

Solving Intuitionistic Fuzzy Transport Equations by Intuitionistic Fuzzy Laplace Transforms

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Abstract

In this paper, we use an intuitionistic fuzzy Laplace transforms for solving intuitionistic fuzzy hyperbolic equations precisely the transport equation with intuitionistic fuzzy data under strongly generalized H-differentiability concept. For this purpose, the intuitionistic fuzzy transport equation is converted to the intuitionistic fuzzy boundary value problem (IFBVP) based on the intuitionistic fuzzy laplace transform. The related theorems and properties are proved in detail. Finally, we solve an example to illustrate this method.

Keywords: Intuitionistic fuzzy number, Intuitionistic fuzzy Laplace transform, Intuitionistic fuzzy ,Intuitionistic fuzzy differential equations, Intuitionistic fuzzy valued function.

1 Introduction

A simple approach to model propagation phenomena that emerge naturally under uncertainty is to use intuitionistic fuzzy partial differential equations (IFPDE). Problems implying time t as an independent variable usually involve parabolic or hyperbolic equations. An archetype of the intuitionistic fuzzy hyperbolic equations is the transport equation, which can appear in many applications such as fluid mechanics, the dynamics of particles interacting with matter (neutrons in a fissile material, photons in a planetary or stellar atmosphere, electrons and holes in a semiconductor, etc).

The notion of intuitionistic fuzzy set was first presented by Atanassov [1, 2] as a generalization of the notion of fuzzy set, that is introduced by Zadeh (1965) [13], the authors in [3, 4] are discussed Fuzzy laplace transform and Solving fuzzy Duffing's equation by the

laplace transform decomposition, the idea of intuitionistic fuzzy metric space and Fuzzy differential systems under generalized metric spaces approach are presented in [7, 9], while in [] the theorem of the existence of the solution for intuitionistic fuzzy transport equations are proved.

This work is motivated by the solution of a intuitionistic fuzzy transport equation using the intuitionistic fuzzy Laplace transform. This appears to be one of the first attempts to solve one of the first attempts to solve these well known intuitionistic fuzzy partial differential equations under a strongly generalized H-differentiability.

This paper is organized as follows: We provide some preliminaries that we will use all along this work, a result for intuitionistic fuzzy laplace transform is discussed in section 3, furthermore, the transport equation with intuitionistic fuzzy data is presented and solved with intuitionistic fuzzy laplace transform method in section 4, and the conclusion is made in section 5.

2 Preliminaries

An intuitionistic fuzzy set $A \in X$ is given by

$$A = \{(x, u_A(x), v_A(x)) | x \in X\}$$

Where the function $u_A(x), v_A(x) : X \rightarrow [0, 1]$ define respectively the degree of membership and degree of non-membership of the element $x \in X$ to the set A , which is a subset of X , and for every $x \in X$, $0 \leq u_A(x) + v_A(x) \leq 1$

Obviously, every fuzzy set has the form

$$\{(x, u_A(x), u_{A^c}(x)) | x \in X\}$$

Let us $J = [a, b] \subset \mathbb{R}$ be a compact interval. We denote by

$$\mathbb{IF}_1 = \{(u, v) | \mathbb{R} \rightarrow [0, 1]^2, \forall x \in \mathbb{R}; 0 \leq u(x) + v(x) \leq 1\}$$

the collection of all intuitionistic fuzzy number. An element $\langle u, v \rangle$ of \mathbb{IF}_1 is said an intuitionistic fuzzy number if it satisfies the following conditions

- $\langle u, v \rangle$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.
- The membership function u is fuzzy convex i.e $u(\lambda x_1 + (1 - \lambda)x_2) \geq \min(u(x_1), u(x_2))$.
- The non-membership function v is fuzzy concave i.e $v(\lambda x_1 + (1 - \lambda)x_2) \leq \max(v(x_1), v(x_2))$.
- u is upper semi-continuous and v is lower semi-continuous
- $Supp\langle u, v \rangle = cl\{x \in \mathbb{R} : |v(x) < 1\}$ is bounded.

