Rank 2 preservers on symmetric matrices with zero trace

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Abstract. Let $F$ be a field, $V_1$ and $V_2$ be vector spaces of matrices over $F$ and let $\rho$ be the rank function. If $T: V_1 \rightarrow V_2$ is a linear map, and $k$ a fixed positive integer, we say that $T$ is a rank $k$ preserver if for any matrix $A \in V_1$, $\rho(A) = k$ implies $\rho(T(A)) = k$. In this paper, we characterize those rank 2 preservers on symmetric matrices with zero trace under certain conditions.

1 Introduction

Let $F^{\text{aza}}$ be the algebra of all $n \times n$ matrices over a field $F$. Let $sl_n(F)$ denote the subspace of $F^{\text{aza}}$ consisting of all matrices with zero trace. In [1], Botta, Pierce and Watkins obtained a useful result concerning the structure of nonsingular linear mapping on $sl_n(F)$ that preserve nilpotent matrices where $F$ is infinite. In [2], Li and Pierce characterized linear mappings on $sl_n(F)$ that preserve nonzero nilpotent matrices with rank at most $k$ where $k$ is a fixed positive integer less than $n$ and $F$ is algebraically closed of characteristic zero. Then, Watkins characterized linear mappings from $sl_n(F)$ to $F^{\text{aza}}$ that preserve rank one matrices where $F$ is an algebraically closed field of characteristic not equal to 2. He applied this result to determine the structure of bilinear mappings on $F^{\text{aza}}$ that have certain rank-preserving properties in [3] and [4] respectively.

Let $S_n(F)$ be the vector space of all $n \times n$ symmetric matrices over $F$ and $Z_0(S_n(F))$ be its subspace consisting of all symmetric matrices with zero trace. Let $n \geq 4$ and $F$ be a field of characteristic greater than 3. Motivated by work of Lim [5] in the characterization of linear rank one preservers on matrices with zero trace, we characterize those rank 2 preservers on symmetric matrices with zero trace under certain conditions in this paper and will discuss some consequences of this characterization in our next paper.

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2 Some definitions and preliminary results

Let $U$ be a vector space over $F$. We use tensor language in our investigation. This provides us with a larger context. We denote by $U^{(2)}$ the second symmetric product space over $U$ and denoted by $x \cdot y$, $x, y \in U$, the decomposable elements of $U^{(2)}$. For each $u$ in $U$, let $u^2$ denote $u \cdot u$.

A scalar product on $U$ is a function which assigns a scalar $(x, y) \in F$ to each ordered pair of vectors $x, y \in U$ such that for any $x, y, z \in U$ and any $c \in F$

(i) $(x + y, z) = (x, z) + (y, z)$
(ii) $(cx, y) = c(x, y)$
(iii) $(x, y) = (y, x)$

We say $x$ is orthogonal to $y$ or $x$ and $y$ are orthogonal if $(x, y) = 0$. Let $S$ be a set of vectors in $U$. Then $S$ is called an orthogonal set if $(x, y) = 0$ for all $x, y \in S, x \neq y$. If in addition, $(x, x) = 1$ for every $x \in S$, then $S$ is called an orthonormal set.

Now we let $U$ be equipped with a scalar product $(,):U \times U \rightarrow F$ and $U$ has an orthonormal basis $\mathbf{E}$. Let $Z_0(U^{(2)})$ be the subset of $U^{(2)}$ that consists of all vectors of the form

$$\sum_{1 \leq i \leq j \leq n} a_{ij} u_i \cdot u_j$$

where $\{u_1, ..., u_n\}$ is an arbitrary finite subset of $\mathbf{E}$ and $a_{ij}$ $(1 \leq i \leq j \leq n)$ are arbitrary scalars in $F$ such that \(\sum_{i=1}^{n} a_{ii} = 0\). Clearly $Z_0(U^{(2)})$ is a subspace of $U^{(2)}$ and we call $Z_0(U^{(2)})$ the space of traceless 2nd order symmetric tensors over $U$.

