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Rank 2 preservers on symmetric matrices with zero trace

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Abstract. Let F be a field, V_1 and V_2 be vector spaces of matrices over F and let ρ be the rank function. If $T: V_1 \to V_2$ is a linear map, and k a fixed positive integer, we say that T is a rank k preserver if for any matrix $A \in V_1$, $\rho(A) = k$ implies $\rho(T(A)) = k$. In this paper, we characterize those rank 2 preservers on symmetric matrices with zero trace under certain conditions.

1 Introduction

Let $F^{n\times n}$ be the algebra of all $n\times n$ matrices over a field F. Let $sl_n(F)$ denote the subspace of $F^{n\times n}$ consisting of all matrices with zero trace. In [1], Botta, Pierce and Watkins obtained a useful result concerning the structure of nonsingular linear mapping on $sl_n(F)$ that preserve nilpotent matrices where F is infinite. In [2], Li and Pierce characterized linear mappings on $sl_n(F)$ that preserve nonzero nilpotent matrices with rank at most k where k is a fixed positive integer less than n and F is algebraically closed of characteristic zero. Then, Watkins characterized linear mappings from $sl_n(F)$ to $F^{n\times n}$ that preserve rank one matrices where F is an algebraically closed field of characteristic not equal to 2. He applied this result to determine the structure of bilinear mappings on $F^{n\times n}$ that have certain rank-preserving properties in [3] and [4] respectively.

Let $S_n(F)$ be the vector space of all $n\times n$ symmetric matrices over F and $Z_0(S_n(F))$ be its subspace consisting of all symmetric matrices with zero trace. Let $n \geq 4$ and F be a field of characteristic greater than 3. Motivated by work of Lim [5] in the characterization of linear rank one preservers on matrices with zero trace, we characterize those rank 2 preservers on symmetric matrices with zero trace under certain conditions in this paper and will discuss some consequences of this characterization in our next paper.

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2 Some definitions and preliminary results

Let U be a vector space over F. We use tensor language in our investigation. This provides us with a larger context. We denote by $U^{(2)}$ the *second symmetric product space* over U and denoted by $x \cdot y,\ x, y \in U$, the decomposable elements of $U^{(2)}$. For each u in U, let u^2 denote $u \cdot u$.

A *scalar product* on U is a function which assigns a scalar $(x, y) \in F$ to each ordered pair of vectors $x, y \in U$ such that for any $x, y, z \in U$ and any $c \in F$

(i) $(x + y, z) = (x, z) + (y, z)$

(ii) $(cx, y) = c(x, y)$

(iii) $(x, y) = (y, x)$

We say x is *orthogonal* to y or x and y are orthogonal if $(x, y) = 0$. Let S be a set of vectors in U. Then S is called an *orthogonal set* if $(x, y) = 0$ for all $x, y \in S, x \neq y$. If in addition, $(x, x) = 1$ for every $x \in S$, then S is called an *orthonormal set*.

Now we let U be equipped with a scalar product $(,) : U \times U \to F$ and U has an orthonormal basis $\mathcal{E}$. *Let* $Z_0(U^{(2)})$ be the subset of $U^{(2)}$ that consists of all vectors of the form

$$\sum_{1 \le i \le j \le n} a_{ij} u_i \cdot u_j$$

where $\{u_1, \dots, u_n\}$ is an arbitrary finite subset of $\mathcal{E}$ and a_{ij} $(1 \le i \le j \le n)$ are arbitrary scalars in F such that $\sum_{i=1}^{n} a_{ii} = 0$. Clearly $Z_0(U^{(2)})$ is a subspace of $U^{(2)}$ and we call $Z_0(U^{(2)})$ the *space of traceless* 2^{nd} *order symmetric tensors* over U.

Proposition 2.1 *If* $\{e_i : i \in A\}$ *where* $A \supseteq \{1, 2\}$ *is an orthonormal basis for U, then* $B = \{e_i \cdot e_j, e_1^2 - e_k^2 : i \neq j, k \neq 1 \text{ and } 1, i, j, k \in A\}$ *is a basis for* $Z_0(U^{(2)})$.