Definition 2.1 An intuitionistic fuzzy number in parametric form is a pair of functions

$$\langle u, v \rangle = ((\underline{\langle u, v \rangle}^+, \overline{\langle u, v \rangle}^+), (\underline{\langle u, v \rangle}^-, \overline{\langle u, v \rangle}^-))$$

wich satisfy the following requirements:

- $\underline{\langle u, v \rangle}^+(\alpha)$ is a bounded monotonic increasing continuous function,
- $\overline{\langle u, v \rangle}^+(\alpha)$ is a bounded monotonic decreasing continuous function,
- $\underline{\langle u, v \rangle}^-(\alpha)$ is a bounded monotonic increasing continuous function,
- $\overline{\langle u, v \rangle}^-(\alpha)$ is a bounded monotonic decreasing continuous function,
- $\underline{\langle u, v \rangle}^-(\alpha) \leq \overline{\langle u, v \rangle}^-(\alpha)$ and $\underline{\langle u, v \rangle}^+(\alpha) \leq \overline{\langle u, v \rangle}^+(\alpha)$ ($\forall \alpha \in [0, 1]$).

For $\alpha \in [0, 1]$ and $\langle u, v \rangle \in \mathbb{IF}_1$, the upper and lower α -cuts of $\langle u, v \rangle$ are defined by

$$[\langle u, v \rangle]^\alpha = \{x \in \mathbb{R} : v(x) \leq 1 - \alpha\}$$

and

$$[\langle u, v \rangle]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$$

Remark 1 If $\langle u, v \rangle \in \mathbb{IF}_1$, so we can see $[\langle u, v \rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v \rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.

We define $0_{(1,0)} \in \mathbb{IF}_1$ as

$$0_{(1,0)}(t) = \begin{cases} (1, 0) & t = 0 \\ (0, 1) & t \neq 0 \end{cases}$$

Let $\langle u, v \rangle, \langle u', v' \rangle \in \mathbb{IF}_1$ and $\lambda \in \mathbb{R}$, we define the following operations by :

$$(\langle u, v \rangle \oplus \langle u', v' \rangle)(z) = (\sup_{z=x+y} \min(u(x), u'(y)), \inf_{z=x+y} \max(v(x), v'(y)))$$

$$\lambda \langle u, v \rangle = \begin{cases} \langle \lambda u, \lambda v \rangle & \text{if } \lambda \neq 0 \\ 0_{(1,0)} & \text{if } \lambda = 0 \end{cases}$$

For $\langle u, v \rangle, \langle z, w \rangle \in \mathbb{IF}_1$ and $\lambda \in \mathbb{R}$, the addition and scaler-multiplication are defined as follows

$$[\langle u, v \rangle \oplus \langle z, w \rangle]^\alpha = [\langle u, v \rangle]^\alpha + [\langle z, w \rangle]^\alpha,$$

$$[\langle u, v \rangle \oplus \langle z, w \rangle]_\alpha = [\langle u, v \rangle]_\alpha + [\langle z, w \rangle]_\alpha,$$

$$[\lambda \langle z, w \rangle]^\alpha = \lambda [\langle z, w \rangle]^\alpha$$

$$[\lambda \langle z, w \rangle]_\alpha = \lambda [\langle z, w \rangle]_\alpha.$$

Definition 2.2 Let $\langle u, v \rangle$ an element of \mathbb{IF}_1 and $\alpha \in [0, 1]$, we define the following sets:

$$[\langle u, v \rangle]_l^+(\alpha) = \inf\{x \in \mathbb{R} | u(x) \geq \alpha\},$$

$$[\langle u, v \rangle]_r^-(\alpha) = \inf\{x \in \mathbb{R} | v(x) \leq 1 - \alpha\},$$

$$[\langle u, v \rangle]_l^+(\alpha) = \sup\{x \in \mathbb{R} | u(x) \geq \alpha\},$$

$$[\langle u, v \rangle]_r^-(\alpha) = \sup\{x \in \mathbb{R} | v(x) \leq 1 - \alpha\}.$$