**Proposition 2.1** If $\{e_i : i \in A\}$ where $A \supseteq \{1, 2\}$ is an orthonormal basis for $U$, then $B = \{e_i \cdot e_j, e_i^2 - e_j^2 : i \neq j, k \neq 1 \text{ and } 1, i, j, k \in A\}$ is a basis for $Z_0(U^{(2)})$.

**Proof.**
Clearly $B$ is a linearly independent set. Hence it is sufficient to show that $B$ spans $Z_0(U^{(2)})$.

Let $x \in Z_0(U^{(2)})$. Then $x = \sum_{1 \leq i \leq j \leq n} a_{ij} u_i \cdot u_j$ and $\sum_{i=1}^{n} a_{ii} = 0$ where $\{u_1, ..., u_n\}$ is a finite subset of $\{e_i : i \in A\}$ and $a_{ij}$ $(1 \leq i \leq j \leq n)$ are scalars in $F$. It follows that

$$x = \sum_{1 \leq i \leq j \leq n} a_{ij} u_i \cdot u_j - \sum_{k=1}^{n} a_{kk} (e_i^2 - u_k^2).$$

Therefore, $B$ spans $Z_0(U^{(2)})$. $\square$

Let $Z_0(S_n(F))$ denote the subspace of $S_n(F)$ such that for any $A \in Z_0(S_n(F))$, $\text{tr}(A) = 0$. If $U$ is a finite dimensional vector space with an orthonormal basis $\{e_i : i = 1, ..., n\}$, then $Z_0(U^{(2)})$ is isomorphic in a natural way to $Z_0(S_n(F))$ by the restricted...
isomorphism $\varphi|_{Z_0(U^{(2)})}$ where $\varphi$ is the isomorphism from $U^{(2)}$ to $S_n(F)$ defined by

$$\varphi(e_i \cdot e_j) = E_{ij} + E_{ji}, 1 \leq i \leq j \leq n.$$  

**Remark.** If $U$ is a Euclidean space, then there does not exist any rank 1 vector in $Z_0(U^{(2)})$.

Let $J_k$ denote the set of vectors in $U^{(2)}$ of the form $\sum_{i=1}^k \lambda_i x_i^2$, where $x_1, \ldots, x_k$ are linearly independent vectors and $\lambda_1, \ldots, \lambda_k \in F \setminus \{0\}$. For each vector $u \in U$, let $u \cdot U = \{u \cdot v : v \in U\}$.

**Lemma 2.2** Let $M$ be a subspace of $U^{(2)}$ such that $M \subseteq \{0\} \cup J_1 \cup J_2$. Then either

(i) $M \subseteq W^{(2)}$ for some subspace $W$ of $U$ that is 2 dimension or

(ii) $M \subseteq u \cdot U$ for some $u \in U \setminus \{0\}$.

**Proof.** If $M \cap J_2 = \emptyset$, then clearly $M = \{x^2\}$ for some $x$ in $U$. Let $A_i = au_i^2 + u_2^2 \in J_2 \cap M$.

Assume that $M \not\subseteq V^{(2)}$ where $V = \langle u_1, u_2 \rangle$. Then it is clear that there exists $A_2 = bu_2^2 + v^2 \in J_2 \cap M$ where $u_1, u_2, u_3$ are linearly independent. Clearly $v \in \langle u_1, u_2, u_3 \rangle$, otherwise $A_1 + A_2 \in J_1$, a contradiction. Let $v = cu_1 + du_2 + eu_3$ where $c,d,e \in F$.

Since for any $\lambda \in F$,

$$\lambda A_1 + A_2 = (\lambda a + c^2)u_1^2 + (\lambda + d^2)u_2^2 + (b + e^2)u_3^2 + 2cdu_1 \cdot u_2 + 2ceu_1 \cdot u_3 + 2deu_2 \cdot u_3 \in J_1 \cup J_2,$$

it follows that

$$\begin{vmatrix}
\lambda a + c^2 & cd & ce \\
\lambda & \lambda + d^2 & de \\
ce & de & b + e^2
\end{vmatrix}
= a(b + e^2)\lambda^2 + b(ad^2 + c^2)\lambda = 0$$

