Coefficient problems for classes $H_q(\varphi)$ and $L_q(\varphi)$

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Abstract. A class of analytic functions is denoted by M. Furthermore, $S \subset M$ includes of analytic, normalized and univalent functions. The main -subclasses of S are starlike functions, S^* and convex functions, C. Recently, many mathematicians studied about the q-derivative operator. Inspired by the ideas from some previous works, we introduce another two new subclasses of M. The coefficient problems in particular the upper bounds of the Fekete-Szegö (F-S) functional for these subclasses were obtained.

1 Introduction

In 1908, Jackson [1] was initiated the ideas of application of q-calculus. Later, in [2], the authors started to apply these ideas to geometric function theory.

First, we consider the functions $g(\delta)$ of the form

$$g(\delta) = \delta + \sum_{k=2}^{\infty} a_k \delta^k \tag{1}$$

These functions are analytic in the open unit disc $U = \{\delta \in \mathbb{C}: |\delta| < 1\}$. The class of functions of the form (1) is denoted by M.

From ([1], [2]), they defined

$$D_q(g(\delta)) = \frac{g(q\delta) - g(\delta)}{(q-1)\delta}, \quad q \neq 1, \delta \neq 0, 0 < q < 1, \tag{2}$$

$$D_q(g(0)) = g'(0).$$

Since we have $g(\delta) = \delta + \sum_{k=2}^{\infty} a_k \delta^k$, then we can deduce the following result:

$$D_q(g(\delta)) = D_q\left(\delta + \sum_{k=2}^{\infty} a_k \delta^k\right) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \delta^{k-1}, \tag{3}$$

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where $[k]_q = \dfrac{1-q^k}{1-q}$ and note that q approaches 1, then $[k]_q$ approaches k.

For an example, let a function $s(\delta) = k\delta$,

$$D_q(s(\delta)) = D_q(k\delta) = \frac{kq\delta - k\delta}{(q-1)\delta} = k, \tag{4}$$

$$\lim_{q\to 1}\left(D_q(s(\delta))\right) = \lim_{q\to 1}(k) = k = s'(\delta) \tag{5}$$

where s' is the ordinary derivative.

Now, we consider analytic functions m and n, we say that m is subordinate to n, written as $m \prec n$, if there exists an analytic function h, $h(0)=0$ and $|h(\delta)|<1$ such that $m(\delta) = n(w(\delta))$. If $m \in S$, then $m \prec n$ if and only if $m(0) = n(0)$ and $m(U) \subseteq n(U)$.

An earlier time, certain researchers discussed about the coefficient estimate which is F-S functional for various type of subclasses of M. After that, some of them started to study the subclasses associated the q-derivative operator (see [3-7]). For example, in [3], they discussed about the bounds of F-S functional of starlike and convex functions regarding the q-derivative. While in [4], the authors found the F-S functional for the function belongs to the class of symmetric and conjugate points. Paper [5] shows the results from some others interesting subclasses, while the authors in [6] discussed q-starlike and q-convex functions of complex order. Lastly, in [7], the authors studied about bi-univalent functions with q-derivative operator.

In this paper, we introduce the subclasses $H_q(\beta)$ and $L_q(\beta)$ of the class M for $0 \le \beta < 1$ which as follows

$$H_q(\beta) = \left\{ g \in M : \operatorname{Re}\left(\frac{\delta D_q g(\delta)}{g(\delta)} + \frac{\alpha q \delta^2 D_q\left(D_q g(\delta)\right)}{g(\delta)} \right) > \beta, \alpha \ge 0, \delta \in U \right\}, \tag{6}$$

$$L_q(\beta) = \left\{ g \in M : \operatorname{Re}\left(\left(\frac{\delta D_q g(\delta)}{g(\delta)} \right)^{\alpha} \left(1 + \frac{\delta q D_q\left(D_q g(\delta)\right)}{g(\delta)} \right)^{1-\alpha} \right) > \beta, \alpha \ge 0, \delta \in U \right\}. \tag{7}$$

We remark that

$$\lim_{q\to 1} H_q(\beta) = H(\beta),$$

$$\lim_{q\to 1} L_q(\beta) = L(\beta).$$

From the inspiration of the principle of subordination and q-derivative of a function $g \in M$, we now give the definitions. Let P be the class of all function φ. This class is analytic and univalent in U. φ is convex with $\varphi(0)=1$ and $\operatorname{Re}(\varphi(\delta)) > 0$ for $\delta \in U$.

Function $g \in H_q(\varphi)$ if it satisfies

$$\frac{\delta D_q g(\delta)}{g(\delta)} + \frac{\alpha q \delta^2 D_q(D_q g(\delta))}{g(\delta)} \prec \varphi(\delta), \quad (\varphi \in P). \tag{8}$$

Function $g \in L_q(\varphi)$ if it satisfies

$$\left(\frac{\delta D_q g(\delta)}{g(\delta)}\right)^{\alpha} \left(1 + \frac{\delta q D_q\left(D_q g(\delta)\right)}{g(\delta)} \right)^{1-\alpha} \prec \varphi(\delta), \quad (\varphi \in P). \tag{9}$$

We remark that

i. $\lim_{q\to 1} H_q(\varphi) = H(\varphi)$ and $\lim_{q\to 1} L_q(\varphi) = L(\varphi)$,

ii. $\lim_{q\to 1} H_q\left(\frac{1+\delta}{1-\delta}\right) = H$ and $\lim_{q\to 1} L_q\left(\frac{1+\delta}{1-\delta}\right) = L$.

