

An alternative approach in finding the stationary queue length distribution of a queueing system with negative customers

Siew Khew Koh^{1*}, *Ching Herny Chin*¹, *Yi Fei Tan*², and *Tan Ching Ng*³

¹Centre for Mathematical Sciences, Universiti Tunku Abdul Rahman, Jalan Sungai Long, Bandar Sungai Long, 43000 Kajang, Selangor, Malaysia

²Multimedia University, Persiaran Multimedia, 63100 Cyberjaya, Selangor, Malaysia

³Universiti Tunku Abdul Rahman, Jalan Sungai Long, Bandar Sungai Long, 43000 Kajang, Selangor, Malaysia

Abstract. A single-server queueing system with negative customers is considered in this paper. One positive customer will be removed from the head of the queue if any negative customer is present. The distribution of the interarrival time for the positive customer is assumed to have a rate that tends to a constant as time t tends to infinity. An alternative approach will be proposed to derive a set of equations to find the stationary probabilities. The stationary probabilities will then be used to find the stationary queue length distribution. Numerical examples will be presented and compared to the results found using the analytical method and simulation procedure. The advantage of using the proposed alternative approach will be discussed in this paper.

1 Introduction

The development of queueing system with negative customers can be dated back to the work of Gelenbe [1, 2]. It is known as G-queue in acknowledgement of Gelenbe's introduction of the model. The simplest form of killing strategy in G-queue is removal of one positive customer at the head (RCH) or at the end (RCE) immediately by a negative customer that arrives to the queue. Various removal disciplines of a discrete-time queueing system were studied by Atencia and Moreno [3]. Since the works of Gelenbe (see [1, 2] and [4-7]), G-queue has drawn great attention from researchers to apply the concept in diverse fields, such as manufacturing system, industrial engineering, computer and communication networks. Survey on the analysis and applications of G-queue can be referred to [7-10].

In Harrison and Pitel [11], an $M/M/1$ G-queue was considered and the sojourn time distribution was found by using Laplace transform. They [12] extended the idea to a tandem pair G-queue and the response time distribution was computed. The expressions to find the equilibrium queue length distribution of an $M/G/1$ queue were also provided by them in [13]. Similar to Harrison and Pitel [11], Shin [14] derived expressions to find the sojourn time distribution by taking into consideration various killing disciplines on a multi-server G-

*Corresponding author: kohsk@utar.edu.my

queue. The original concept of Gelenbe's work was generalised by Zhu and Zhang [15] in which negative customer may also receive service when arrives to the queue. Abbas and Assani [16] investigated the performance of the system that is affected by negative customers in a $G|M/1$ queue. Recently, the concept of G-queue has also been applied by Teoh [17] in modelling the framework for delay in aircraft arrival. Most of the early work on G-queue focus on continuous-time queueing system. The discrete-time counterparts have recently received increasing attention due to its application in digital communication systems and mobile network. For discrete-time G-queue, the readers may refer to [18-22].

An alternative approach will be presented in this paper to find the G-queue's stationary queue length distribution. RCH killing discipline is considered and the interarrival time of the positive customers is assumed to have a constant asymptotic rate as time t tends to infinity. We call such distribution a CAR distribution. Similar model has been considered in [23]. However, the service time for positive customer in [23] is assumed to have a non-exponential distribution while the interarrival time remains as exponentially distributed. The same approach has been successfully applied to the models in [23-27]. The numerical result will be compared to the result computed using the analytical method presented in [24]. Analytical method could be preferred over numerical method if we want an exact result. However, numerical method is more versatile. Moreover when exponential assumption of both the interarrival and service time distributions is relaxed, the expressions derived using the analytical method may have a complicated form and will not be easy to solve. Hence, it is worthwhile to study the proposed method in this paper since simple probabilistic approach is used to derive the equations for the stationary probabilities. The values of the stationary probabilities are then solved by using simple recursive method. Model considered will be discussed in Section 2. The steps for deriving stationary probability equations will be shown in Section 3 and followed by the method to solve these equations in Section 4. Numerical example is shown in Section 5 and in the last section, conclusion remarks will be presented.

2 Model description

A single-server queue with RCH killing discipline is considered. Poisson arrival rate of negative customers is denoted as γ . The service time for positive customers is also assumed to have exponential distribution with rate μ . Let $f(t)$ be the probability density function of the interarrival time of positive customers and $\bar{F}(t)$ is the respective survival function. When $t \rightarrow 0$, $f(t)/\bar{F}(t)$ tends to a constant. The service provided to the positive customer is on first come first serve basis.