for any $\lambda \in F$. Since $|F| \geq 3$, we have

$$b + e^2 = 0 \quad (1)$$

and

$$ad^2 + c^2 = 0. \quad (2)$$

From (1) we have

$$A_2 = (cu_1 + du_2) \cdot (cu_1 + du_2 + 2eu_3). \quad (3)$$

From (2) we have $ad^{-1} + d^{-1}c = 0$ and hence

$$A_1 = (cu_1 + du_2) \cdot (ac^{-1}u_1 + d^{-1}u_2). \quad (4)$$

From (3) and (4) we have $A_1 = u \cdot y_1$, $A_2 = u \cdot y_2$ where $u, y_1, y_2$ are linearly independent.

Suppose now $A = x^2 \in J_1 \cap M$. Then $A + A_1, A + A_2 \in J_1 \cap J_2$ imply that $x \in \langle u, y_1 \rangle \cap \langle u, y_2 \rangle = \langle u \rangle$ and hence $A \in u \cdot U$.

Suppose that $B = \lambda_1 y_1^2 + \lambda_2 y_2^2 \in J_2 \cap M$. For each $\lambda \in F$, let $C_\lambda = u \cdot z_\lambda$, where $z_\lambda = y_1 + \lambda y_2$. Then $C_\lambda \in M$. Clearly there exists a subset $D$ of $F$ with 2 elements such
that $\langle v_i, v_j \rangle \neq \langle u, z_\lambda \rangle$ for all $\lambda \in D$. Hence $\dim \langle v_i, v_j, u, z_\lambda \rangle \geq 3$ for all $\lambda \in D$. By our previous argument, for any $\lambda \in D$, there exists $w_\lambda \in U$ such that $B, C_\lambda \in w_\lambda \cdot U$.

Suppose $B \neq u \cdot U$. Then $B = \alpha z_i \cdot z_j$ for some $\alpha \in F \setminus \{0\}$ where $i$ and $j$ are distinct element in $D$. Let $H = u \cdot z_k$ where $k \in F \setminus \{i, j\}$. Since $\dim \langle u, z_i, z_j, z_k \rangle = 3$,

$B, H \in v \cdot U$ for some $v \in U$ by previous argument. This yields a contradiction since $B = \alpha z_i \cdot z_j$ and $H = u \cdot z_k$ do not have a common factor. Therefore $B = u \cdot U$.

3 Rank 2 preservers

Let $U$ and $W$ be vector spaces over $F$. We always assume that $U$ has an orthonormal basis, $\{e_i : i \in A\}$, with respect to a scalar product $\langle , \rangle : U \times U \to F$, where $A \supseteq \{1, 2, \ldots, n\}$ if $A$ has at least $n$ elements. For each vector $u \in U$, let $\langle u \rangle^\perp = \{v \in U : \langle v, u \rangle = 0\}$.

**Lemma 3.1** Let $T : Z_0(U^{(2)}) \to W^{(2)}$ be a rank 2 preserver. If $V$ is a subspace of $U$ such that $V \subseteq \langle u \rangle^\perp$ for some $u \in U \setminus V$, then $\dim T(u \cdot V) = \dim u \cdot V$. Moreover, if $\dim V \geq 3$, then $T(u \cdot V) \subseteq w \cdot W$ for some $w \in W \setminus \{0\}$.

**Proof.**
Suppose $T(u \cdot v_1) = T(u \cdot v_2)$ for some $v_1, v_2 \in V$. Then $T(u \cdot (v_1 - v_2)) = 0$ and this implies that $v_1 = v_2$, since $T$ is a rank 2 preserver. Hence $\dim T(u \cdot V) = \dim u \cdot V$. If $\dim V \geq 3$, then $\dim(T(u \cdot V)) \geq 3$. Since $T(u \cdot V)$ is a subspace of $W^{(2)}$ contained in $J_2 \cup \{0\}$, it follows by Lemma 2.2 that $T(u \cdot V) \subseteq w \cdot W$ for some $w \in W \setminus \{0\}$. □

**Theorem 3.2** Let $T$ be a rank 2 preserver from $Z_0(U^{(2)})$ to $W^{(2)}$. If $\dim U \geq 4$, then one of the following holds:

(i) $T = \lambda P_2(x)\big|_{Z_0(U^{(2)})}$ for some $\lambda \in F \setminus \{0\}$ and some one-to-one linear mapping $f : U \to W$ where $P_2(x)$ is a second induced power of $f$ such that $P_2(x)(x \cdot y) = f(x) \cdot f(y)$;

(ii) $\text{Im} T \subseteq w \cdot W$ for some $w \in W \setminus \{0\}$. 