Proof.
Clearly B is a linearly independent set. Hence it is sufficient to show that B spans $Z_0(U^{(2)})$.

Let $x \in Z_0(U^{(2)})$. Then $x = \sum_{1 \le i \le j \le n} a_{ij} u_i \cdot u_j$ and $\sum_{i=1}^{n} a_{ii} = 0$ where $\{u_1, \dots, u_n\}$ is a finite subset of $\{e_i : i \in A\}$ and a_{ij} $(1 \le i \le j \le n)$ are scalars in F. It follows that

$$x = \sum_{1 \le i \le j \le n} a_{ij} u_i \cdot u_j - \sum_{k=1}^{n} a_{kk} (e_1^2 - u_k^2).$$

Therefore, B spans $Z_0(U^{(2)})$. □

Let $Z_0(S_n(F))$ denote the subspace of $S_n(F)$ such that for any $A \in Z_0(S_n(F))$, $tr(A) = 0$. If U is a finite dimensional vector space with an orthonormal basis $\{e_i : i = 1, \dots, n\}$, then $Z_0(U^{(2)})$ is isomorphic in a natural way to $Z_0(S_n(F))$ by the restricted isomorphism $\varphi|_{Z_0(U^{(2)})}$ where φ is the isomorphism from $U^{(2)}$ to $S_n(F)$ defined by $\varphi(e_i \cdot e_j) = E_{ij} + E_{ji}, 1 \le i \le j \le n$.

Remark. If U is a Euclidean space, then there does not exist any rank 1 vector in $Z_0\left(U^{(2)}\right)$.

Let J_k denote the set of vectors in $U^{(2)}$ of the form $\sum_{i=1}^{k} \lambda_i x_i^2$, where $x_1, ..., x_k$ are linearly independent vectors and $\lambda_1, ..., \lambda_k \in F \setminus \{0\}$. For each vector $u \in U$, let $u \cdot U = \{u \cdot v : v \in U\}$.

Lemma 2.2 *Let M be a subspace of* $U^{(2)}$ *such that* $M \subseteq \{0\} \cup J_1 \cup J_2$. *Then either*

(i) $M \subseteq W^{(2)}$ *for some subspace W of U that is* 2 *dimension* or

(ii) $M \subseteq u \cdot U$ *for some* $u \in U \setminus \{0\}$.

Proof.

If $M \cap J_2 = \varnothing$, then clearly $M = \langle x^2 \rangle$ for some x in U. Let $A_1 = au_1^2 + u_2^2 \in J_2 \cap M$. Assume that $M \not\subset V^{(2)}$ where $V = \langle u_1, u_2 \rangle$. Then it is clear that there exists $A_2 = bu_3^2 + v^2 \in J_2 \cap M$ where u_1, u_2, u_3 are linearly independent. Clearly $v \in \langle u_1, u_2, u_3 \rangle$, otherwise $A_1 + A_2 \in J_4$, a contradiction. Let $v = cu_1 + du_2 + eu_3$ where $c, d, e \in F$.

Since for any $\lambda \in F$,

$$\lambda A_1 + A_2 = (\lambda a + c^2)u_1^2 + (\lambda + d^2)u_2^2 + (b + e^2)u_3^2$$
$$+2cdu_1 \cdot u_2 + 2ceu_1 \cdot u_3 + 2deu_2 \cdot u_3 \in J_1 \cup J_2,$$

it follows that

$$\begin{vmatrix} \lambda a + c^2 & cd & ce \\ cd & \lambda + d^2 & de \\ ce & de & b + e^2 \end{vmatrix}$$
$$= a(b + e^2)\lambda^2 + b(ad^2 + c^2)\lambda$$
$$= 0$$

for any $\lambda \in F$. Since $|F| \ge 3$, we have

$$b + e^2 = 0 \tag{1}$$

and

$$ad^2 + c^2 = 0. \tag{2}$$

From (1) we have

$$A_2 = (cu_1 + du_2) \cdot (cu_1 + du_2 + 2eu_3). \tag{3}$$

From (2) we have $adc^{-1} + d^{-1}c = 0$ and hence

$$A_1 = (cu_1 + du_2) \cdot (ac^{-1}u_1 + d^{-1}u_2). \tag{4}$$

From (3) and (4) we have $A_1 = u \cdot y_1$, $A_2 = u \cdot y_2$ where u, y_1, y_2 are linearly independent.