Remark 2 $[\langle u, v \rangle]_\alpha = [[\langle u, v \rangle]_l^+(\alpha), [\langle u, v \rangle]_r^+(\alpha)]$,

$$[\langle u, v \rangle]^\alpha = [[\langle u, v \rangle]_l^-(\alpha), [\langle u, v \rangle]_r^-(\alpha)]$$

On the space \mathbb{IF}^1 we will consider the following metric,

$$d_\infty(\langle u, v \rangle, \langle z, w \rangle) = \frac{1}{4} \sup_{0 < \alpha \leq 1} \| \langle u, v \rangle_r^+(\alpha) - \langle z, w \rangle_r^+(\alpha) \|$$

$$+ \frac{1}{4} \sup_{0 < \alpha \leq 1} \| \langle u, v \rangle_l^+(\alpha) - \langle z, w \rangle_l^+(\alpha) \|$$

$$+ \frac{1}{4} \sup_{0 < \alpha \leq 1} \| \langle u, v \rangle_r^-(\alpha) - \langle z, w \rangle_r^-(\alpha) \|$$

$$+ \frac{1}{4} \sup_{0 < \alpha \leq 1} \| \langle u, v \rangle_l^-(\alpha) - \langle z, w \rangle_l^-(\alpha) \|$$

Where $\| \cdot \|$ denotes the usual Euclidean norm in \mathbb{R}^n .

Theorem 2.3 (5) The metric space $(\mathbb{IF}_1, d_\infty)$ is complete.

Definition 2.4 Let be $F : J \rightarrow \mathbb{IF}_1$ and $x_0 \in (a, b)$. It is said that F is strongly generalized differentiable on x_0 , if $\exists F'^+(x_0), \exists F'^-(x_0) \in E^1$, such that

- for all $h > 0$ sufficiently small, $\exists F^+(x_0 + h) - F^+(x_0), F^+(x_0) - F^+(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \rightarrow 0} \frac{F^+(x_0+h) - F^+(x_0)}{h} = \lim_{h \rightarrow 0} \frac{F^+(x_0) - F^+(x_0-h)}{h} = F'^+(x_0)$$

Or

- for all $h > 0$ sufficiently small, $\exists F^+(x_0) - F^+(x_0+h), F^+(x_0-h) - F^+(x_0)$ and the limits

$$\lim_{h \rightarrow 0} \frac{F^+(x_0) - F^+(x_0+h)}{-h} = \lim_{h \rightarrow 0} \frac{F^+(x_0-h) - F^+(x_0)}{-h} = F'^+(x_0)$$

Or

- for all $h > 0$ sufficiently small, $\exists F^+(x_0+h) - F^+(x_0), F^+(x_0-h) - F^+(x_0)$ and the limits

$$\lim_{h \rightarrow 0} \frac{F^+(x_0) - F^+(x_0-h)}{h} = \lim_{h \rightarrow 0} \frac{F^+(x_0-h) - F^+(x_0)}{-h} = F'^+(x_0)$$

- for all $h > 0$ sufficiently small, $\exists F^+(x_0) - F^+(x_0+h), F^+(x_0) - F^+(x_0-h)$ and the limits

$$\lim_{h \rightarrow 0} \frac{F^+(x_0) - F^+(x_0+h)}{-h} = \lim_{h \rightarrow 0} \frac{F^+(x_0) - F^+(x_0-h)}{-h} = F'^+(x_0)$$

Lemma 2.1 For $x_0 \in \mathbb{R}$, the fuzzy differential equation $\tilde{y}'(x) = \tilde{f}(x, y)$, $\tilde{y}(x_0) = \tilde{y}_0 \in \mathbb{IF}_1$, where $g : \mathbb{R} \times \mathbb{IF}_1 \rightarrow \mathbb{IF}_1$ is supposed to be continuous, if equivalent to one of the integral equations :

$$\tilde{y}(x) = \tilde{y}_0 \oplus \int_{x_0}^x f(t, \tilde{y}(t)) dt, \forall x \in [x_0, x_1],$$

or

$$\tilde{y}(x) = \tilde{y}_0 \ominus (-1) \int_{x_0}^x f(t, \tilde{y}(t)) dt, \forall x \in [x_0, x_1],$$