Proof.
Note that \( \langle e_i \rangle^\perp \) is a subspace of \( U \), \( \dim \langle e_i \rangle^\perp \geq 3 \) and \( e_i \notin \langle e_i \rangle^\perp \). In view of Lemma 3.1, \( T(\langle e_i \rangle^\perp) \subseteq w_i \cdot W \) for some \( w_i \in W \setminus \{0\}, i \in A \). Now we have either \( \{w_i : i \in A\} \) is a pairwise linearly independent set or \( \langle w_j \rangle = \langle w_j \rangle \) for some distinct \( i, j \). We will consider these two cases separately.

**Case 1:** \( \{w_i : i \in A\} \) is a pairwise linearly independent set.

Since \( T(e_1 \cdot e_2) \in w_1 \cdot W \), \( T(e_2 \cdot e_1) \in w_2 \cdot W \) and \( w_1, w_2 \) are linearly independent, we have \( T(e_1 \cdot e_2) = \alpha_{12} w_1 \cdot w_2 \) for some \( \alpha_{12} \in F \setminus \{0\} \). Likewise,
\[
T(e_j \cdot e_j) = \alpha_{jj} w_i \cdot w_j ,
\]
where \( \alpha_{jj} \in F \setminus \{0\} \) for all distinct \( i, j \). Clearly
\[
\alpha_{ij} = \alpha_{ji} .
\]

Now we claim that \( \{w_i : i \in A\} \) is a linearly independent set. Suppose the contrary. Let \( w_i = \sum_{i \in A \setminus \{1\}} a_i w_i \) for some \( a_i \in F \). Then from (5), we have
\[
T\left( e_2 \cdot \left( \frac{1}{\alpha_{12}} e_1 - \sum_{i \in A \setminus \{1,2\}} \frac{a_i}{\alpha_{12}} e_i \right) \right) = w_2 \cdot \left( w_i - \sum_{i \in A \setminus \{1,2\}} a_i w_i \right)
\]
is of rank \( \leq 1 \), a contradiction. So, \( \{w_i : i \in A\} \) is a linearly independent set. Let
\[
M = \langle (e_1 + e_2) \cdot (e_1 - e_2), (e_1 + e_2) \cdot e_3, (e_1 + e_2) \cdot e_4, \rangle.
\]
Since \( \langle e_1 + e_2, e_3, e_4 \rangle \) is a 3-dimensional subspace of \( \langle e_1 + e_2 \rangle^\perp \) and \( e_1 + e_2 \notin \langle e_1 - e_2, e_3, e_4 \rangle \), it follows by Lemma 3.1 that \( \dim M = 3 \) and \( T(M) \subseteq u \cdot W \) for some \( u \in W \setminus \{0\} \).