Suppose now $A = x^2 \in J_1 \cap M$. Then $A + A_1, A + A_2 \in J_1 \cap J_2$ imply that $x \in \langle u, y_1 \rangle \cap \langle u, y_2 \rangle = \langle u \rangle$ and hence $A \in u \cdot U$.

Suppose that $B = \lambda_1 v_1^2 + \lambda_2 v_2^2 \in J_2 \cap M$. For each $\lambda \in F$, let $C_\lambda = u \cdot z_\lambda$, where $z_\lambda = y_1 + \lambda y_2$. Then $C_\lambda \in M$. Clearly there exists a subset D of F with 2 elements such that $\langle v_1, v_2\rangle \neq \langle u, z_\lambda\rangle$ for all $\lambda \in D$. Hence $\dim\langle v_1, v_2, u, z_\lambda\rangle \geq 3$ for all $\lambda \in D$. By our previous argument, for any $\lambda \in D$, there exists $w_\lambda \in U$ such that

$$B, C_\lambda \in w_\lambda \cdot U .$$

Suppose $B \neq u \cdot U$. Then

$$B = \alpha z_i \cdot z_j$$

for some $\alpha \in F \setminus \{0\}$ where i and j are distinct element in D. Let

$$H = u \cdot z_k$$

where $k \in F \setminus \{i, j\}$. Since $\dim\langle u, z_i, z_j, z_k\rangle = 3$,

$$B, H \in v \cdot U$$

for some $v \in U$ by previous argument. This yields a contradiction since $B = \alpha z_i \cdot z_j$ and $H = u \cdot z_k$ do not have a common factor. Therefore $B = u \cdot U$. □

3 Rank 2 preservers

Let U and W be vector spaces over F. We always assume that U has an orthonormal basis, $\{e_i : i \in \boldsymbol{A}\}$, with respect to a scalar product $(,): U \times U \to F$, where $\boldsymbol{A} \supseteq \{1,2,\ldots,n\}$ if $\boldsymbol{A}$ has at least n elements. For each vector $u \in U$, let $\langle u\rangle^{\perp} = \{v \in U : (v,u) = 0\}$.

Lemma 3.1 *Let* $T : Z_0(U^{(2)}) \to W^{(2)}$ *be a rank 2 preserver. If* V *is a subspace of* U *such that* $V \subseteq \langle u\rangle^{\perp}$ *for some* $u \in U \setminus V$, *then* $\dim T(u \cdot V) = \dim u \cdot V$. *Moreover, if* $\dim V \geq 3$, *then* $T(u \cdot V) \subseteq w \cdot W$ *for some* $w \in W \setminus \{0\}$.

Proof.
Suppose $T(u \cdot v_1) = T(u \cdot v_2)$ for some $v_1, v_2 \in V$. Then $T(u \cdot (v_1 - v_2)) = 0$ and this implies that $v_1 = v_2$, since T is a rank 2 preserver. Hence $\dim T(u \cdot V) = \dim u \cdot V$. If $\dim V \geq 3$, then $\dim(T(u \cdot V)) \geq 3$. Since $T(u \cdot V)$ is a subspace of $W^{(2)}$ contained in $J_2 \cup \{0\}$, it follows by Lemma 2.2 that $T(u \cdot V) \subseteq w \cdot W$ for some $w \in W \setminus \{0\}$. □

Theorem 3.2 Let T be a rank 2 preserver from $Z_0(U^{(2)})$ to $W^{(2)}$. If $\dim U \geq 4$, then one of the following holds:

(i) $T = \lambda P_2(f)\big|_{Z_0(U^{(2)})}$ for some $\lambda \in F \setminus \{0\}$ and some one-to-one linear mapping $f : U \to W$ where $P_2(f)$ is a second induced power of f such that $P_2(f)(x \cdot y) = f(x) \cdot f(y)$;

(ii) $\operatorname{Im} T \subseteq w \cdot W$ for some $w \in W \setminus \{0\}$.