3 Stationary probabilities

A set of equations will be derived in this section. These equations will be used to compute for the stationary probabilities. First, segment the time axis into equal size of length Δt . Let $\tau_k = ((k - 1)\Delta t, k\Delta t)$ be the k th interval for $k = 1, 2, 3, \dots$. Denote λ_k as the hazard rate function and

$$\lambda_k = \frac{f(k\Delta t)}{\bar{F}(k\Delta t)}, \quad 1 \leq k \leq J \tag{1}$$

where J is an integer such that $\lambda_j \approx \lim_{k \rightarrow \infty} \lambda_k$. $\lambda_k \Delta t$ will be the approximate probability that given no positive customer arrives in $\tau_1, \tau_2, \dots, \tau_{k-1}$, a positive customer arrives in the interval τ_k .

Suppose that at time $t = 0$, a positive customer arrives and the service provided starts immediately. Let M_k be the state number of the service process at the end of the interval τ_k . M_k is defined as follows:

$$M_k = \begin{cases} 0, & \text{if there is no customer in the system; or} \\ & \text{a negative arrival in } \tau_k, k \geq 1; \text{ or} \\ & \text{a completion of service occurs in } \tau_k, k \geq 1. \\ 1, & \text{if no completion of service occurs in } \tau_k, k \geq 1. \end{cases} \quad (2)$$

Next, let Λ_k and Γ_k be the state numbers for the arrivals of the positive and negative customers, respectively. Then,

$$\Lambda_k = \begin{cases} 0, & \text{if } k = 0; \text{ or} \\ & \text{a positive arrival in } \tau_k, k \geq 1. \\ \min(k, J), & \text{if no positive arrival in } \tau_k, k \geq 1. \end{cases} \quad (3)$$

$$\Gamma_k = \begin{cases} 0, & \text{if } k = 0; \text{ or} \\ & \text{no negative arrival in } \tau_k, k \geq 1. \\ 1, & \text{if a negative arrival in } \tau_k, k \geq 1. \end{cases} \quad (4)$$

Denote m_k as the queue length at the end of the interval τ_k . Let $Z_k = (m_k, \Lambda_k, \Gamma_k, M_k)$ be the state vector at the end of τ_k that the queue length is m_k and

- the state of the positive arrival is Λ_k ;
- the state of the negative arrival is Γ_k ; and
- the state of the service processes is M_k .

Suppose $Z_{k-1} = (m_k - 1, j - 1, 0, 1)$, then the list of possible events that could occur at the end of τ_k with the resulting Z_k is as follows:

1. An arrival of positive customer to the system and $Z_k = (m_k + 1, 0, 0, 1)$.
2. An arrival of negative customer to the system and $Z_k = (m_k - 1, j^*, 1, 0)$.
3. A completion of service occurs and $Z_k = (m_k - 1, j^*, 0, 0)$.
4. None of the above three events occurs, hence $Z_k = (m_k, j^*, 0, 1)$.

where $j^* = \min(j, J)$. If $m_k = 0$ at the end of τ_{k-1} , then the corresponding state vector is $Z_{k-1} = (0, j - 1, 0, 0)$. Hence, the list of possible events with the resulting Z_k is:

1. An arrival of positive customer to the system and $Z_k = (1, 0, 0, 0)$.
2. An arrival of negative customer to the system and $Z_k = (0, j^*, 1, 0)$.
3. No positive or negative arrivals and $Z_k = (0, j^*, 0, 0)$.

Let the probability at the end of τ_k be $p_{m_j r k}^{(k)}$ and the state vector $Z_k = (m, j, r, k)$. Assume that $p_{m_j r k} = \lim_{k \rightarrow \infty} p_{m_j r k}^{(k)}$ exists. If $Z_{k-1} = (2, j - 1, 0, 1)$ and a positive customer arrives in τ_k , we will get

$$P_{3001}^{(k)} = P_{2(j-1)01}^{(k-1)} (\lambda_j \Delta t). \tag{5}$$

When $k \rightarrow \infty$, $P_{3001} \cong P_{2(j-1)01} (\lambda_j \Delta t)$. Similarly, by considering the possible events that could occur in τ_k , the following set of equations will be obtained.