Now, we may find the bounds of the F-S functional for $g \in H_q(\varphi)$ and $g \in L_q(\varphi)$. Before that, the following lemma is needed.

2 Lemma

Lemma 2.1. [8] If $p(\delta) = 1 + c_1\delta + c_2\delta^2 + \cdots$ is a function with positive real part in U and μ is a complex number, then

$$\left|c_2 - \mu c_1^{\,2}\right| \le \left\{1 ; \left|2\mu - 1\right|\right\}. \tag{10}$$

The result is sharp for functions given by

$$p(\delta) = \frac{1+\delta}{1-\delta} \text{ and } p(z) = \frac{1+\delta^2}{1-\delta^2}.$$

3 Results

First, we determine the bound of F-S functional for $g \in H_q(\varphi)$.

Theorem 3.1. Consider $g \in H_q(\varphi)$ with $\varphi(\delta) = 1 + B_1\delta + B_2\delta^2 + \cdots \in P$. Then

$$\left|a_3 - \mu a_2^{\,2}\right| \le \frac{B_1}{q\left(1+\alpha[3]_q\right)[2]_q} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{q\left(1+\alpha[2]_q\right)} - \frac{\mu B_1\left(1+\alpha[3]_q\right)[2]_q}{q\left(1+\alpha[2]_q\right)^2}\right|\right\}. \tag{11}$$

Proof.
If $g \in H_q(\varphi)$, then

$$\frac{\delta D_q g(\delta)}{g(\delta)} + \frac{\alpha q \delta^2 D_q\left(D_q g(\delta)\right)}{g(\delta)} = \varphi\left(w(\delta)\right) \tag{12}$$

Define the function $p(\delta)$ by

$$p(\delta) = \frac{1+w(\delta)}{1-w(\delta)} = 1 + p_1\delta + p_2\delta^2 + \cdots \tag{13}$$

Note that w is a Schwarz function, thus $p(\delta)$ is in the class P. Let

$$d(\delta) = \frac{\delta D_q g(\delta)}{g(\delta)} + \frac{\alpha q \delta^2 D_q(D_q g(\delta))}{g(\delta)} = 1 + d_1\delta + d_2\delta^2 + \cdots \tag{14}$$

From equations (12), (13) and (14), we get

$$d(\delta) = \varphi\left(\frac{p(\delta)-1}{p(\delta)+1}\right). \tag{15}$$

Since

$$\frac{p(\delta)-1}{p(\delta)+1} = \frac{1}{2}\left[p_1\delta + \left(p_2 - \frac{p_1^2}{2}\right)\delta^2 + \left(p_3 + \frac{p_1^2}{4} - p_1p_2\right)\delta^3 + \cdots\right] \tag{16}$$

thus, we get

$$\varphi\left(\frac{p(\delta)-1}{p(\delta)+1}\right) = 1 + \frac{1}{2}B_1p_1\delta + \left[\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2\right]\delta^2 + \cdots \tag{17}$$

From the equations (14) and (17), we obtain

$$d_1 = \frac{1}{2}B_1p_1 \tag{18}$$

and

$$d_2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2 . \tag{19}$$

Next, we can get

$$\frac{\delta D_q g(\delta)}{g(\delta)} + \frac{\alpha q\delta^2 D_q(D_q g(\delta))}{g(\delta)} = 1 + (1+\alpha[2]_q)qa_2\delta + \{(1+\alpha[3]_q)[2]_q a_3 - (1+\alpha[2]_q)a_2^2\}q\delta^2 + \cdots \tag{20}$$

Then from (14), we get

$$d_1 = \left(1+\alpha[2]_q\right)qa_2 \tag{21}$$

and

$$d_2 = \left(1+\alpha[3]_q\right)q[2]_q a_3 - \left(1+\alpha[2]_q\right)qa_2^2 \tag{22}$$

or equivalently

$$a_2 = \frac{B_1p_1}{2q\left(1+\alpha[2]q\right)} \tag{23}$$

and

$$a_3 = \frac{B_1}{2q\left(1+\alpha[3]_q\right)[2]_q}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_2p_1^2}{4q\left(1+\alpha[3]_q\right)[2]_q} + \frac{B_1^2p_1^2}{4q^2\left(1+\alpha[3]_q\right)\left(1+\alpha[2]_q\right)[2]_q} . \tag{24}$$

Therefore

$$a_3 - \mu a_2^2 = \frac{B_1}{2q\left(1+\alpha[3]_q\right)[2]_q}\left(p_2 - vp_1^2\right) \tag{25}$$

where

$$v = \frac{1}{2} - \frac{B_2}{2B_1} - \frac{B_1}{2q\left(1+\alpha[2]_q\right)} + \frac{\mu B_1\left(1+\alpha[3]_q\right)[2]_q}{2q\left(1+\alpha[2]_q\right)^2} . \tag{26}$$

By using Lemma 2.1, then the result is followed. This finalises the proof of Theorem 3.1.

Similar approaches are applied to the class $L_q(\varphi)$ and the result as follows.

Theorem 3.2. Consider $g \in L_q(\varphi)$ with $\varphi(z) = 1 + B_1\delta + B_2\delta^2 + \cdots \in P$. Then

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{qs_3[2]_q} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{tB_1}{q^2[2]_q s_2^2 s_3} + \frac{\mu[2]_q s_3}{2qs_2^2}\right|\right\} \tag{27}$$

where

$$t = \alpha - \frac{\alpha(\alpha-1)}{2} q\left(1+[2]_q^2\right) + (\alpha-1)[2]_q\left(\alpha q - [2]_q\right) \text{ and } s_k = \alpha + (1-\alpha)[k]_q .$$

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