Proof.
Note that $\langle e_i \rangle^{\perp}$ is a subspace of U, $\dim\langle e_i \rangle^{\perp} \geq 3$ and $e_i \notin \langle e_i \rangle^{\perp}$. In view of Lemma 3.1, $T\left(e_i \cdot \langle e_i \rangle^{\perp}\right) \subseteq w_i \cdot W$ for some $w_i \in W \setminus \{0\}$, $i \in A$. Now we have either $\{w_i : i \in A\}$ is a pairwise linearly independent set or $\langle w_i \rangle = \langle w_j \rangle$ for some distinct i, j. We will consider these two cases separately.

Case 1: $\{w_i : i \in A\}$ is a pairwise linearly independent set.

Since $T(e_1 \cdot e_2) \in w_1 \cdot W$, $T(e_2 \cdot e_1) \in w_2 \cdot W$ and w_1, w_2 are linearly independent, we have $T(e_1 \cdot e_2) = \alpha_{12} w_1 \cdot w_2$ for some $\alpha_{12} \in F \setminus \{0\}$. Likewise,

$$T(e_i \cdot e_j) = \alpha_{ij} w_i \cdot w_j\,, \tag{5}$$

where $\alpha_{ij} \in F \setminus \{0\}$ for all distinct i, j. Clearly

$$\alpha_{ij} = \alpha_{ji}\,. \tag{6}$$

Now we claim that $\{w_i : i \in A\}$ is a linearly independent set. Suppose the contrary. Let $w_1 = \sum_{i \in A \setminus \{1\}} a_i w_i$ for some $a_i \in F$. Then from (5), we have

$$\begin{aligned} T\left(e_2 \cdot \left(\frac{1}{\alpha_{12}} e_1 - \sum_{i \in A \setminus \{1,2\}} \frac{a_i}{\alpha_{i2}} e_i\right)\right) &= w_2 \cdot \left(w_1 - \sum_{i \in A \setminus \{1,2\}} a_i w_i\right) \\ &= a_2 w_2^2 \end{aligned}$$

is of rank ≤ 1, a contradiction. So, $\{w_i : i \in A\}$ is a linearly independent set. Let

$$M = \langle (e_1 + e_2) \cdot (e_1 - e_2), (e_1 + e_2) \cdot e_3, (e_1 + e_2) \cdot e_4 \rangle.$$

Since $\langle e_1 - e_2, e_3, e_4 \rangle$ is a 3-dimensional subspace of $\langle e_1 + e_2 \rangle^{\perp}$ and $e_1 + e_2 \notin \langle e_1 - e_2, e_3, e_4 \rangle$, it follows by Lemma 3.1 that $\dim T(M) = 3$ and $T(M) \subseteq u \cdot W$ for some $u \in W \setminus \{0\}$.

Let

$$\begin{aligned} T((e_1 + e_2) \cdot (e_1 - e_2)) &= u \cdot u_1\,, \\ T((e_1 + e_2) \cdot e_3) &= u \cdot u_2\,, \\ T((e_1 + e_2) \cdot e_4) &= u \cdot u_3\,, \end{aligned}$$

where $\dim\langle u_1, u_2, u_3 \rangle = 3$. In view of (5), we have

$$\begin{aligned} T((e_1 + e_2) \cdot e_3) &= T(e_1 \cdot e_3) + T(e_2 \cdot e_3) \\ &= (\alpha_{13} w_1 + \alpha_{23} w_2) \cdot w_3\,, \end{aligned}$$

$$\begin{aligned} T((e_1 + e_2) \cdot e_4) &= T(e_1 \cdot e_4) + T(e_2 \cdot e_4) \\ &= (\alpha_{14} w_1 + \alpha_{24} w_2) \cdot w_4\,. \end{aligned}$$

Hence

$$\begin{aligned} u \cdot u_2 &= (\alpha_{13} w_1 + \alpha_{23} w_2) \cdot w_3\,, \\ u \cdot u_3 &= (\alpha_{14} w_1 + \alpha_{24} w_2) \cdot w_4\,. \end{aligned}$$