3 The Intuitionistic Fuzzy Laplace Transform Method

Theorem 3.1 Let $u(x, t)$ be an intuitionistic fuzzy-valued function on $[a, \infty)$ represented by $(\underline{u}(x, t, \alpha), \bar{u}(x, t, \alpha), \underline{\underline{u}}(x, t, \alpha), \bar{\bar{u}}(x, t, \alpha))$, for all fixed $\alpha \in [0, 1]$, assume that $\underline{u}(x, t, \alpha), \bar{u}(x, t, \alpha), \underline{\underline{u}}(x, t, \alpha), \bar{\bar{u}}(x, t, \alpha)$ are Riemann-integrable on $[a, b]$ for every $b \geq a$, and assume that for positive constants $\underline{M}(\alpha), \bar{M}(\alpha), \underline{\underline{M}}(\alpha)$ and $\bar{\bar{M}}(\alpha)$ such that

$$\begin{aligned} \int_a^\infty |\underline{u}(x, t, \alpha)| dt &\leq \underline{M}(\alpha), \\ \int_a^\infty |\bar{u}(x, t, \alpha)| dt &\leq \bar{M}(\alpha) \\ \int_a^\infty |\underline{\underline{u}}(x, t, \alpha)| dt &\leq \underline{\underline{M}}(\alpha), \\ \int_a^\infty |\bar{\bar{u}}(x, t, \alpha)| dt &\leq \bar{\bar{M}}(\alpha) \end{aligned}$$

Then $u(x, t)$ is an improper intuitionistic fuzzy Riemann-integrable on $[a, \infty)$ and the improper intuitionistic fuzzy Riemann-integral is an intuitionistic fuzzy number.

Furthermore, we have:

$$\int_a^\infty u(x, t) dt = (\int_a^\infty \underline{u}(x, t) dt, \int_a^\infty \bar{u}(x, t) dt, \int_a^\infty \underline{\underline{u}}(x, t) dt, \int_a^\infty \bar{\bar{u}}(x, t) dt).$$

Definition 3.2 Let $u = u(x, t)$ be continuous intuitionistic fuzzy-valued function on $[0, \infty)$, suppose that $u(x, t)e^{-st}$ is intuitionistic fuzzy Riemann-integrable $[0, \infty)$, then we denote by $U(x, s) = \int_0^\infty e^{-st} u(x, t) dt$ the intuitionistic fuzzy Laplace transform, such as : $U(x, s) = L\{u(x, t)\} = \int_0^\infty e^{-st} u(x, t) dt \quad s > 0$.

From theorem 3.1 we have

$$\int_a^\infty u(x, t) e^{-st} dt = (\int_a^\infty \underline{u}(x, t) e^{-st} dt, \int_a^\infty \bar{u}(x, t) e^{-st} dt, \int_a^\infty \underline{\underline{u}}(x, t) e^{-st} dt, \int_a^\infty \bar{\bar{u}}(x, t) e^{-st} dt).$$

Therefore by using the definition of classical Laplace transform, we can present this definition of the intuitionistic fuzzy laplace transform based on the α - cut of the intuitionistic fuzzy valued function as follows :

$$\begin{aligned} U(x, s) &= L\{u(x, t)\} \\ &= ([\underline{L}\{u(x, t, \alpha)\}, \bar{L}\{u(x, t, \alpha)\}]; [\underline{\underline{L}}\{u(x, t, \alpha)\}, \bar{\bar{L}}\{u(x, t, \alpha)\}]) \\ &= ([\int_0^\infty e^{-st} \underline{u}(x, t, \alpha) dt, \int_0^\infty e^{-st} \bar{u}(x, t, \alpha) dt]; \\ &\quad [\int_0^\infty e^{-st} \underline{\underline{u}}(x, t, \alpha) dt, \int_0^\infty e^{-st} \bar{\bar{u}}(x, t, \alpha) dt]) \end{aligned}$$

