, r^{n+c-4} S, r^{n+c-3} S \} \{ r^k - 1, r^k, \ldots, r^{n-1}, r^{k-1} S, r^k S, \ldots, r^n S \}
\]

\[
= \{ r^{n+c-k-6} S, r^{n+c-k-5} S, \ldots, r^{2n+c-4} S \} \cup \{ r^{n+c-k-6} S, r^{n+c-k-5} S, \ldots, r^{2n+c-4} S \}
\]

\[= R_6 \].

Note that \( A_1 A_2 \ldots A_k = R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup R_6 \). Observe that

\[
R_2 = \{ r^{n-c+4} S, r^{n-c+9} S, \ldots, r^{2n+c-k+8} S \} \cup \{ r^{n-c+4} S, r^{n-c+9} S, \ldots, r^{2n+c-k+8} S \},
\]
where
\[ |\{r^{n-c+8}, r^{n-c+9}, ..., r^{2n+c-k-8}\}| = |\{r^{n-c+8}s, r^{n-c+9}s, ..., r^{2n+c-k-8}s\}| \]
\[ = n + 2c - k - 15 \]
\[ = n + k^2 - 2k - 15 \geq n \]
for \( k \in \{8, 9, ..., n\} \). Therefore, we have \( R_2 = Q_{2n} \). Hence, there exists a complete decomposition of \( D_{2n} \) of order \( k \) for \( k \in \{8, 9, ..., n\} \).

The research has been carried out under Fundamental Research Grant Scheme project FRGS/1/2017/STG06/UTAR/02/3 provided by Ministry of Higher Education of Malaysia.

References