When $m = 0$,

$$P_{mj00} \cong \left(P_{(m+1)(j-1)00} + P_{(m+1)(j-1)01} + P_{(m+1)(j-1)10} \right) (\mu \Delta t) \quad \text{for } 2 \leq j \leq J-1, \tag{6}$$

$$+ \left(P_{m(j-1)00} + P_{m(j-1)10} \right) (1 - \lambda_j \Delta t - \gamma \Delta t)$$

$$P_{mJ00} \cong \left(P_{(m+1)(J-1)00} + P_{(m+1)(J-1)01} + P_{(m+1)(J-1)10} \right) (\mu \Delta t) \tag{7}$$

$$+ \left(P_{(m+1)J00} + P_{(m+1)J01} + P_{(m+1)J10} \right) (\mu \Delta t) ,$$

$$+ \left(P_{m(J-1)00} + P_{m(J-1)10} + P_{mJ00} + P_{mJ10} \right) (1 - \lambda_J \Delta t - \gamma \Delta t)$$

$$P_{mj10} \cong \left(P_{(m+1)(j-1)00} + P_{(m+1)(j-1)01} + P_{(m+1)(j-1)10} \right) (\gamma \Delta t) \quad \text{for } 2 \leq j \leq J-1, \tag{8}$$

$$+ \left(P_{m(j-1)00} + P_{m(j-1)10} \right) (\gamma \Delta t)$$

$$P_{mJ00} \cong \left(P_{(m+1)(J-1)00} + P_{(m+1)(J-1)01} + P_{(m+1)(J-1)10} \right) (\gamma \Delta t) \tag{9}$$

$$+ \left(P_{(m+1)J00} + P_{(m+1)J01} + P_{(m+1)J10} \right) (\gamma \Delta t)$$

$$+ \left(P_{m(J-1)00} + P_{m(J-1)10} + P_{mJ00} + P_{mJ10} \right) (\gamma \Delta t)$$

For $m \geq 0$,

$$P_{m100} \cong P_{(m+1)001} \mu \Delta t, \tag{10}$$

$$P_{m110} \cong P_{(m+1)001} \gamma \Delta t. \tag{11}$$

When $m = 1$,

$$P_{m001} \cong \sum_{j=1}^J \left(P_{(m-1)j00} + P_{(m-1)j10} \right) (\lambda_{\min(j+1, I)} \Delta t). \tag{12}$$

For $m \geq 1$,

$$P_{mj00} \cong \left(P_{(m+1)(j-1)00} + P_{(m+1)(j-1)01} + P_{(m+1)(j-1)10} \right) \mu \Delta t \quad \text{for } 2 \leq j \leq J-1, \tag{13}$$

$$P_{mJ00} \cong \left(P_{(m+1)(J-1)00} + P_{(m+1)(J-1)01} + P_{(m+1)(J-1)10} \right) \mu \Delta t \tag{14}$$

$$+ \left(P_{(m+1)J00} + P_{(m+1)J01} + P_{(m+1)J10} \right) \mu \Delta t$$

$$P_{mj10} \cong \left(P_{(m+1)(j-1)00} + P_{(m+1)(j-1)01} + P_{(m+1)(j-1)10} \right) \gamma \Delta t \quad \text{for } 2 \leq j \leq J-1, \tag{15}$$

$$P_{mJ10} \cong \left(P_{(m+1)(J-1)00} + P_{(m+1)(J-1)01} + P_{(m+1)(J-1)10} \right) \gamma \Delta t \tag{16}$$

$$+ \left(P_{(m+1)J00} + P_{(m+1)J01} + P_{(m+1)J10} \right) \gamma \Delta t$$

$$p_{m101} \cong p_{m001} (1 - \lambda_1 \Delta t - \gamma \Delta t - \mu \Delta t), \tag{17}$$

$$p_{mj01} \cong \left(p_{m(j-1)00} + p_{m(j-1)01} + p_{m(j-1)10} \right) (1 - \lambda_j \Delta t - \gamma \Delta t - \mu \Delta t) \quad \text{for } 2 \leq j \leq J - 1 \tag{18}$$

$$p_{mJ01} \cong \left(p_{m(J-1)00} + p_{m(J-1)01} + p_{m(J-1)10} \right) (1 - \lambda_J \Delta t - \gamma \Delta t - \mu \Delta t) + \left(p_{mJ00} + p_{mJ01} + p_{mJ10} \right) (1 - \lambda_J \Delta t - \gamma \Delta t - \mu \Delta t) \tag{19}$$

For $m \geq 2$,

$$p_{m001} \cong \sum_{i=1}^I \left(p_{(m-1)j00} + p_{(m-1)j01} + p_{(m-1)j10} \right) \lambda_{\min(j+1, J)} \Delta t + p_{(m-1)001} \lambda_1 \Delta t \tag{20}$$