Theorem 3.3 Let $u'(x, t)$ be an integrable fuzzy-valued function, and $u(x, t)$ is the primitive of $u'(x, t)$ on $[0, \infty)$. Then

$$\begin{aligned} L[\frac{\partial u}{\partial t}(x, t)] &= s.L[u(x, t)] \ominus u(x, 0) \\ &= s.U(x, s) \ominus u(x, 0) \end{aligned}$$

Where u is (i)-differentiable.

Or

$$\begin{aligned} L[\frac{\partial u}{\partial t}(x, t)] &= -u(x, 0) \ominus (-s.L[u(x, t)]) \\ &= -u(x, 0) \ominus (-s.U(x, s)). \end{aligned}$$

Where u is (ii)-differentiable.

Proof 1 We denote by $(\underline{u}(x, t, \alpha), \bar{u}(x, t, \alpha), \underline{\underline{u}}(x, t, \alpha), \bar{\bar{u}}(x, t, \alpha))$ the parametric forms of the $u(x, t)$ and $u'(x, t)$ respectively.

$$\begin{aligned} L[\frac{\partial u}{\partial t}(x, t)] &= \int_0^\infty e^{-sT} \frac{\partial u}{\partial t}(x, t) dT \\ &= s \int_0^\infty u(x, t) e^{-sT} dT \ominus u(x, 0) \\ &= sL\{u(x, t)\} \ominus u(x, 0) \\ &= sU(x, s) \ominus u(x, 0). \end{aligned}$$

since u is (i)-differentiable.

Now we assume that u is the (ii)-differentiable, for $\alpha \in [0, 1]$ we have :

$$\begin{aligned} L\left[\frac{\partial u}{\partial t}(x, t)\right] &= -u(x, 0) \ominus (-s \int_0^\infty u(x, t) e^{-st} dt) \\ &= -u(x, 0) \ominus (-s.L\{u(x, t)\}) \\ &= -u(x, 0) \ominus (-s.U(x, s)) \end{aligned}$$

Theorem 3.4 Let $u(x, t)$ et $v(x, t)$ be continuous intuitionistic fuzzy-valued functions suppose that C_1 and C_2 are constant, then

$$\begin{aligned} L[(C_1 \odot u(x, t)) \oplus (C_2 \odot v(x, t))] &= \\ (C_1 \odot L[u(x, t)]) \oplus (C_2 \odot L[v(x, t)]) &. \end{aligned}$$

Proof 2 Let $L[(C_1 \odot u(x, t)) \oplus (C_2 \odot v(x, t))]$

$$\begin{aligned} &= \int_0^\infty ((C_1 \odot u(x, t)) \oplus (C_2 \odot v(x, t))) \odot e^{-sx} dx \\ &= \int_0^\infty C_1 \odot u(x, t) \odot e^{-sx} dx \oplus \int_0^\infty C_2 \odot v(x, t) \odot e^{-sx} dx \\ &= (C_1 \odot \int_0^\infty u(x, t) \odot e^{-sx} dx) \oplus (C_2 \odot \int_0^\infty v(x, t) \odot e^{-sx} dx) \\ &= C_1 \odot L[u(x, t)] \oplus C_2 \odot L[v(x, t)] \end{aligned}$$

Hence

$$\begin{aligned} L[(C_1 \odot u(x, t)) \oplus (C_2 \odot v(x, t))] &= \\ (C_1 \odot L[u(x, t)]) \oplus (C_2 \odot L[v(x, t)]) &. \end{aligned}$$

Remark 3 Let $u(x, t)$ be continuous intuitionistic fuzzy-valued function on $[0, \infty)$ and $\lambda \geq 0$, then $L[\lambda \odot u(x, t)] = \lambda \odot L[u(x, t)]$.