Since $\dim\langle w_1, w_2, w_3, w_4 \rangle = 4$, we have $\langle u \rangle = \langle \alpha_{13} w_1 + \alpha_{23} w_3 \rangle = \langle \alpha_{14} w_1 + \alpha_{24} w_2 \rangle$. Hence $\frac{\alpha_{23}}{\alpha_{13}} = \frac{\alpha_{24}}{\alpha_{14}}$. Likewise

$$\frac{\alpha_{ik}}{\alpha_{jk}}=\frac{\alpha_{il}}{\alpha_{jl}} \tag{7}$$

for all distinct i, j,k,l . Now let

$$N=\langle (e_1-e_2)\cdot(e_1+e_2),(e_1-e_2)\cdot e_3,(e_1-e_2)\cdot e_4\rangle .$$

Since $\langle e_1+e_2,e_3,e_4\rangle$ is a 3-dimensional subspace of $\langle e_1-e_2\rangle^{\perp}$ and $e_1-e_2\notin\langle e_1+e_2,e_3,e_4\rangle$, it follows by Lemma 3.1 that $\dim T(N)=3$ and $T(N)\subseteq v\cdot W$, $v\in W\setminus\{0\}$. Let

$$T((e_1-e_2)\cdot(e_1+e_2))=v\cdot v_1,$$
$$T((e_1-e_2)\cdot e_3)=v\cdot v_2,$$
$$T((e_1-e_2)\cdot e_4)=v\cdot v_3.$$

where $\dim\langle v_1,v_2,v_3\rangle=3$. On the other hand, In view of (5)

$$\begin{aligned}T((e_1-e_2)\cdot e_3)&=T(e_1\cdot e_3)-T(e_2\cdot e_3)\\&=(\alpha_{13}w_1-\alpha_{23}w_2)\cdot w_3,\\T((e_1-e_2)\cdot e_4)&=T(e_1\cdot e_4)-T(e_2\cdot e_4)\\&=(\alpha_{14}w_1-\alpha_{24}w_2)\cdot w_4.\end{aligned}$$

Hence

$$v\cdot v_2=(\alpha_{13}w_1-\alpha_{23}w_2)\cdot w_3,$$
$$v\cdot v_3=(\alpha_{14}w_1-\alpha_{24}w_2)\cdot w_4.$$

Since $\dim\langle w_1,w_2,w_3,w_4\rangle=4$, we have

$$\langle v\rangle=\langle \alpha_{13}w_1-\alpha_{23}w_3\rangle=\langle \alpha_{14}w_1-\alpha_{14}w_2\rangle .$$

Now we have

$$T((e_1+e_2)\cdot(e_1-e_2))\in(\alpha_{13}w_1+\alpha_{23}w_2)\cdot W$$

and

$$T((e_1-e_2)\cdot(e_1+e_2))\in\langle \alpha_{13}w_1-\alpha_{23}w_2\rangle\cdot W .$$

Since $\alpha_{13}w_1+\alpha_{23}w_2$ and $\alpha_{13}w_1-\alpha_{23}w_2$ are linearly independent, we have

$$\begin{aligned}T(e_1^2-e_2^2)&=T((e_1+e_2)\cdot(e_1-e_2))\\&=\lambda_{12}^{(3)}(\alpha_{13}w_1+\alpha_{23}w_2)\cdot(\alpha_{13}w_1-\alpha_{23}w_2)\\&=\lambda_{12}^{(3)}(\alpha_{13}^2w_1^2-\alpha_{23}^2w_2^2)\end{aligned}$$

where $\lambda_{12}^{(3)}\in F\setminus\{0\}$. Likewise

$$T(e_i^2-e_j^2)=\lambda_{ij}^{(k)}(\alpha_{ik}^2w_i^2-\alpha_{jk}^2w_j^2) \tag{8}$$

where $\lambda_{ij}^{(k)}\in F\setminus\{0\}$ for all distinct i, j,k . As a consequence