Let
\[
T((e_1 + e_2) \cdot (e_1 - e_2)) = u \cdot u_1 ,
\]
\[
T((e_1 + e_2) \cdot e_3) = u \cdot u_2 ,
\]
\[
T((e_1 + e_2) \cdot e_4) = u \cdot u_3 ,
\]
where \( \dim \langle u_1, u_2, u_3 \rangle = 3 \). In view of (6), we have
\[
T(e_1 + e_2) = T(e_1 \cdot e_2) + T(e_1 \cdot e_3)
\]
\[
= (\alpha_{13} w_1 + \alpha_{23} w_2) \cdot w_2 ,
\]
\[
T(e_1 + e_2) = T(e_1 \cdot e_4) + T(e_1 \cdot e_4)
\]
\[
= (\alpha_{14} w_1 + \alpha_{24} w_2) \cdot w_2 .
\]
Hence
\[
u \cdot u_2 = (\alpha_{13} w_1 + \alpha_{23} w_2) \cdot w_3 ,
\]
\[
u \cdot u_3 = (\alpha_{14} w_1 + \alpha_{24} w_2) \cdot w_4 .
\]
Since \( \dim \langle w_1, w_2, w_3, w_4 \rangle = 4 \), we have \( \langle u \rangle = \langle \alpha_{13} w_1 + \alpha_{23} w_2, \rangle = \langle \alpha_{14} w_1 + \alpha_{24} w_2 \rangle \). Hence
\[
\frac{\alpha_{23}}{\alpha_{13}} = \frac{\alpha_{24}}{\alpha_{14}} .
\]
Likewise
\[
\frac{\alpha_{ik}}{\alpha_{jk}} = \frac{\alpha_{il}}{\alpha_{jl}} \tag{7}
\]

for all distinct \(i, j, k, l\). Now let
\[
N = \langle (e_1 - e_2) \cdot (e_1 + e_2), (e_1 - e_2) \cdot e_3, (e_1 - e_2) \cdot e_4 \rangle.
\]

Since \(\langle e_1 + e_2, e_3, e_4 \rangle\) is a 3-dimensional subspace of \(\langle e_1 - e_2 \rangle^\perp\) and \(e_1 - e_2 \notin \langle e_1 + e_2, e_3, e_4 \rangle\), it follows by Lemma 3.1 that \(\dim T(N) = 3\) and \(T(N) \subseteq v \cdot W, \; v \in W \setminus \{0\}\). Let
\[
T((e_1 - e_2) \cdot (e_1 + e_2)) = v \cdot v_1,
\]
\[
T((e_1 - e_2) \cdot e_3) = v \cdot v_2,
\]
\[
T((e_1 - e_2) \cdot e_4) = v \cdot v_3.
\]

where \(\dim \langle v_1, v_2, v_3 \rangle = 3\). On the other hand, in view of (5)
\[
T((e_1 - e_2) \cdot e_3) = T(e_1 \cdot e_3) - T(e_2 \cdot e_3) = (\alpha_{13} w_1 - \alpha_{23} w_2) \cdot w_3,
\]
\[
T((e_1 - e_2) \cdot e_4) = T(e_1 \cdot e_4) - T(e_2 \cdot e_4) = (\alpha_{14} w_1 - \alpha_{24} w_2) \cdot w_4.
\]

Hence
\[
v \cdot v_2 = (\alpha_{13} w_1 - \alpha_{23} w_2) \cdot w_3,
\]
\[
v \cdot v_3 = (\alpha_{14} w_1 - \alpha_{24} w_2) \cdot w_4.
\]

Since \(\dim \langle w_1, w_2, w_3, w_4 \rangle = 4\), we have
\[
\langle v \rangle = \langle \alpha_{13} w_1 - \alpha_{23} w_3 \rangle = \langle \alpha_{14} w_1 - \alpha_{24} w_3 \rangle.
\]

Now we have
\[
T((e_1 + e_2) \cdot (e_1 - e_2)) \in (\alpha_{13} w_1 + \alpha_{23} w_2) \cdot W
\]
and
\[
T((e_1 - e_2) \cdot (e_1 + e_2)) \in (\alpha_{13} w_1 - \alpha_{23} w_2) \cdot W.
\]