4 Stationary queue length distribution

In this section, the method to solve for the values of Equations (6) to (20) will be presented. First, we set M to be an integer that is large enough such that $p_{(M+1)jrk} \cong 0$ for all j, r, k . If $p_{(M+1)jrk} \cong 0$, Equations (10), (11) and (13) to (16) will be all equal to zero when $m = M$, and by substituting these expressions into Equations (18) and (19), we will get

$$p_{MJ01} \cong p_{M(j-1)01} (1 - \lambda_j \Delta t - \gamma \Delta t - \mu \Delta t) \quad \text{for } 2 \leq j \leq J - 1, \tag{21}$$

$$p_{MJ01} \cong \left(p_{M(J-1)01} + p_{MJ01} \right) (1 - \lambda_J \Delta t - \gamma \Delta t - \mu \Delta t). \tag{22}$$

Substitute Equation (17) into (21) when $j = 2$, we will get an expression of p_{M201} in terms of p_{M001} . Repeat the process of substituting $p_{M(j-1)01}$ into (21) for $3 \leq j \leq J - 1$ and into p_{MJ01} in (22), we will be able to express all p_{Mj01} in terms of p_{M001} for $1 \leq j \leq J$. Let

$$\left(p_{Mj01} \mid p_{M001} \right) \tag{23}$$

be the set of equations that expresses p_{Mj01} in terms of p_{M001} for $1 \leq j \leq J$. By substituting (23) into (10), (11) and (13) to (16) when $m = M - 1$, all p_{Mj00} and p_{Mj10} are expressed in terms of p_{M001} as well. It can be seen that the process of substitution can be continue to Equation (12), (19) and (20) for $m = M$ until all the p_{Mjrk} are expressed in terms of p_{M001} .

The substitution process are then repeated for $m = M - 2, M - 3, \dots, 1, 0$ where

$$\left(p_{mjrk} \mid p_{M001} \right) \tag{24}$$

are the set of equations that expresses p_{mjrk} in terms of p_{M001} . From the inspection, two equations in $\left(p_{0jrk} \mid p_{M001} \right)$ appear to be linearly dependent. Hence, we may substitute one of the equations in $\left(p_{0jrk} \mid p_{M001} \right)$ with

$$\sum_{m=1}^M \sum_j \sum_r \sum_k p_{mjrk} \cong 1. \tag{25}$$

Solving the system of equations, we will get the value of p_{M001} and substituting this value into (24), all stationary probabilities p_{mjrk} will be obtained. These values of p_{mjrk} can then be used to find the stationary queue length distribution by computing the summation

$$P_m = \sum_i \sum_r \sum_k p_{mjrk} \tag{26}$$

for all m .

5 Numerical example

Let the rate of negative arrivals be $\gamma = 0.4$. Service time of the positive customer is exponentially distributed with the rate $\mu = 0.5$. The interarrival time of the positive customers is assumed to have a gamma distribution with parameter $(\kappa, \theta) = (2.6, 0.5)$. Table 1 shows the result obtained by the proposed numerical method and the analytical method in Chin [24]. The values found under the proposed numerical method in Table 1 has also made use of the K -layers Geometrical Linear extrapolation method introduced in [24]. A simulation procedure is also performed to verify the results.

Table 1. Stationary queue length distribution computed from the proposed numerical method, generating function method and simulation procedure $[(\kappa, \theta) = (2.6, 0.5), \gamma = 0.4, \mu = 0.5]$.

Probability that queue length = m			
Queue length, m	Proposed numerical method	Analytical method in [24]	Simulation procedure
0	0.185518	0.18552	0.185721
1	0.211733	0.211733	0.212092
2	0.156691	0.156691	0.15691
3	0.115957	0.115957	0.116106
4	0.085812	0.085813	0.085800
5	0.063504	0.063505	0.063434
6	0.046996	0.046996	0.046892
7	0.034778	0.034779	0.034696
8	0.025737	0.025738	0.025625
9	0.019047	0.019047	0.018924
10	0.014095	0.014095	0.014008
⋮	⋮	⋮	⋮
35	7.53E-06	7.59E-06	3.22E-05

The results in Table 1 shows that the stationary queue length distribution obtained by the proposed numerical method is close to the one computed using the analytical method in [24]. Both the results are verified by the results obtained using a simulation procedure.

6 Conclusion

In this paper, an alternative numerical method has been used to find the stationary queue length distribution. The numerical method was first proposed in Koh [25] and has been successfully applied in some other queueing systems with different features. In Koh et al. [23], the proposed method was applied in a model similar to this paper but the interarrival time of the positive customer remains exponentially distributed whereas the service time is relaxed to the one with CAR distribution. The work can be extended to queueing system with both the interarrival time and service time having a more general distribution. The proposed numerical method can also be applied in discrete-time queue with negative customers which has been receiving increasing attention in the recent years. In Koh [25], the proposed numerical method works well for a discrete-time queue. The drawback of this method is that

small Δt leads to large value of J and hence dimensionality problem may be encountered. This problem can be solved by extrapolating Δt to zero.

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