$$\lambda_{ij}^{(k)}\alpha_{ik}^2=\lambda_{ij}^{(l)}\alpha_{il}^2 \tag{9}$$

$$\lambda_{ij}^{(k)}=\lambda_{ji}^{(k)} \tag{10}$$

for all distinct i, j,k,l . Now let

$$u_1=e_1+e_2+e_3,$$
$$u_2=e_1-e_2,$$
$$u_3=e_1-e_3,$$
$$u_4=e_4.$$

Let $H=\left\langle u_1\cdot u_2, u_1\cdot u_3, u_1\cdot u_4\right\rangle$. Since $\left\langle u_2,u_3,u_4\right\rangle$ is a 3 -dimensional subspace of $\left\langle u_1\right\rangle^{\perp}$ and $u_1\notin\left\langle u_2,u_3,u_4\right\rangle$, it follows by Lemma 3.1 that $\dim T(H)=3$ and $T(H)\subseteq v\cdot W$ for some $v\in W\setminus\{0\}$. In view of (5), (7) and (8), we have

$$\begin{aligned} T(u_1\cdot u_2) &= T(e_1^2-e_2^2+e_1\cdot e_3-e_2\cdot e_3)\\ &= \lambda_{12}^{(3)}\left(\alpha_{13}^2w_1^2-\alpha_{23}^2w_2^2\right)+\alpha_{13}w_1\cdot w_3-\alpha_{23}w_2\cdot w_3\\ &= (\alpha_{13}w_1-\alpha_{23}w_2)\cdot(\lambda_{12}^{(3)}\alpha_{13}w_1+\lambda_{12}^{(3)}\alpha_{23}w_2+w_3)\\ T(u_1\cdot u_3) &= T(e_1^2-e_3^2+e_1\cdot e_2-e_3\cdot e_2)\\ &= \lambda_{13}^{(3)}\left(\alpha_{12}^2w_1^2-\alpha_{32}^2w_3^2\right)+\alpha_{12}w_1\cdot w_2-\alpha_{23}w_3\cdot w_2\\ &= (\alpha_{12}w_1-\alpha_{32}w_3)\cdot(\lambda_{13}^{(2)}\alpha_{12}w_1+\lambda_{13}^{(2)}\alpha_{32}w_3+w_2) \end{aligned}$$

Since $(\alpha_{13}w_1-\alpha_{23}w_2)$, $(\alpha_{12}w_1-\alpha_{32}w_3)$ and $(\lambda_{12}^{(3)}\alpha_{13}w_1+\lambda_{12}^{(3)}\alpha_{23}w_2+w_3)$ are pairwise linearly independent and $T(u_1\cdot u_2), T(u_1\cdot u_3)$ have a common factor, we have

$$\left\langle \lambda_{12}^{(3)}\alpha_{13}w_1+\lambda_{12}^{(3)}\alpha_{23}w_2+w_3\right\rangle=\left\langle \lambda_{13}^{(2)}\alpha_{12}w_1+w_2+\lambda_{13}^{(2)}\alpha_{32}w_3\right\rangle.$$

Hence

$$\frac{\lambda_{12}^{(3)}\alpha_{13}}{\lambda_{12}^{(2)}\alpha_{12}}=\frac{\lambda_{12}^{(3)}\alpha_{23}}{1}=\frac{1}{\lambda_{13}^{(2)}\alpha_{32}}.$$

In view of (6), we have

$$\frac{1}{\lambda_{12}^{(3)}}=\lambda_{13}^{(2)}\alpha_{23}\alpha_{32}=\lambda_{13}^{(2)}\alpha_{32}^2,$$

$$\lambda_{13}^{(2)}=\frac{\alpha_{13}}{\alpha_{12}\alpha_{23}}.$$

Likewise,

$$\frac{1}{\lambda_{ij}^{(k)}}=\lambda_{ik}^{(j)}\alpha_{kj}^2 \tag{11}$$

$$\lambda_{ij}^{(k)}=\frac{\alpha_{ij}}{\alpha_{ik}\alpha_{kj}} \tag{12}$$

for all distinct i,j,k. In view of (12) and (7), we have

$$\lambda_{ij}^{(k)}=\frac{1}{\alpha_{ik}}\left(\frac{\alpha_{ij}}{\alpha_{kj}}\right)=\frac{1}{\alpha_{ik}}\left(\frac{\alpha_{il}}{\alpha_{kl}}\right)=\lambda_{il}^{(k)} \tag{13}$$