Since \(\alpha_{13} w_1 + \alpha_{23} w_2\) and \(\alpha_{13} w_1 - \alpha_{23} w_2\) are linearly independent, we have
\[
T(e_1^2 - e_2^2) = T((e_1 + e_2) \cdot (e_1 - e_2)) = \lambda_{12}^{(3)} (\alpha_{13} w_1 + \alpha_{23} w_2) \cdot (\alpha_{13} w_1 - \alpha_{23} w_2) = \lambda_{12}^{(3)} (\alpha_{13}^2 w_1^2 - \alpha_{23}^2 w_2^2)
\]
where \(\lambda_{12}^{(3)} \in F \setminus \{0\}\). Likewise
\[
T(e_1^2 - e_2^2) = \lambda_{ij}^{(k)} (\alpha_{ik}^2 w_i^2 - \alpha_{jk}^2 w_j^2) \tag{8}
\]
where \(\lambda_{ij}^{(k)} \in F \setminus \{0\}\) for all distinct \(i, j, k\). As a consequence
\[
\lambda_{ij}^{(k)} \alpha_{ik}^2 = \lambda_{ij}^{(l)} \alpha_{il}^2 \tag{9}
\]
\[
\lambda_{ij}^{(k)} = \lambda_{ji}^{(k)} \tag{10}
\]
for all distinct \(i, j, k, l\). Now let
\[
u_1 = e_1 + e_2 + e_3,
\]
\[
u_2 = e_1 - e_2,
\]
\[
u_3 = e_1 - e_3,
\]
\[
u_4 = e_4.
\]
Let $H = \langle u_1 \cdot u_2, u_1 \cdot u_3, u_1 \cdot u_4 \rangle$. Since $\langle u_2, u_3, u_4 \rangle$ is a 3-dimensional subspace of $\langle u_1 \rangle^\perp$ and $u_1 \notin \langle u_2, u_3, u_4 \rangle$, it follows by Lemma 3.1 that $\dim(T(H)) = 3$ and $T(H) \subseteq v \cdot W$ for some $v \in W \setminus \{0\}$. In view of (5), (7) and (8), we have

\[ T(u_1 \cdot u_2) = T(e_1^2 - e_2^2 + e_1 \cdot e_3 - e_2 \cdot e_3) = \lambda_{12}^{(3)} (\alpha_{13}^2 w_1^2 - \alpha_{23}^2 w_2^2) + \alpha_{13} w_1 \cdot w_3 - \alpha_{23} w_2 \cdot w_3. \]

\[ T(u_1 \cdot u_3) = T(e_1^2 - e_3^2 + e_1 \cdot e_2 - e_3 \cdot e_2) = \lambda_{13}^{(3)} (\alpha_{13}^2 w_1^2 - \alpha_{23}^2 w_3^2) + \alpha_{13} w_1 \cdot w_2 - \alpha_{23} w_3 \cdot w_2. \]

Since $(\alpha_{13} w_1 - \alpha_{23} w_2), (\alpha_{13} w_1 - \alpha_{23} w_3)$ and $(\lambda_{12}^{(3)} \alpha_{13} w_1 + \lambda_{13}^{(3)} \alpha_{23} w_2 + w_3)$ are pairwise linearly independent and $T(u_1 \cdot u_2), T(u_1 \cdot u_3)$ have a common factor, we have

\[ \left\langle \lambda_{12}^{(3)} \alpha_{13} w_1 + \lambda_{13}^{(3)} \alpha_{23} w_2 + w_3 \right\rangle = \left\langle \lambda_{13}^{(3)} \alpha_{13} w_1 + w_2 + \lambda_{13}^{(3)} \alpha_{23} w_3 \right\rangle. \]

Hence

\[ \frac{\lambda_{12}^{(3)} \alpha_{13}}{\lambda_{13}^{(3)} \alpha_{23}} = \frac{1}{\lambda_{13}^{(3)} \alpha_{23}}. \]

In view of (6), we have

\[ \frac{1}{\lambda_{12}^{(3)}} = \lambda_{13}^{(2)} \alpha_{23} \alpha_{32} = \lambda_{13}^{(2)} \alpha_{32}^2, \]

\[ \frac{1}{\lambda_{13}^{(3)}} = \frac{\alpha_{13}}{\alpha_{12} \alpha_{23}}. \]