Proof 3 Intuitionistic fuzzy transform $\lambda \odot u(x, t)$ is denoted as

$$L[\lambda \odot u(x, t)] = \int_0^\infty \lambda \odot u(x, t) \odot e^{-sx} dx$$

and also we have

$$\int_0^\infty \lambda \odot u(x, t) \odot e^{-sx} dx = \lambda \odot \int_0^\infty u(x, t) \odot e^{-sx} dx$$

$$\text{Then } L[\lambda \odot u(x, t)] = \lambda \odot L[u(x, t)]$$

Theorem 3.5 Let $u(x, t)$ be continuous intuitionistic fuzzy-valued function and $L[u(x, t)] = F(t)$, then $L[e^{sx} \odot u(x, t)] = F(t - s)$ where e^{sx} is real value function and $t - s > 0$.

Proof 4 $L[e^{sx} \odot u(x, t)] = \int_0^\infty e^{sx-tx} \odot u(x, t) dx$

$$\begin{aligned} &= \left(\int_0^\infty e^{sx-tx} \underline{u}(x, t) dx, \int_0^\infty e^{sx-tx} \bar{u}(x, t) dx, \right. \\ &\quad \left. \int_0^\infty e^{sx-tx} \underline{\underline{u}}(x, t) dx, \int_0^\infty e^{sx-tx} \bar{\bar{u}}(x, t) dx \right) \\ &= \int_0^\infty e^{-(t-s)x} \odot u(x, t) dx \\ &= F(t - s). \end{aligned}$$

4 Application

Let us consider the following intuitionistic fuzzy non-homogeneous transport equation

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + u = 0 \\ u(x, 0) = \langle [1 + \alpha, 3 - \alpha], [2 - 2\beta, 2 + 2\beta] \rangle \\ u(0, t) = \langle [\alpha, 2 - \alpha], [1 - 2\beta, 1 + \beta] \rangle \end{cases} \quad (1)$$

By applying intuitionistic fuzzy Laplace transforms method, we get

$$L\left[\frac{\partial u}{\partial x}\right] + L\left[\frac{\partial u}{\partial t}\right] + L[u] = 0$$

i.e

$$\begin{cases} \ell\left[\frac{\partial u}{\partial x}\right] + \ell\left[\frac{\partial u}{\partial t}\right] + \ell[u] = 0 \\ \ell\left[\frac{\partial \underline{u}}{\partial x}\right] + \ell\left[\frac{\partial \underline{u}}{\partial t}\right] + \ell[\underline{u}] = 0 \\ \ell\left[\frac{\partial \bar{u}}{\partial x}\right] + \ell\left[\frac{\partial \bar{u}}{\partial t}\right] + \ell[\bar{u}] = 0 \\ \ell\left[\frac{\partial \underline{\underline{u}}}{\partial x}\right] + \ell\left[\frac{\partial \underline{\underline{u}}}{\partial t}\right] + \ell[\underline{\underline{u}}] = 0 \\ \ell\left[\frac{\partial \bar{\bar{u}}}{\partial x}\right] + \ell\left[\frac{\partial \bar{\bar{u}}}{\partial t}\right] + \ell[\bar{\bar{u}}] = 0 \end{cases} \quad (2)$$

Therefore,

$$\begin{cases} \frac{\partial}{\partial x} \underline{U}(x, s) + s \underline{U}(x, s) - \underline{u}(x, 0) + \underline{U}(x, s) = 0 \\ \frac{\partial}{\partial x} \bar{U}(x, s) + s \bar{U}(x, s) - \bar{u}(x, 0) + \bar{U}(x, s) = 0 \\ \frac{\partial}{\partial x} \underline{\underline{U}}(x, s) + s \underline{\underline{U}}(x, s) - \underline{\underline{u}}(x, 0) + \underline{\underline{U}}(x, s) = 0 \\ \frac{\partial}{\partial x} \bar{\bar{U}}(x, s) + s \bar{\bar{U}}(x, s) - \bar{\bar{u}}(x, 0) + \bar{\bar{U}}(x, s) = 0 \end{cases} \quad (3)$$