for all distinct i,j,k,l. In view of (10) and (13), we find that

$$\lambda_{ij}^{(k)} \text{ have a common value } \lambda^{(k)} \tag{14}$$

for all $i,j\notin\{k\}$. Now we define a linear mapping $f:U\to W$ by

$$f(e_i)=\begin{cases}\dfrac{1}{\lambda}w_3, i=3\\[2ex] \alpha_{i3}w_i, i\neq 3\end{cases}$$

where $\lambda = \lambda^{(3)}$. For any distinct elements $i, j, 3$ in A, in view of (5), (8), (9), (11), (12) and (14), we have

$$\begin{aligned}
&\lambda P_2(f)(e_i \cdot e_j) \\
&= \lambda^{(3)} f(e_i) \cdot f(e_j) \\
&= \lambda_{ij}^{(3)} \alpha_{i3} \alpha_{j3} w_i \cdot w_j \\
&= \alpha_{ij} w_i \cdot w_j \\
&= T(e_i \cdot e_j) ,
\end{aligned}$$

$$\begin{aligned}
&\lambda P_2(f)(e_i \cdot e_3) \\
&= \lambda f(e_i) \cdot f(e_3) \\
&= \lambda(\alpha_{i3} w_i) \cdot \left(\frac{1}{\lambda} w_3 \right) \\
&= \alpha_{i3} w_i \cdot w_3 \\
&= T(e_i \cdot e_3) ,
\end{aligned}$$

$$\begin{aligned}
&\lambda P_2(f)(e_i^2 - e_3^2) \\
&= \lambda^{(3)} f(e_i)^2 - \lambda^{(3)} f(e_3)^2 \\
&= \lambda_{ij}^{(3)} \alpha_{i3}^2 w_i^2 - \lambda^{(3)} \left(\frac{1}{\lambda^{(3)}} \right)^2 w_3^2 \\
&= \lambda_{ij}^{(k)} \alpha_{ik}^2 w_i^2 - \frac{1}{\lambda^{(3)}} w_3^2 \qquad \text{where } k \notin \{i, j, 3\} \\
&= \lambda_{ij}^{(k)} \alpha_{ik}^2 w_i^2 - \frac{1}{\lambda_{ik}^{(3)}} w_3^2 \\
&= \lambda_{ij}^{(k)} \alpha_{ik}^2 w_i^2 - \lambda_{i3}^{(k)} \alpha_{3k}^2 w_3^2 \\
&= \lambda^{(k)} (\alpha_{ik}^2 w_i^2 - \alpha_{3k}^2 w_3^2) \\
&= T(e_1^2 - e_3^2) .
\end{aligned}$$

Therefore $T = \lambda P_2(f)\big|_{Z_0(U^{(2)})}$ and f is injective.

Case 2: $\langle w_i \rangle = \langle w_j \rangle$ for some distinct i, j .

Let $w = w_1$. Without loss of generality, we may assume that $w_1 = w_2$. We first show that

$$T\left(e_i \cdot \langle e_i \rangle^{\perp}\right) \subseteq w \cdot W$$

for all $i \in A$.

If w_3 and w are linearly independent, then

$$T(e_1 \cdot e_3) = \alpha w \cdot w_3$$

for some $\alpha \in F \setminus \{0\}$. Similarly, $T(e_2 \cdot e_3) = \beta w \cdot w_3$ for some $\beta \in F \setminus \{0\}$. Hence

$$T(e_3 \cdot (\beta e_1 - \alpha e_1)) = 0 .$$

This contradicts the hypothesis that $\rho(T(e_3 \cdot (\beta e_1 - \alpha e_1))) = 2$. Thus $\langle w_3 \rangle = \langle w \rangle$. Likewise, $\langle w_k \rangle = \langle w \rangle$, for all $k \in A$. Therefore, we can conclude that

$$T\left(e_i \cdot \langle e_i \rangle^{\perp}\right) \subseteq w \cdot W$$

for all $i \in A$.