Likewise,

\[ \frac{1}{\lambda_{ij}^{(k)}} = \lambda_{ik}^{(j)} \alpha_{ij}^2 \]

\[ \lambda_{ij}^{(k)} = \frac{\alpha_{ij}}{\alpha_{ik} \alpha_{jk}} \]

for all distinct $i, j, k$. In view of (12) and (7), we have

\[ \lambda_{ij}^{(k)} = \frac{1}{\alpha_{ik}} \left( \frac{\alpha_{ij}}{\alpha_{jk}} \right) = \frac{1}{\alpha_{ik}} \left( \frac{\alpha_{ji}}{\alpha_{jk}} \right) = \lambda_{ij}^{(k)} \]

for all distinct $i, j, k, l$. In view of (10) and (13), we find that

\[ \lambda_{ij}^{(k)} \]

have a common value $\lambda^{(k)}$ for all $i, j \notin \{k\}$. Now we define a linear mapping $f: U \rightarrow W$ by

\[ f(e_i) = \begin{cases} \frac{1}{\lambda} w_3, & i = 3 \\ \alpha_{13} w_i, & i \neq 3 \end{cases} \]
where \( \lambda = \lambda^{(3)} \). For any distinct elements \( i, j, 3 \) in \( A \), in view of (5), (8), (9), (11), (12) and (14), we have

\[
\lambda P_2(f)(e_i \cdot e_j) \\
= \lambda^{(3)} f(e_i) \cdot f(e_j) \\
= \lambda^{(3)} \alpha_i \alpha_j w_i \cdot w_j \\
= \alpha_i w_i \cdot w_j \\
= T(e_i \cdot e_j),
\]

\[
\lambda P_2(f)(e_i \cdot e_3) \\
= \lambda f(e_i) \cdot f(e_3) \\
= \lambda (\alpha_i w_i) \cdot \left( \frac{1}{\lambda^{(3)}} w_3 \right) \\
= \alpha_i w_i \cdot w_3 \\
= T(e_i \cdot e_3),
\]

\[
\lambda P_2(f)(e_i^2 - e_i^3) \\
= \lambda^{(3)} f(e_i)^2 - \lambda^{(3)} f(e_i)^2 \\
= \lambda^{(3)} \alpha_i^2 w_i^2 - \lambda^{(3)} \left( \frac{1}{\lambda^{(3)}} \right)^2 w_i^2 \\
= \lambda^{(k)} \alpha_i^2 w_i^2 - \frac{1}{\lambda^{(3)_k}} w_i^2 \quad \text{where } k \neq \{i, j, 3\}
\]

\[
= \lambda^{(k)} \alpha_i^2 w_i^2 - \alpha_{i_3}^2 w_3^2 \\
= \lambda^{(k)} (\alpha_i^2 w_i^2 - \alpha_{i_3}^2 w_3^2) \\
= T(e_i^2 - e_i^3).
\]

Therefore \( T = \lambda P_2(f)|_{Z_\alpha(w)_{(v)}} \) and \( f \) is injective.

**Case 2:** \( \langle w_i \rangle = \langle w_j \rangle \) for some distinct \( i, j \).

Let \( w = w_i \). Without loss of generality, we may assume that \( w_i = w_2 \). We first show that

\[
T(e_i \cdot \langle w_i \rangle^\perp) \subseteq w \cdot W
\]

for all \( i \in A \).

If \( w_3 \) and \( w \) are linearly independent, then

\[
T(e_i \cdot e_j) = \alpha w \cdot w_3
\]

for some \( \alpha \in F \setminus \{0\} \). Similarly, \( T(e_i \cdot e_3) = \beta w \cdot w_3 \) for some \( \beta \in F \setminus \{0\} \). Hence

\[
T(e_i \cdot (\beta e_i - \alpha e_3)) = 0.
\]

This contradicts the hypothesis that \( \rho(T(e_i \cdot (\beta e_i - \alpha e_3))) = 2 \). Thus \( \langle w_3 \rangle = \langle w \rangle \). Likewise, \( \langle w_k \rangle = \langle w \rangle \), for all \( k \in A \). Therefore, we can conclude that
for all \(i \in A\).

Now we show that

\[ T\left( e_i \cdot \{e_i\}^\perp \right) \subseteq w \cdot W \]

for all distinct \(i, j\).