Hence,

$$\begin{cases} \frac{\partial}{\partial x} \underline{U}(x, s) + s \underline{U}(x, s) + \underline{U}(x, s) = (1 + \alpha) \\ \frac{\partial}{\partial x} \bar{U}(x, s) + s \bar{U}(x, s) + \bar{U}(x, s) = (3 - \alpha) \\ \frac{\partial}{\partial x} \underline{\underline{U}}(x, s) + s \underline{\underline{U}}(x, s) + \underline{\underline{U}}(x, s) = (2 - 2\beta) \\ \frac{\partial}{\partial x} \bar{\bar{U}}(x, s) + s \bar{\bar{U}}(x, s) + \bar{\bar{U}}(x, s) = (2 + 2\beta) \end{cases} \quad (4)$$

$$\begin{cases} e^{(s+1)x} \frac{\partial}{\partial x} \underline{U}(x, s) + e^{(s+1)x} (s+1) \underline{U}(x, s) = e^{(s+1)x} (1 + \alpha) \\ e^{(s+1)x} \frac{\partial}{\partial x} \bar{U}(x, s) + e^{(s+1)x} (s+1) \bar{U}(x, s) = e^{(s+1)x} (3 - \alpha) \\ e^{(s+1)x} \frac{\partial}{\partial x} \underline{\underline{U}}(x, s) + e^{(s+1)x} (s+1) \underline{\underline{U}}(x, s) = e^{(s+1)x} (2 - 2\beta) \\ e^{(s+1)x} \frac{\partial}{\partial x} \bar{\bar{U}}(x, s) + e^{(s+1)x} (s+1) \bar{\bar{U}}(x, s) = e^{(s+1)x} (2 + 2\beta) \end{cases} \quad (5)$$

So,

$$\begin{cases} \frac{\partial}{\partial x} (e^{(s+1)x} \underline{U}(x, s)) = e^{(s+1)x} (1 + \alpha) \\ \frac{\partial}{\partial x} (e^{(s+1)x} \bar{U}(x, s)) = e^{(s+1)x} (3 - \alpha) \\ \frac{\partial}{\partial x} (e^{(s+1)x} \underline{\underline{U}}(x, s)) = e^{(s+1)x} (2 - 2\beta) \\ \frac{\partial}{\partial x} (e^{(s+1)x} \bar{\bar{U}}(x, s)) = e^{(s+1)x} (2 + 2\beta) \end{cases} \quad (6)$$

Thus,

$$\begin{cases} e^{(s+1)x}\underline{U}(x,s) = \int_0^\infty e^{(s+1)r}(1+\alpha)dr \\ e^{(s+1)x}\overline{U}(x,s) = \int_0^\infty e^{(s+1)r}(3-\alpha)dr \\ e^{(s+1)x}\underline{\underline{U}}(x,s) = \int_0^\infty e^{(s+1)r}(2-2\beta)dr \\ e^{(s+1)x}\overline{\overline{U}}(x,s) = \int_0^\infty e^{(s+1)r}(2+2\beta)dr \end{cases} \quad (7)$$

$$\begin{cases} \underline{U}(x,s) = \frac{1+\alpha}{1+s}e^{(s+1)x}e^{-(s+1)x} + e^{-x(s+1)}C_1 \\ \overline{U}(x,s) = \frac{3-\alpha}{1+s}e^{(s+1)x}e^{-(s+1)x} + e^{-x(s+1)}C_2 \\ \underline{\underline{U}}(x,s) = \frac{2(1-\beta)}{1+s}e^{(s+1)x}e^{-x(s+1)} + e^{-x(s+1)}C_3 \\ \overline{\overline{U}}(x,s) = \frac{2(1+\beta)}{1+s}e^{(s+1)x}e^{-x(s+1)} + e^{-x(s+1)}C_4 \end{cases} \quad (8)$$