Now we show that

$$T\left(e_i^2 - e_j^2\right) \in w \cdot W$$

for all distinct i, j.

Let

$$T(e_1 \cdot e_3) = w \cdot z_1,$$
$$T(e_2 \cdot e_3) = w \cdot z_2,$$
$$T(e_1 \cdot e_4) = w \cdot z_3,$$
$$T(e_2 \cdot e_4) = w \cdot z_4,$$

where $z_1, z_2, z_3, z_4 \in W$. Let

$$M = \langle (e_1 + e_2) \cdot (e_1 - e_2), (e_1 + e_2) \cdot e_3, (e_1 + e_2) \cdot e_4 \rangle .$$

Then $\langle e_1 - e_2, e_3, e_4 \rangle$ is a three dimensional subspace of $\langle e_1 + e_2 \rangle^{\perp}$ and in view of Lemma 3.1, $\dim T(M) = 3$ and $T(M) \subseteq v \cdot W$ for some $v \in W$.

Let

$$T((e_1 + e_2) \cdot (e_1 - e_2)) = v \cdot v_1,$$
$$T((e_1 + e_2) \cdot e_3) = v \cdot v_2,$$
$$T((e_1 + e_2) \cdot e_4) = v \cdot v_3,$$

where $\dim \langle v_1, v_2, v_3 \rangle = 3$. On the other hand,

$$T((e_1 + e_2) \cdot e_3) = T(e_1 \cdot e_3) + T(e_2 \cdot e_3) = (z_1 + z_2) \cdot w .$$

$$T((e_1 + e_2) \cdot e_4) = T(e_1 \cdot e_4) + T(e_2 \cdot e_4) = (z_3 + z_4) \cdot w .$$

Since $\dim T(M) = 3$, $z_1 + z_2$ and $z_3 + z_4$ are linearly independent. Then by comparing the two expressions for $T((e_1 + e_2) \cdot e_3)$ and $T((e_1 + e_2) \cdot e_4)$, clearly $\langle w \rangle = \langle v \rangle$ and we obtain

$$T(e_1^2 - e_2^2) \in w \cdot W .$$

Likewise,

$$T(e_i^2 - e_j^2) \in w \cdot W$$

for all distinct i, j . Now we have proved that

$$T\left(e_i \cdot \langle e_i \rangle^{\perp}\right) \subseteq w \cdot W$$

for all $i \in A$, and

$$T(e_i^2 - e_j^2) \in w \cdot W$$

for all distinct i, j. Therefore $\operatorname{Im} T \subseteq w \cdot W$. □

Remark. We conjecture that Theorem 3.2 is also true when $\dim U = 3$. If $\dim U = 2$, Theorem 3.2 is no longer true.

Example Let U be a 2-dimensional Euclidean space with an orthonormal basis $\{e_1, e_2\}$. Let $T : Z_0\left(U^{(2)}\right) \to U^{(2)}$ be a linear mapping such that $T(e_1^2 - e_2^2) = e_1 \cdot e_2$ and $T(e_1 \cdot e_2) = e_1^2 - e_2^2$. Then clearly T is a rank 2 preserver. However T is neither of the form (i) nor the form (ii) in Theorem 3.2.

Now we state Theorem 3.2 in matrix language when the dimension of U and W are finite.

Corollary 3.3 *Let* n *and* m *be positive integers and let* L *be a rank* 2 *preserver from* $Z_0(S_n(F))$ *to* $S_m(F)$. *If* $n \geq 4$, *then one of the following holds:*

(i) *There exists a rank* n $m \times n$ *matrix* P *such that* $L(A) = \lambda PAP^t$ *for all* $A \in Z_0(S_n(F))$ *where*
$\lambda \in F \setminus \{0\}$;

(ii) *There exists a nonsingular* m*-square matrix* Q *such that*

$$\operatorname{Im} L \subseteq \left\{ Q \begin{pmatrix} c_1 & c_2 & \cdots & c_m \\ c_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_m & 0 & \cdots & 0 \end{pmatrix} Q^t : c_j \in F, j = 1,2,\ldots,m \right\}.$$

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