Let

\[
T(e_i \cdot e_i) = w \cdot z_i, \\
T(e_i \cdot e_j) = w \cdot z_j, \\
T(e_i \cdot e_k) = w \cdot z_k, \\
T(e_i \cdot e_l) = w \cdot z_l,
\]

where \(z_1, z_2, z_3, z_4 \in W\). Let

\[
M = \left\langle (e_1 + e_2) \cdot (e_1 - e_2), (e_1 + e_2) \cdot e_3, (e_1 + e_2) \cdot e_4 \right\rangle.
\]

Then \(\langle e_1 - e_2, e_1, e_4 \rangle\) is a three dimensional subspace of \(\langle e_1 + e_2 \rangle^\perp\) and in view of Lemma 3.1, \(\dim T(M) = 3\) and \(T(M) \subseteq v \cdot W\) for some \(v \in W\). Let

\[
T((e_i + e_2) \cdot (e_1 - e_2)) = v \cdot v_1, \\
T((e_1 + e_2) \cdot e_3) = v \cdot v_2, \\
T((e_1 + e_2) \cdot e_4) = v \cdot v_3,
\]

where \(\dim \langle v_1, v_2, v_3 \rangle = 3\). On the other hand,

\[
T((e_1 + e_2) \cdot e_3) = T(e_i \cdot e_3) + T(e_2 \cdot e_3) \\
= (z_i + z_2) \cdot w.
\]

\[
T((e_1 + e_2) \cdot e_4) = T(e_i \cdot e_4) + T(e_2 \cdot e_4) \\
= (z_i + z_4) \cdot w.
\]

Since \(\dim T(M) = 3\), \(z_i + z_2\) and \(z_i + z_4\) are linearly independent. Then by comparing the two expressions for \(T((e_1 + e_2) \cdot e_3)\) and \(T((e_1 + e_2) \cdot e_4)\), clearly \(\langle w \rangle = \langle v \rangle\) and we obtain

\[
T(e_i^2 - e_j^2) \subseteq w \cdot W.
\]

Likewise,

\[
T(e_i^2 - e_j^2) \subseteq w \cdot W
\]

for all distinct \(i, j\). Now we have proved that

\[
T\left( e_i \cdot \{e_i\}^\perp \right) \subseteq w \cdot W
\]

for all \(i \in A\), and

\[
T(e_i^2 - e_j^2) \subseteq w \cdot W
\]

for all distinct \(i, j\). Therefore \(\text{Im} T \subseteq w \cdot W\). \(\square\)

**Remark.** We conjecture that Theorem 3.2 is also true when \(\dim U = 3\). If \(\dim U = 2\), Theorem 3.2 is no longer true.
**Example** Let $U$ be a 2-dimensional Euclidean space with an orthonormal basis $\{e_1, e_2\}$.

Let $T : Z_0\left(U^{(2)}\right) \to U^{(2)}$ be a linear mapping such that $T(e_1^2 - e_2^2) = e_1 \cdot e_2$ and $T(e_1 \cdot e_2) = e_1^2 - e_2^2$. Then clearly $T$ is a rank 2 preserver. However $T$ is neither of the form (i) nor the form (ii) in Theorem 3.2.

Now we state Theorem 3.2 in matrix language when the dimension of $U$ and $W$ are finite.

**Corollary 3.3** Let $n$ and $m$ be positive integers and let $L$ be a rank 2 preserver from $Z_0(S_n(F))$ to $S_m(F)$. If $n \geq 4$, then one of the following holds:

(i) There exists a rank $n \times n$ matrix $P$ such that $L(A) = \lambda P A P^t$ for all $A \in Z_0(S_n(F))$ where $\lambda \in F \setminus \{0\}$;

(ii) There exists a nonsingular $m$-square matrix $Q$ such that

$$\text{Im } L \subseteq \begin{Bmatrix} Q \begin{pmatrix} c_1 & c_2 & \cdots & c_m \\ c_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_m & 0 & \cdots & 0 \end{pmatrix} Q^t : c_j \in F, j = 1, 2, \ldots, m \end{Bmatrix}.$$

The author would like to thank the referees for valuable comments.

**References**