afterwards, we find the values of the constants C_1, C_2, C_3 and C_4 , by replacing the value of the parametric form of $u(0, t)$ in the system 8, we have

$$\begin{cases} \underline{U}(x,s) = \alpha e^{-(s+1)x} + (1 - e^{-(s+1)x})\frac{(1+\alpha)}{1+s} \\ \overline{U}(x,s) = (2 - \alpha)e^{-(s+1)x} + (1 - e^{-(s+1)x})\frac{(3-\alpha)}{1+s} \\ \underline{\underline{U}}(x,s) = (1 - 2\beta)e^{-(s+1)x} + (1 - e^{-(s+1)x})\frac{2(1-\beta)}{1+s} \\ \overline{\overline{U}}(x,s) = (1 + \beta)e^{-(s+1)x} + 2(1 - e^{-(s+1)x})\frac{(1+\beta)}{1+s} \end{cases} \quad (9)$$

Now, after using the inverse classical Laplace transform, $\underline{u}(x, s), \overline{u}(x, s), \underline{\underline{u}}(x, s)$ and $\overline{\overline{u}}(x, s)$ are calculated as follows

$$\begin{cases} \underline{u}(x,s) = \alpha e^{-x}\ell^{-1}[e^{-sx}] + (1 + \alpha)\ell^{-1}[\frac{1}{1+s}] - (1 + \alpha)e^{-x}\ell^{-1}[\frac{e^{-sx}}{1+s}] \\ \overline{u}(x,s) = (2 - \alpha)e^{-x}\ell^{-1}[e^{-sx}] + (3 - \alpha)\ell^{-1}[\frac{1}{1+s}] - (3 - \alpha)e^{-x}\ell^{-1}[\frac{e^{-sx}}{1+s}] \\ \underline{\underline{u}}(x,s) = (1 - 2\beta)e^{-x}\ell^{-1}[e^{-sx}] + 2(1 - \beta)\ell^{-1}[\frac{1}{1+s}] - 2(1 - \beta)e^{-x}\ell^{-1}[\frac{e^{-sx}}{1+s}] \\ \overline{\overline{u}}(x,s) = (1 + \beta)e^{-x}\ell^{-1}[e^{-sx}] + 2(1 + \beta)\ell^{-1}[\frac{1}{1+s}] - 2(1 + \beta)e^{-x}\ell^{-1}[\frac{e^{-sx}}{1+s}] \end{cases} \quad (10)$$

Hence solutions of (1) is as follows:

$$\begin{cases} \underline{u}(x,s) = \alpha e^{-x}\delta(t-x) + (1 + \alpha)e^{-t}(1 - u(t-x)) \\ \overline{u}(x,s) = (2 - \alpha)e^{-x}\delta(t-x) + (3 - \alpha)e^{-t}(1 - u(t-x)) \\ \underline{\underline{u}}(x,s) = (1 - 2\beta)e^{-x}\delta(t-x) + 2(1 - \beta)e^{-t}(1 - u(t-x)) \\ \overline{\overline{u}}(x,s) = (1 + \beta)e^{-x}\delta(t-x) + 2(1 + \beta)e^{-t}(1 - u(t-x)) \end{cases}$$

where

$$u(t-x) = \begin{cases} 0, & t \leq x \\ 1, & t > x \end{cases}$$

5 Conclusion

We remark that $\underline{u}(x, s) \leq \overline{u}(x, s)$ and $\underline{\underline{u}}(x, s) \leq \overline{\overline{u}}(x, s)$ also the functions $\underline{u}(x, s), \underline{\underline{u}}(x, s)$ are increasing with respect to α and the functions $\overline{u}(x, s), \overline{\overline{u}}(x, s)$ are decreasing with respect to α .

So, this we shown that $(\underline{u}(x, s), \overline{u}(x, s), \underline{\underline{u}}(x, s), \overline{\overline{u}}(x, s))$ is the parametric form of the solution of the problem (1).

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